Endogenous growth economic model with delay in the control variable.
Dynamic programming approach

Corina G. Averbuj
Centro de Investigación en Economía Teórica y Matemática Aplicada-
Escuela de Economía y Negocios Universidad Nacional de San Martín.

In this working paper, a generalized neoclassic model is studying, where the consumption is endogenous and a temporal delay between the production and the real investment is proposed.

1 Introduction

From the work of Solow (1956), where the economic growth model assumed a constant saving rate and consumption was exogenous, economic theory has been developed in order to understand the macroeconomic temporal dynamics; one way of approaching this study is to propose, as in general equilibrium theory, ordered preferences (utility functions) on which savings and investment decisions are made.

Thus, more general endogenous growth models emerged in the neoclassical theory, where the problem consisted in finding a consumption trajectory that maximized some utility, subject to constraints given by a dynamic system that shows the temporal evolution of the distribution of production between consumption and investment (Fleming, Rishel, 1975).

Understanding that the dynamics in investment is central to the study of aggregate fluctuations, our fundamental interest will be to study optimal control models, where the investment is defined as an operator $\Upsilon_{(t,\mathbf{X}(t))}(\cdot)$, dependent on all the factors that make the construction of capital over time and that also "disaggregates" productive capital according to its maturity time. Thus the functions will not only show the capacity of the economy to make investments effective, but also their productive diversity.

1.1 The problem

This working paper has been developed to study a class of state contrained optimal control problem with distributed delay in the control variable and the associated Hamilton-Jacobi- Bellman equation, extending (Averbuj-Martinez, 2014).Then, our main objetive will be maximize a functional of the form $J(c) = \int_0^{+\infty} u(c)e^{-\rho t}dt$, with the optimal control of one dimensional delay differential equation as:

$$f\left[k(t)\right] - c(t) = \int_t^{t+T} \Upsilon_{(t,\mathbf{X}(t))}(\tau)d\tau$$

$$k' = -(\gamma+\varepsilon)k + \int_{t-T}^{t} \Upsilon_{(\tau,\mathbf{X}(\tau))}(t)e^{-\phi(t-\tau)}d\tau$$

$$k(0) = k_0, \qquad \int_{s}^{s+T} \Upsilon_{(s,\mathbf{X}(s))}(\tau)d\tau = \alpha_0(s) \ \ -T \leq s < 0$$

where $u(\cdot)$ is the utility, $f(\cdot)$ is the production function, $c(\cdot)$ is the consumption, $k(\cdot)$ is the productive capital, $\alpha(\cdot)$ is the investment per cápita, T is the fixed temporal delay.

If $\Upsilon_{(t,\mathbf{X}(t))}(\tau) = \frac{1}{T}\alpha(t)$, we can rewrite our problem. Given the initial conditions, $\eta = (k_0, \alpha_0)$, the objetive operator to maximize will be:

$$J(\eta;\alpha) := \int_0^\infty U([f(k(t;\eta,\alpha) - \alpha(t)])e^{-\rho t}dt$$

under the admisibles strategies set:

$$I(\eta) := \left\{\alpha \in L^2_{loc}(R^+, R) : k(t;\eta,\alpha) > 0, \alpha(t) \leq f(k(t;\eta,\alpha)), t \geq 0 \ ae\right\}$$

And the state equation is given by:

$$k' = -(\gamma+\varepsilon)k + \frac{1}{T}\int_{-T}^{0} \alpha(t+s)e^{\phi s}ds \qquad \text{(eq1)}$$

$$k(0) = k_0, \qquad \alpha(s) = \alpha_0(s) \ \ -T \leq s < 0 \quad \text{(c-i)}$$

$$\text{(c1) } k(t) > 0, \ t \geq 0, \qquad \text{(c2) } \alpha(t) \leq f\left[k(t)\right]$$

Problem 1 *The presence of the delay in (eq1) renders applying the dynamic programming techniques to the problem in its current form impossible. Then the optimal control problem is embedded in a suitable Hilbert space* $H = \mathbb{R} \times W^{1,2}([-T,0], \mathbb{R})$ *and the delay differential equation restates as an abstract evolution equation in* H*, then we are ready to approach by dynamic programming and we want to prove that* $V(\eta) := Sup_{\alpha \in I(\eta)} J(\eta;\alpha)$ *is a viscosity solution of HJM equation.*

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