A LIMITING OPTIMAL DESIGN PROBLEM GOVERNED BY NON-LOCAL DIFFUSION

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Abstract: We study a $p-$fractional optimal design problem under volume constraint taking special care of the case when p is large, obtaining in the limit profile a free boundary problem modelled by the Hölder Infinity Laplacian operator. A necessary and sufficient condition is imposed in order to obtain the uniqueness of solutions to the limiting problem, and, under such a condition, we find precisely the optimal configuration for the limit problem. We also prove the sharp $C^{0,s}_{\text{loc}}$ regularity for any limiting solution.

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1 INTRODUCTION

In this work we study a minimizing problem for the p-fractional energy (for $p \geq 2$) with a positive data g prescribed outside $\Omega \subset \mathbb{R}^N$ (a bounded and smooth domain) and a restriction on the maximum volume of the support of the involved functions inside Ω. From a mathematical point of view, fixed $0 < \alpha < \mathcal{L}^N(\Omega)$, we consider the following optimization problem:

$$\mathfrak{L}_p^s[\alpha] = \min\left\{[v]_{W^{s,p}(\mathbb{R}^N)} \mid v \in W^{s,p}\left(\mathbb{R}^N\right),\ v = g \text{ in } \mathbb{R}^N \setminus \Omega \text{ and } \mathcal{L}^N(\{v > 0\} \cap \Omega) \leq \alpha\right\}, \qquad (\mathfrak{P}_p^s)$$

where for $0 < s < 1$ fixed, $W^{s,p}\left(\mathbb{R}^N\right) := \left\{u \in L^p(\mathbb{R}^N) : \frac{|u(y)-u(x)|}{|y-x|^{\frac{N}{p}+s}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N)\right\}$ are the *Fractional Sobolev Spaces* and $[u]_{W^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^p}{|y-x|^{N+sp}}dxdy\right)^{\frac{1}{p}}$ is the well-known *Gagliardo semi-norm*. For a modern study about Fractional Sobolev Spaces we recommend the survey [3].

Notice that any minimizer u_p to $(\mathfrak{P}_p^s)$ is a weak solution to the following Dirichlet problem driven by *Fractional p-Laplacian operator*

$$\begin{cases} -\left(-\Delta_{\mathbb{R}^N}\right)_p^s u_p(x) &= 0 & \text{in} & \{u_p > 0\} \cap \Omega \\ u_p(x) &= g(x) & \text{on} & \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$\left(-\Delta_{\mathbb{R}^N}\right)_p^s u_p(x) := C_{N,s,p}.\text{P.V.} \int_{\mathbb{R}^N} \frac{|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))}{|x-y|^{N+ps}}dy.$$

Particularly, we are interested in the asymptotic behaviour, as $p \to \infty$, of optimal shapes to problem $(\mathfrak{P}_p^s)$. The limiting configurations for $p \to \infty$ have been inspired by the works of the second author in the local setting, see [4] and [5] for more details. The non-local character of our problem was also motivated by [6] and references therein. Motivated by formal considerations, we are led to consider the following limiting configuration:

$$\mathfrak{L}_\infty^s[\alpha] = \min\left\{[v]_{C^{0,s}(\mathbb{R}^N)} \mid v \in W^{s,\infty}\left(\mathbb{R}^N\right),\ v = g \text{ in } \mathbb{R}^N \setminus \Omega \text{ and } \mathcal{L}^N(\{v > 0\} \cap \Omega) \leq \alpha\right\}. \qquad (\mathfrak{P}_\infty^s)$$

Finally, we prove that any sequence of minimizers u_p to $\mathfrak{P}_p^s$ converges, up to a subsequence, to a solution u_∞ of the limiting problem $\mathfrak{P}_\infty^s$. Furthermore, we find the associated equation that u_∞ verifies (in an appropriated sense) in its positivity region, $\Omega_\infty^+ := \{u_\infty > 0\} \cap \Omega$, i.e.,

$$-\mathcal{L}_\infty^s[u_\infty](x) := -\left(\sup_{y\in\mathbb{R}^N} \frac{u_\infty(y)-u_\infty(x)}{|x-y|^s} + \inf_{y\in\mathbb{R}^N} \frac{u_\infty(y)-u_\infty(x)}{|x-y|^s}\right) = 0 \quad \text{in} \quad \Omega_\infty^+,$$

where $\mathcal{L}_\infty^s[\cdot]$ is the *Hölder Infinity Laplacian operator*, which is a non-linear, non-local and non-integral operator (compare with [1]).

2 MAIN THEOREMS

Next, we will present our main results which can be found in the manuscript [2].

Theorem 1 ([2, Theorem 1.1]) *Let u_p be a minimizer to ($\mathfrak{P}_p^s$). Then, up to a subsequence, $u_p \to u_\infty$ as $p \to \infty$, uniformly in Ω and weakly in $W^{s,q}(\Omega)$ for all $1 < q < \infty$, where u_∞ minimizes ($\mathfrak{P}_\infty^s$). Furthermore, the extremal values also converge*

$$\mathfrak{L}_p^s[\alpha] \to \mathfrak{L}_\infty^s[\alpha] \quad \text{as} \quad p \to \infty.$$

Finally, the limit $u_\infty \in C^{0,s}_{loc}(\Omega)$ and fulfils

$$-\mathcal{L}_\infty^s[u_\infty](x) = 0 \quad \text{in} \quad \{u_\infty > 0\} \cap \Omega,$$

in the viscosity sense.

We also provide a uniqueness result to limit solutions. For this end, a key ingredient is the following geometric compatibility condition on the data:

$$\alpha \le \mathcal{L}^N \left(\bigcup_{y \in \mathbb{R}^N \setminus \Omega} B_{\left(\frac{g(y)}{[g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}}\right)^{\frac{1}{s}}}(y) \cap \Omega \right). \qquad \textbf{(Comp. Assump.)}$$

It is worth to highlighting that such a condition (**Comp. Assump.**) turns out to be necessary and sufficient in order to obtain uniqueness of solutions to the limit problem.

Theorem 2 ([2, Theorem 1.2]) *Let v_∞ be given by $v_\infty(x) = \sup\limits_{\mathbb{R}^N \setminus \Omega} \left(g(y) - \mathfrak{H}^\sharp |x-y|^s \right)_+$.*

1. *Assume that* (**Comp. Assump.**) *holds. Let $\mathfrak{H}^\sharp$ be the unique positive number such that*

$$\Omega^\sharp := \bigcup_{y \in \mathbb{R}^N \setminus \Omega} B_{\left(\frac{g(y)}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}}}(y) \cap \Omega \quad \text{fulfils} \quad \mathcal{L}^N\left(\Omega^\sharp\right) = \alpha.$$

   Then v_∞ is the unique minimizer for ($\mathfrak{P}_\infty^s$).

2. *On the other hand, if* (**Comp. Assump.**) *does not hold. Then there exists infinitely many minimizers for ($\mathfrak{P}_\infty^s$). Moreover, v_∞ is the minimal solution, in the sense that $v_\infty(x) \le u_\infty(x)$ in Ω for any other minimizer u_∞ to ($\mathfrak{P}_\infty^s$) and verifies*

$$\{v_\infty > 0\} \cap \Omega = \bigcup_{y \in \mathbb{R}^N \setminus \Omega} B_{\frac{g(y)}{\mathfrak{H}^\sharp}}(y) \cap \Omega \quad \text{fulfils} \quad \mathcal{L}^N(\{v_\infty > 0\} \cap \Omega) < \alpha.$$

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REFERENCES

[1] A. CHAMBOLLE, E. LINDGREN AND R. MONNEAU. *A Hölder Infinity Laplacian*. ESAIM Control Optim. Calc. Var. 18 (2012) 799–835.

[2] J.V. DA SILVA AMD J.D. ROSSI. *The limit as $p \to \infty$ in free boundary problems with fractional p-Laplacians*.

[3] E. DI NEZZA, G. PALATUCCI AND E. VALDINOCI. *Hitchhiker's guide to the fractional Sobolev spaces*. Bull. Sci. Math. 136 (2012), no. 5, 521–573.

[4] J.D. ROSSI AND E.V. TEIXEIRA. *A limiting free boundary problem ruled by Aronsson's equation*. Trans. Amer. Math. Soc. 364 (2012), no. 2, 703–719.

[5] J.D. ROSSI AND P. WANG. *The limit as $p \to \infty$ in a two-phase free boundary problem for the p-Laplacian*. Interfaces Free Bound. 18 (2016), 117–137.

[6] E.V. TEIXEIRA AND R. TEYMURAZYAN. *Optimal design problems with fractional diffusion*. J. London Math. Soc. (2) 92 (2015), no. 2, 338–352.