A principle of relatedness for systems with small delays

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A general result by Krasnoselskii [2] establishes that, if we consider a fixed point operator $K : U \subset C_T(\mathbb{R}, \mathbb{R}^N) \to C_T(\mathbb{R}, \mathbb{R}^N)$ associated to the problem

$$u'(t) = g(t, u(t)), \;\; u(0) = u(T),$$

and $P : G \subset \mathbb{R}^N \to \mathbb{R}^N$ is the Poincaré map, then, under appropriate hypotheses, the Leray-Schauder degree of $I - K$ in U coincides with the Brouwer degree of $I - P$ in G.

In this work, we extend this relatedness principle to a system of DDEs

$$u'(t) = g(u(t), u(t - \tau)) + p(t), \tag{1}$$

where $\tau > 0$, $g : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}^N$ is continuously differentiable and $\Omega \subset \mathbb{R}^N$. In this case the Poincaré map is defined in the infinite-dimensional space $C([-\tau, 0], \mathbb{R}^N)$. Based on the result for $\tau = 0$, we shall prove that the principle holds for small values of τ.

As a consequence, we deduce that, for nearly all, i. e. except a countable set, $T > 0$, if $G(u) := g(u, u)$ is an inward pointing field, then the system with $p = 0$ has an equilibrium $e \in \Omega$ and, furthermore, the index of the Poincaré operator of the linearised system for $\tau = 0$ is equal to -1, then problem (1) has at least two (generically three) $T-$periodic solutions, provided that $p \in C(\mathbb{R}, \mathbb{R}^N)$ is $T-$periodic and close to the origin.

Moreover, extending another result by Krasnoselskii [1], we prove that the previous assumptions imply that the equilibrium is unstable.

References

[1] M. Krasnoselskii, *Translations along trajectories of differential equations*, Translations of Mathematical Monographs 19, American Mathematical Society, Providence (1968).

[2] M. Krasnoselskii and P. Zabreiko, *Geometrical methods of nonlinear analysis.* Springer, Berlin (1984).