Abstract: Loss of boundary conditions for two classes of nonlinear parabolic equations with dominating gradient terms: (second-order) fully nonlinear and fractional cases.

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We present a contribution to the study of qualitative properties of viscosity solutions of nonlinear parabolic equations whose rate of growth with respect to the gradient variable makes the corresponding term the dominant one in the equation. Specifically, we show that the phenomenon of *loss of boundary conditions* (LOBC, for short; defined below) occurs for two model problems with prescribed boundary and initial value data.

The first is the fully nonlinear case:

$$u_t - \mathcal{M}^-(D^2u) = |Du|^p \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$u|_{\partial\Omega\times(0,T)} = 0, \qquad u(\cdot,0) = u_0 \in C^1(\overline{\Omega}). \tag{2}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain, $T > 0$; $\mathcal{M}^-$ denotes Pucci's (minimal) extremal operator, $\mathcal{M}^-(X) = \inf\{\mathrm{tr}(AX)|\lambda\mathrm{Id} \le A \le \Lambda\mathrm{Id}\}$, where A and X are symmetric $N \times N$ matrices and $0 < \lambda \le \Lambda$; and $p > 2$. The second model problem is of nonlocal character:

$$u_t + (-\Delta)^s u = |Du|^p \quad \text{in } \Omega \times (0,T), \tag{3}$$

$$u|_{\mathbb{R}^N\setminus\Omega\times(0,T)} = 0, \qquad u(\cdot,0) = u_0 \in C^\beta(\overline{\Omega}), \tag{4}$$

where Ω and T are as before, $(-\Delta)^s$ denotes the well-known fractional Laplacian operator with $s \in (0,1)$, p satisfies

$$s + 1 < p < \frac{s}{1-s}, \tag{5}$$

and $\beta < {}^{p-2s}/_{p-1}$; (5) restricts the value of s to $(0.618\ldots,1)$, where $0.618\ldots$ is the constant sometimes called *reciprocal golden ratio*. These restrictions, however, are related to our methods and may not be essential.

For each of our model problems, we prove that *a)* there exists a small time $T^* > 0$ depending only on the initial condition u_0 (specifically, on $\|u_0\|_{C^1(\overline{\Omega})}$ and $\|u_0\|_{C^\beta(\Omega)}$, respectively) and universal constants, such that the corresponding viscosity solution satisfies the boundary data in the classical sense (pointwise); and *b)* LOBC occurs depending on a largeness condition for u_0 given in terms of an eigenfunction of $\mathcal{M}^-$ and $(-\Delta)^s$, respectively. Joint work with Alexander Quaas.

Definitions and remarks on our methods. We study equations (1) and (3) from the viewpoint of (continuous) viscosity solutions. In this context we employ the notion of *generalized* or *viscosity* boundary conditions, which amount to stating the equations hold up to the boundary wherever (2) or (4), respectively, fail to hold pointwise. More precisely: for, e.g. (1), a subsolution satisfies the first part of (2) in the viscosity sense if

$$\min\left\{u(x,t), u_t(x,t) - \mathcal{M}^-(D^2u(x,t)) - |Du(x,t)|^p\right\} \le 0 \quad \text{for all } (x,t) \in \partial\Omega \times (0,T).$$

The corresponding definitions for supersolutions and for sub- and supersolutions of (3) are similar. This notion is motivated by the optimal control problems underlying (1) and (3), and has been used to obtain the existence of solutions defined globally in time, i.e., solving (1) or (3) for any $T > 0$.

LOBC is said to occur if at some point the inequality for *the equation* holds at the boundary while the inequality for the value of the solution fails.

For the LOBC results (part *b)*), we are adapting an argument for the so-called viscous Hamilton-Jacobi equation, which involves the classical Laplacian. The main difficulty is the lack of regularity of the (viscosity) solutions, which a priori are assumed to be merely continuous. This is overcome by employing regularization by inf-sup-convolution. In the fully nonlinear case, we also employ the divergence form of the Pucci operator for radial solutions, then proceed to the general case by comparison arguments.