Periodic solutions of Euler-Lagrange equations in an anisotropic Orlicz-Sobolev space setting

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Let $\Phi : \mathbb{R}^d \to [0, +\infty)$ be a differentiable, convex function such that $\Phi(0) = 0$, $\Phi(y) > 0$ if $y \neq 0$, $\Phi(-y) = \Phi(y)$, and $\lim_{|y|\to\infty} \frac{\Phi(y)}{|y|} = +\infty$.

For $T > 0$, we assume that $F : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ with F and $\nabla_x F$ Carathéodory functions satisfying

$$|F(t, x)| + |\nabla_x F(t, x)| \leq a(x)b(t), \text{ for a.e. } t \in [0, T],$$

where $a \in C\left(\mathbb{R}^d, [0, +\infty)\right)$ and $0 \leq b \in L^1([0, T], \mathbb{R})$.

Our goal is to set conditions that guarantee existence of solutions of the problem

$$\begin{cases} \frac{d}{dt}\nabla\Phi(u'(t)) = \nabla_x F(t, u(t)), & \text{for a.e. } t \in (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{P_Φ}$$

minimizing the functional

$$I(u) := \int_0^T \Phi(u'(t)) + F(t, u(t))\, dt,$$

on the set $H := \{u \in W^1 L^\Phi | u(0) = u(T)\}$, where $W^1 L^\Phi$ is the anisotropic Orlicz-Sobolev space associated to Φ.

We denote by Φ^* the complementary function of Φ in the sense of convex analysis.

We say that $\Phi \in \Delta_2$, if there exists a constant $C > 0$ such that

$$\Phi(2x) \leq C\Phi(x) + 1, \quad x \in \mathbb{R}^d.$$

We write $\Phi_0 \prec \Phi$ if for every $k > 0$ there exists $C = C(k) > 0$ such that

$$\Phi_1(x) \leq C + \Phi_2(kx), \quad x \in \mathbb{R}^d.$$

The following is our main theorem which contains, as a particular case, known results of existence of periodic solutions of p-laplacian and (p, q)-laplacian systems.

Theorem 1. *Let $\Phi^\star \in \Delta_2$.*

1. *If there exist Φ_0 with $\Phi_0 \prec \Phi$ and a function $d \in L^1([0, T], \mathbb{R})$, with $d \geq 1$, such that*

$$\Phi^\star(d^{-1}(t)\nabla_x F) \leq \Phi_0(x) + 1 \quad \text{and} \quad \lim_{|x|\to\infty} \frac{\int_0^T F(t, x)\, dt}{\Phi_0(2x)} = +\infty,$$

   then I attains a minimum on H.

2. *If u is a minimum and $d\left(u', L^\infty([0, T], \mathbb{R}^d)\right) < 1$, then u is solution of (P_Φ).*

We also discuss conditions under which $d(u', L^\infty([0, T], \mathbb{R}^d)) < 1$ is satisfied.

Now, we are exploring the possibility of addressing Hamiltonian systems by the dual method, instead of the direct one applied to Euler-Lagrange equations, in an anisotropic Orlicz-Sobolev framework.