Global stability analysis of a second order delay differential equation

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It is considered a second order differential equation with one delay

$$\ddot{x} + \gamma x = f(x(t-\tau)), \tag{1}$$

where $f(x) = \alpha x + \beta x^2$, $\gamma, \tau > 0$ and $\alpha, \beta \neq 0$. This model was analyzed in the work of Campbell and LeBlanc (1998) for $\gamma = 2.5$ and $\beta = 0.9$ in relation with a 1:2 resonance. Now, the original model is modified, just to include four parameters, counting the delay τ. Then, equation (1) can be written as

$$\dot{y}_1 = -\gamma y_2 + f(y_2(t-\tau)), \quad \dot{y}_2 = y_1, \tag{2}$$

and the equilibrium points result $Y_1 = (0,0)$ and $Y_2 = \left(0, (\gamma - \alpha)\,\beta^{-1}\right)$. Two approaches were combined to carry out the dynamic study of (2) analytically: one, which is more usual and we call it "in time domain" (Bellman and Cooke, 1963), for simplicity, and other which uses concepts of control theory, named as "frequency domain" (Moiola and Chen, 1996). Both points of view were applied, exploding the advantages of each one. Besides, the Dde-Biftool software (Engelborghs et *al.*, 2002) was employed to check results and show other complex stability issues.

- **Equilibrium points stability**

To analyze the stability of the equilibrium points is necessary to find the zeroes of an equation, which in the case of the trivial one gives $\lambda^2 + \gamma - \alpha e^{-\lambda\tau} = 0$. This follows from the linearization of (2), evaluated at Y_1. If all its roots have negative real parts then Y_1 results asymptotically stable. Changing variables: $z = \lambda\tau$, the last equation becomes $P_1(z) = e^z z^2 + e^z \gamma\tau^2 - \alpha\tau^2 = 0$, this means that one has to locate the roots of an exponential polynomial. Thus, the next result can be proved:

Theorem 1 *It is considered the equation $P_1(z) = 0$, where $\gamma, \tau > 0$ and $\alpha \neq 0$. Let $\tau > 0$ and $r = \left(\frac{\pi}{\tau}\right)^2$. It is supposed that $-\gamma < \alpha < \gamma$, $\gamma \neq k^2 r$, $k \in \mathbb{N} \cup \{0\}$. As $\gamma \in \bigcup_{k=0}^{\infty} (k^2 r, (k+1)^2 r)$, it follows that*
(a) If $k^2 r < \gamma < (k+1)^2 r$ and k is even, then P_1 has all its roots with negative real parts if

$$\gamma - (k+2)^2 r < \alpha < \gamma - k^2 r, \quad 0 < \alpha < -\gamma + (k+1)^2 r.$$

(b) If $k^2 r < \gamma < (k+1)^2 r$ and k is odd, then P_1 has all its roots with negative real parts if

$$\gamma - (k+1)^2 r < \alpha < \gamma - (k-1)^2 r, \quad -\gamma + k^2 r < \alpha < 0.$$

Corollary 1 *The trivial equilibrium of (1) is asymptotically stable if its parameters satisfy the conditions given in Theorem 1.*

- **Hopf curves and other singularities related with the trivial equilibrium Y_1**

1. Fixing a value of the parameter τ (in the $\gamma - \alpha$ plane)

To detect the Hopf bifurcations related with Y_1, one must solve the system $-\omega^2 + \gamma - \alpha\cos\omega\tau = 0$, $\alpha \sin\omega\tau = 0$. Therefore, one obtains the Hopf bifurcation curves, which in this case are the straight lines

$$\bar{\alpha}_k(\gamma) = (-1)^k \gamma + (-1)^{k+1} k^2 r, \text{ where } \quad k \in \mathbb{N} \text{ and } r = \left(\frac{\pi}{\tau}\right)^2. \tag{3}$$

Moreover, in the $\gamma - \alpha$ plane infinite points can be found where resonant double Hopf points appear, which result as the intersection of two $\bar{\alpha}_k$ lines, choosing an odd k_1 and an even $k_2 \neq 0$.

2. Fixing a value of the parameter γ (in the $\tau - \alpha$ plane)

Setting $\gamma > 0$, the conditions of (a) and (b) of Theorem 1 are depicted, in general, as areas between hyperbolas. In this case, the Hopf curves which now depend on τ are

$$\alpha_{(k)}(\tau) = (-1)^k\gamma + (-1)^{k+1}k^2r^2, \quad k \in \mathbb{N}.$$

- **Limit cycles stability**

The stability in the birth of a branch of the periodic solutions, arising from a Hopf bifurcation, is established through the named stability or curvature coefficient σ_0 in the frequency domain. Thus, according with the sign of σ_0, the Hopf bifurcation can be classified as supercritical ($\sigma_0 < 0$) or subcritical ($\sigma_0 > 0$). Moreover, one complex number ξ_1, associated with σ_0, is computed. Notation: $\xi_1(\omega) = \xi_{1,\mathcal{O}}(\omega)$ if k is odd or $\xi_1(\omega) = \xi_{1,\mathcal{E}}(\omega)$ when k is even. Then, it has been attained that

$$\xi_{1,\mathcal{O}}(\omega) = \frac{\beta^2}{2\alpha}\frac{5\gamma + 11\alpha}{(\gamma-\alpha)(3\gamma+5\alpha)}, \quad \xi_{1,\mathcal{E}}(\omega) = \frac{5}{6}\frac{\beta^2}{\alpha(\gamma-\alpha)},$$

and $\mathrm{sign}(\sigma_0) = \mathrm{sign}(\xi_1)$. Taking into account the previous formulas of ξ_1, it can be shown that:

Theorem 2 *Given a Hopf bifurcation point of (2), where* $\gamma, \tau > 0$, $\alpha, \beta \neq 0$ *and* $-\gamma < \alpha < \gamma$, *which satisfies (3) with*

1. k odd. Then

(a) If $-\gamma < \alpha < -\frac{3}{5}\gamma$ *or* $-\frac{11}{5} < \alpha < 0$, *the Hopf bifurcation is supercritical.*

(b) If $-\frac{3}{5}\gamma < \alpha < -\frac{5}{11}\gamma$ *or* $0 < \alpha < \gamma$, *the Hopf bifurcation is subcritical.*

2. k even. Then

(a) If $-\gamma < \alpha < 0$, *the Hopf bifurcation is supercritical.*

(b) If $0 < \alpha < \gamma$, *the Hopf bifurcation is subcritical.*

Remark: Thereby, by means of the last theorem, the stability of an emergent orbit, coming from a Hopf bifurcation, can be obtained in the whole parameter space of model (2).

- **Bifurcations of limit cycles**

It has been analyzed deeply the complex dynamic in the neighborhood of a 1:2 resonance. Fold of cycles, associated with the existence of singularities where $\sigma_0 = 0$, period doubling bifurcations, which are typical of a 1:2 resonance and also torus or Neimark-Sacker bifurcations were found analytically and checked numerically through Dde-Biftool.

Procedure for the analytical determination of cycles bifurcations:

1. Fourth order approximations for the orbits, which are born at a Hopf bifurcation point, are built. This is attained through the Graphical Hopf theorem and its generalizations, in the frequency domain.
2. Using the results of the first item and a Tchebyschev collocation method (with polynomials of grade 10-12) a finite approximation of the cycle monodromy operator is achieved (Butcher and Mann, 2009).
3. From the second point outcomes, relevant Floquet multipliers are computed for one periodic orbit. These results allow to check the accuracy of the approximation as well as place its possible bifurcations.
4. In agreement with the third point conclusions, different cycle bifurcations curves are depicted in the parameter space.

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