Two different fractional Stefan problems which are convergent to the same classical Stefan problem

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Abstract: Two fractional Stefan problems are considered. The first one by using the Caputo[1] derivative of order $\alpha \in (0,1)$:

$$\begin{array}{lll} (i) & {}_0^C D_t^\alpha u(x,t) = \lambda^2 \frac{\partial}{\partial x^2} u(x,t), & 0 < x < s(t),\ 0 < t < T, \\ (ii) & u(x,0) = f(x), & 0 \le x \le b = s(0), \\ (iii) & u(0,t) = g(t), & 0 < t \le T, \\ (iv) & u(s(t),t) = 0, & 0 < t \le T, \\ (v) & {}_0^C D^\alpha s(t) = -u_x(s(t),t), & 0 < t \le T, \end{array} \tag{1}$$

and the second one by using the Riemann-Liouville[2] derivative of order $\alpha \in (0,1)$:

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t} u(x,t) = \lambda \frac{\partial}{\partial x} \left({}_0^{RL} D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t) \right), & 0 < x < s(t),\ 0 < t < T, \\ (ii) & u(x,0) = f(x), & 0 \le x \le b = s(0), \\ (iii) & u(0,t) = g(t), & 0 < t \le T, \\ (iv) & u(s(t),t) = 0, & 0 < t \le T, \\ (v) & \frac{d}{dt} s(t) = - {}_0^{RL} D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t) \Big|_{(s(t),t)}, & 0 < t \le T. \end{array} \tag{2}$$

In the limit case ($\alpha = 1$) both problems coincide with the same classical Stefan problem. Some works foccused in problems like these are [1, 3, 4, 6, 7]. Fractional diffusion equations like $(1-i)$ or $(2-i)$), are linked to the modeling of diffusive processes in heterogeneous media, such called sub or super diffusive processes (see [2, 5, 8]).

The Riemann–Liouville fractional derivative ${}_0^{RL} D_t^{1-\alpha}$ is the left inverse operator of the fractional Riemann–Liouville integral ${}_0 I_t^{1-\alpha}$[3], so we can apply ${}_0^{RL} D_t^{1-\alpha}$ to both sides of equation $(1-i)$ obtaining the fractional diffusion equation $(2-i)$. But, what

[1] The Caputo derivative is defined by

$${}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial}{\partial t} u(x,\tau)}{(t-\tau)^\alpha} d\tau,$$

[2] The Riemann–Liouville derivative is defined by

$${}_0^{RL} D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha} d\tau,$$

[3] The fractional Riemann–Liouville integral is defined by

$${}_0 I_t^\beta f(x,t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(x,\tau)}{(t-\tau)^{1-\beta}} d\tau,$$

defined for every $\beta > 0$, and Γ is the Gamma function.

happen with the fractional Stefan conditions $(2-v)$ and $(1-v)$?
For example, if we apply ${}_0^{RL}D_t^{1-\alpha}$ to both sides of the Stefan condition $(1-v)$ we get

$$\frac{d}{dt}s(t) = -{}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(s(t),t)$$

which is not exactly condition $(2-v)$, unless $\alpha=1$. In fact, the right side of $(2-v)$ is

$$\begin{aligned} -\ {}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)\Big|_{(s(t),t)} &= -\lim_{x\to s(t)} {}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t) \\ &= -\lim_{x\to s(t)}\frac{\partial}{\partial t}\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\frac{\partial}{\partial x}u(x,\tau)d\tau. \end{aligned} \tag{3}$$

The aim of this paper is to show explicit solutions (in fact, similarity solutions which are given in terms of Wright functions) to problems (1) and (2) respectively, and prove that they are different, which clearly implies that the "fractional Stefan conditions" $(2-v)$ and $(1-v)$ are different and that for fractional derivatives some limits like (3) are not commutative.

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