THE TWO MEMBRANES PROBLEM FOR FULLY NONLINEAR OPERATORS (ABSTRACT)

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The two membranes problem was first studied by Vergara-Caffarelli [VC71] in the context of variational inequalities to describe the equilibrium position of two elastic membranes in contact with each other that are not allowed to cross. He considered the linear elliptic case, in which the problem can be reduced to the classical obstacle problem by looking at the difference between the two functions representing the position of each membrane.

Nearly 35 years later, Silvestre [Sil] studied the problem for a nonlinear operator in divergence form. He obtained the optimal $C^{1,1}$ regularity of solutions together with a characterization of the regularity of the free boundary, that is the boundary of the set where the two functions coincide. The strategy in his proof was to show that the difference of the two functions satisfies an obstacle problem for the linearized operator, for which the regularity theory of the solutions and the free boundary are well known. An important remark is that in both of this cases the operator governing the behavior of each function is the same.

In a recent paper, Caffarelli, De Silva and Savin [CDS16] considered the two membranes problem for (possibly nonlocal) different operators, i.e. they consider the case in which one of the membranes (say the lower one) satisfies an equation that has higher order with respect to the other one. Here, heuristically, the lower order operator can be treated as a perturbation and some regularity for the lower membrane is obtained. Regularity from the upper membrane can then be deduced by solving an obstacle problem (with the lower membrane as obstacle) and obtaining estimates for solutions of nonlocal obstacle type problems in which the obstacle is not smooth. Repeating this arguments, the optimal regularity is achieved.

We also point out that the problem has been studied by several authors in the general case of N membranes, see [CV85], [CCVC05], [ARS05].

Here, motivated by a model from mathematical finance, we consider a version of the two membrane problem for two different fully nonlinear operators. It is worth pointing out that, for the case of two different operators of the same order, the only result available (to the best of the authors' knowledge) is the Hölder regularity obtained in [CDS16] (see the Introduction there for a discussion of the difficulties of this problem). In this paper we prove $C^{1,\alpha}$ regularity of the solution pair for (concave or convex) operators satisfying a sort of compatibility condition (see (0.2) below) and $C^{1,1}$ regularity for the case of the Pucci extremal opeartors, which is optimal. Moreover, we give an explicit example that shows that no regularity can be expected to hold for the free boundary in general.

The precise statemente of the problem we consider is the following: given two functions $u_0, v_0 \in C^\gamma(\partial B_1)$ and $f, g \in C^\gamma(B_1)$ for some $\gamma \in (0,1)$, we want to study the solutions u and v of

$$\begin{cases} u & \geq & v & \text{in } B_1 \\ F(D^2u) & \leq & f(x) & \text{in } B_1 \\ G(D^2v) & \geq & g(x) & \text{in } B_1 \\ F(D^2u) & = & f(x) & \text{in } B_1 \cap \Omega \\ G(D^2v) & = & g(x) & \text{in } B_1 \cap \Omega \end{cases} \tag{0.1}$$

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where

$$\Omega := \{u > v\},$$

F is convex and

$$G(X) = -F(-X). \tag{0.2}$$

Equation (0.1) models a so called “bid and ask” situation in which we have an asset, a seller (represented by u) and a buyer (represented by v). The price of the asset is random and the transaction will only take place when u and v “agree on a price”, i.e. when $u = v$. Moreover, we want to model the expected earnings of u and v, assuming that their strategy is optimal.

One can think of this problem as having two different (although related) features: on one hand, we have an “obstacle type” situation, in which u tries to maximize gain with v being an obstacle and vice versa (v minimizing cost and u being an obstacle), hence the constraint $u \geq v$. But perhaps more interesting is the special relation between u and v. Because of the “bid and ask” nature of the model, the Bellman type equations that govern the behavior of our solutions are closely related (recall (0.2)) and it is precisely this feature which opens a way to get regularity even though the operators are different. The main result in our work is the following:

Theorem 0.1. *Let u and v solve* (0.1) *in the viscosity sense with $F = \mathcal{M}^+$ and $G = \mathcal{M}^-$. Then u and v are $C^{1,1}$ in $B_{1/4}$ and*

$$\|D^2u\|_{L^\infty(B_{1/4})}, \|D^2v\|_{L^\infty(B_{1/4})} \leq C$$

where C depends only on $n, \lambda, \Lambda, \|f\|_{C^\gamma(B_1)}, \|g\|_{C^\gamma(B_1)}, \|v\|_{L^\infty(B_1)}$ and $\|u\|_{L^\infty(B_1)}$.

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