Existence and Uniqueness for Parabolic Problems with Caputo Time Derivative.

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In this talk we are interested in evolution equations whose general form is

$$\partial_t^\alpha u + F(x,t,u,Du,D^2u,\mathcal{I}(u)) = 0 \quad \text{in } Q, \tag{0.1}$$

where $Q := \mathbb{R}^N \times (0,+\infty)$, complemented with the initial condition

$$u = u_0 \quad \text{in } \mathbb{R}^N \times \{0\}, \tag{0.2}$$

with $u_0 \in C(\mathbb{R}^N)$ bounded. In (0.1), the function $F \in C(\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N\times N} \times \mathbb{R})$ is an elliptic nonlinearity. It is also considered as integro-differential operator in the space variable which, at the introductory level can be considered through the model form

$$\mathcal{I}(u,x) = \int_{\mathbb{R}^N} [u(x+z) - u(x) - \mathbf{1}_B \langle Du(x), z\rangle]\nu(dz),$$

where ν is a nonnegative measure whose basic assumption is the *Lévy integrability condition*

$$\int_{\mathbb{R}^N} \min\{1,|z|^2\}\nu(dz) < +\infty.$$

The main particularity of (0.1) is the presence of an integro-differential operator in the time variable instead of the classical first-order time derivative arising in the standard parabolic approach. More specifficaly, for $\alpha \in (0,1)$ fixed, D_t^α denotes the Caputo fractional time derivative of α-th order defined as

$$\partial_t^\alpha u(t) = \Gamma(1-\alpha)^{-1} \int_0^t (t-\xi)^{-\alpha} u'(\xi) d\xi,$$

where Γ denotes the Gamma function.

We are interested in the well-posedness of fully nonlinear Cauchy problem (0.1)-(0.2) in which the time derivative is of Caputo type. We address this question in the framework of vicosity solutions, obtaining the existence via Perron's method, and comparison for bounded sub and supersolutions by a suitable regularization through inf and sup convolution in time. As an application, we prove the steady-state large time behavior in the case of proper nonlinearities and provide a rate of convergence by using the Mittag-Leffler operator.