Chapter 2
Shooting type methods

As mentioned in the Introduction, different tools have been developed for the study of boundary value problems. In this chapter, we shall start with a very simple one: the *shooting method*. Roughly speaking, the method consists in solving firstly an initial value with a free parameter λ and then trying to find an appropriate value of λ such that the obtained solution satisfies the desired boundary condition. This task requires some qualitative analysis; as we shall see, in some cases it is possible to know in advance the behavior of the solutions of the initial value problem, accordingly to the variations of the parameter λ.

2.1 Set the angle and shoot

Let us start by considering the second order equation

$$u''(t) = f(t, u(t)) \tag{2.1}$$

with homogeneous Dirichlet conditions

$$u(0) = u(1) = 0. \tag{2.2}$$

The idea of the shooting method is really simple: in first place, for fixed $\lambda \in \mathbb{R}$, we may solve equation (2.1) with initial conditions

$$u(0) = 0, \qquad u'(0) = \lambda. \tag{2.3}$$

If f satisfies the standard requirements, i. e. it is continuous and locally Lipschitz in u, then the solution u_λ is well defined and unique. Thus, it suffices to find some value of the shooting parameter λ such that $u_\lambda(1) = 0$. In other words, we look for a zero of the function ϕ defined by

$$\phi(\lambda) := u_\lambda(1).$$

Figure 1

This explains the title of this section: indeed, the procedure consists in adjusting the value of the parameter λ until an appropriate shooting angle is obtained, thus hitting the point $(1,0)$. This reminds that old computer game called *Gorilla*, in which the players throw bananas at each other, setting the velocity and the angle before each throw (a modern and more sophisticated version of this game is the nowadays very popular *Angry birds*).

But there is always a "but": in this case, it is worth noticing that the solutions of the initial value problem are not necessarily defined up to the value $t=1$, so ϕ may not be defined for all values of λ. However, there is a very important fact about ϕ that we certainly know: it is continuous. In some situations, it is possible to assert that $dom(\phi)=\mathbb{R}$; for example, this is the case when f grows at most linearly in its second variable, that is:

$$|f(t,u)| \leq a|u|+b$$

for some constants a and b (see Appendix A for details).

In particular, if f is bounded then the shooting method works very well, as we shall see in the following example.

2.1.1 A back-and-forth example: the pendulum equation

Our first example in order to illustrate the above described technique consists in a well known problem, the pendulum equation

$$u''(t)+\sin u(t) = p(t) \qquad 0<t<1$$

where the forcing term $p:[0,1]\to\mathbb{R}$ is continuous. This equation is very famous and has been the subject of many relevant works; furthermore, there are still interesting open problems concerning the periodic conditions (for an account of the history and open problems on the pendulum equation, see e.g. the excellent survey [7]. More shall be said about the subject later on, in Chapter 4). However, the situation is completely different when dealing with the Dirichlet conditions (2.2); as we shall see, the existence of solutions can be easily proven by the shooting method.

Indeed, if u_λ is the unique solution satisfying the initial value conditions (2.3), then integration yields

$$u_\lambda'(t) = \lambda + \int_0^t [p(s) - \sin u_\lambda(s)]\, ds$$

and setting $R := \int_0^1 |p(t)|\, dt + 1$ we deduce that u_λ is monotone for $|\lambda| \geq R$. More precisely,

- $\lambda \geq R \Rightarrow u_\lambda'(t) \geq 0$ for all t.
- $\lambda \leq -R \Rightarrow u_\lambda'(t) \leq 0$ for all t.

In particular, u_R is nondecreasing and u_{-R} is nonincreasing; together with the fact that $u_\lambda(0) = 0$ for all λ, this implies

$$\phi(R) \geq 0 \geq \phi(-R).$$

Thus, by Bolzano's Theorem we conclude that ϕ vanishes in $[-R, R]$.

It is clear that the same procedure can be applied to the more general equation (2.1) for arbitrary bounded f. But this boundedness condition is very restrictive: it would be desirable to demonstrate that the shooting method can be applied to a more general situation. This is the goal of the next subsection.

2.1.2 A priori bounds

From the previous example we may conclude, as a general rule, that the success of the shooting method relies very strongly on the fact that we know some properties of the associated flow of the differential equation: at least, we need to be sure that the solutions of the initial value problem for an appropriate set of parameters are defined up to the endpoint of the interval. As mentioned, from standard results in the theory of ODEs this is guaranteed for all λ when the nonlinearity has linear growth; however, there are plenty of cases in which this restriction is not fulfilled and the shooting method is still applicable. In particular, in some cases it might be of much help to count with *a priori* bounds of the solutions.

The philosophy behind this idea is again very simple: if we know in advance that the solutions of a certain problem are bounded by some constant R, then we may replace the nonlinearity f by a bounded one, say $\tilde{f}$, such that $\tilde{f}(t,u) = f(t,u)$ for $|u| \leq R$. Obviously, this has to be done in such a way that the solutions of the modified problem $u'' = \tilde{f}(t,u)$ are also bounded by R, so they are in fact solutions of the original problem. Let us see some simple examples.

Example 2.1. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and locally Lipschitz in its second variable and assume there exists a positive constant $R > 0$ such that

$$f(t,-R) < 0 < f(t,R) \qquad \text{for all } t \in [0,1]. \tag{2.4}$$

Then the Dirichlet problem (2.1)-(2.2) has at least one solution u with $\|u\|_\infty \leq R$.

Condition (2.4) is a particular case of the so-called *Hartman condition*. Here, it seems reasonable to define

$$\tilde{f}(t,u) := \begin{cases} f(t,u) & \text{if } |u| \leq R \\ f(t,R) & \text{if } u > R \\ f(t,-R) & \text{if } u < -R. \end{cases}$$

This 'cut off' operation might look a bit drastic, although it is true that $\tilde{f}$ is still a continuous and locally Lipschitz function. Moreover, it is bounded, so the shooting method provides a solution u of the problem $u''(t) = \tilde{f}(t,u)$ satisfying (2.2): thus, it suffices to prove that $|u(t)| \leq R$ for all t. Indeed, assume for example that u achieves its maximum value at some t_0 with $u(t_0) > R$, then $t_0 \in (0,1)$ and

$$u''(t_0) = \tilde{f}(t_0, u(t_0)) = f(t_0, R) > 0.$$

This is a contradiction, since $u(t_0)$ is a maximum. The proof that $u(t) \geq -R$ for all t is analogous. Note that, in this case, it is not necessarily true that all the solutions of the original problem are bounded by R: for our purposes, it was enough to prove it for the cut-off function $\tilde{f}$.

Example 2.2. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and locally Lipschitz in its second variable and assume that f is nondecreasing with respect to u, namely

$$f(t,u) \leq f(t,v) \quad \text{for all } t \in [0,1], u, v \in \mathbb{R}, \, u \leq v.$$

Then the Dirichlet problem (2.1)-(2.2) has a unique solution.

This is a more tricky problem, although it can be also solved by elementary arguments. In first place, it is worthy to observe a remarkable novelty with respect to the preceding examples: the solution is unique. So it seems a good idea to understand the role of the monotonicity condition, in order to see how uniqueness follows out from it.

To this end, note that if u and v are solutions of the problem then the function $w := u - v$ satisfies

$$w''(t)w(t) = [f(t,u(t)) - f(t,v(t))].[u(t) - v(t)] \geq 0.$$

Moreover, $w(0) = w(1) = 0$, so integration of the previous inequality yields

$$0 \leq \int_0^1 w''(t)w(t)\,dt = -\int_0^1 w'(t)^2\,dt.$$

This implies that $w' \equiv 0$ which, in turn, implies $w \equiv 0$.

So we have proven uniqueness, but... how can we deduce now the *existence* of solutions? As we shall see in the next chapters, this is a particular case of a rather general class of problems in which, roughly speaking, "uniqueness implies existence". In this specific context, we shall replace f by a 'cut off' function $\tilde{f}$ as before, although in the present case we don't have a value R given in advance. Thus, we have to choose it in an accurate way.

That's the key idea of *a priori* bounds: sometimes it is possible, before knowing whether or not the problem has a solution, to prove that the image of such a solution, if it exists, lies in a certain bounded set. Here, if $u(t)$ solves (2.1)-(2.2) then we may write

$$u''(t) = f(t,u(t)) - f(t,0) + f(t,0)$$

and as $u(0) = 0$ we obtain:

$$u''(t)u(t) = [f(t,u(t)) - f(t,0)]u(t) + f(t,0)u(t) \geq f(t,0)u(t).$$

Integration at both sides of the inequality yields

$$\int_0^1 u'(t)^2\,dt \leq -\int_0^1 f(t,0)u(t)\,dt \leq \|u\|_\infty \int_0^1 |f(t,0)|\,dt. \tag{2.5}$$

Next, observe that

$$|u(t)| = \left|\int_0^t u'(s)\,ds\right| \leq \int_0^t |u'(s)|\,ds \leq \int_0^1 |u'(s)|\,ds$$

for all t. Now take the maximum at the left hand side to conclude, using the Cauchy-Schwarz's inequality, that

$$\|u\|_\infty \leq \left(\int_0^1 u'(t)^2\,dt\right)^{1/2}.$$

Combined with (2.5), this last inequality implies:

$$\|u\|_\infty \leq \left(\int_0^1 u'(t)^2\,dt\right)^{1/2} \leq \int_0^1 |f(t,0)|\,dt := R.$$

If we set $\tilde{f}$ exactly as in the previous example, then the modified problem has a solution u. But now comes the most interesting part: the bound R was obtained *using only the monotonicity of f*. As $\tilde{f}$ is also nondecreasing in its second variable and $\tilde{f}(t,0) = f(t,0)$, we deduce that $\|u\|_\infty$ is bounded by the same R and thus a solution of the original problem.

Example 2.3. Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be continuous and locally Lipschitz in its second variable and assume that f has *linear growth* in u, that is: $|f(t,u)| \leq c|u| + d$ for some constants $c,d \geq 0$. We claim that if $c < 1$ then the Dirichlet problem (2.1)-(2.2) has a solution. Indeed, in this case an *a priori* bound for an arbitrary solution $u(t)$ is obtained as follows. As before, let us multiply by u and integrate to obtain

$$\int_0^1 u'(t)^2\,dt = -\int_0^1 f(t,u(t))u(t)\,dt \le c\int_0^1 u(t)^2\,dt + d\int_0^1 |u(t)|\,dt.$$

The computations in the previous example now yield:

$$\|u\|_\infty^2 \le c\|u\|_\infty^2 + d\|u\|_\infty$$

and hence $\|u\|_\infty \le \frac{d}{1-c}$. Fix $R \ge \frac{d}{1-c}$ and let $\tilde{f}$ be defined as before. If R is large enough, then the inequality $|\tilde{f}(t,u)| \le c|u| + d$ is satisfied, so the solutions of the truncated problem are, in particular, solutions of the original one.

Remark 2.1. The condition $c < 1$ can be improved: indeed, using the *Poincaré inequality* $\int_0^1 u(t)^2\,dt \le \frac{1}{\pi^2}\int_0^1 u'(t)^2\,dt$ (see Appendix), it is easy to verify that a sufficient condition for the existence of solutions is that $c < \pi^2$. This condition is already sharp, as it can be seen for example when $f(t,u) = \sin(\pi t) - \pi^2 u$. In this case, if we suppose that the problem has a solution $u(t)$, then multiplying by $\sin(\pi t)$ at both sides and integrating we obtain:

$$\int_0^1 \sin^2(\pi t)\,dt = \int_0^1 [u''(t) + \pi^2 u(t)]\sin(\pi t)\,dt = 0,$$

a contradiction.

2.2 From scalar equations to systems

Beside the extensions obtained in the last two examples, the application of the shooting method for the case of a bounded nonlinearity might have been perhaps a bit disappointing for those who were expecting something really spectacular. In some sense, the resolution was just *too* simple: notice, for example, that our argument has relied only on the fact that solutions with large absolute value of the first derivative at the initial point $t = 0$ are monotone. This is, indeed, very simple, although it is not clear how it can be extended for a *system* of equations.

The goal of the present section consists in solving, instead of a scalar equation, a system of two equations. In other words, we shall consider still equation (2.1) but now with $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ and look for a solution $u : [0,1] \to \mathbb{R}^2$ satisfying the Dirichlet conditions (2.2).

Of course, one may ask why we should restrict ourselves to the 2-dimensional case. In fact, all the results in this section can be extended to higher dimensions; however, the case of 2 equations contains already the main condiments of the non-scalar case and it has the additional advantage that it allows a very elementary and elegant approach. The general case shall be considered later in Chapter 6.

2.2.1 With a little help of my intuition

With the scalar case still in mind, let us assume now that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and locally Lipschitz in u and define, for $\lambda \in \mathbb{R}^2$, $u_\lambda(t)$ as the unique solution of the initial value problem (2.1)-(2.3). As before,

$$u_\lambda'(t) = \lambda + \int_0^t f(s,u(s))\,ds,$$

but now we shall go ahead with a second integration step to obtain:

$$u_\lambda(t) = \lambda + \int_0^t \int_0^\tau f(s,u_\lambda(s))\,ds\,d\tau.$$

As far as u_λ is defined up to $t = 1$, we may define again $\phi(\lambda) := u_\lambda(1)$ and hence

$$\phi(\lambda) = \lambda + S,$$

where S (for "something") satisfies: $|S| \le \|f\|_\infty$.

The bad new is that, in this context, we cannot use Bolzano's theorem, since we are dealing now with a 2-dimensional problem. In consequence, the existence of a zero of ϕ is not obvious yet.

Let us start trying a simple argument: in first place, it is clear that if $R > \|f\|_\infty$, then $\|\phi(\lambda) - \lambda\| < R$ for all λ. Thus,

$$\|\lambda\|^2 - 2\phi(\lambda)\cdot\lambda + \|\phi(\lambda)\|^2 < R^2$$

and, in particular, when $\|\lambda\| = R$ we deduce:

$$\phi(\lambda)\cdot\lambda > \frac{1}{2}\|\phi(\lambda)\|^2 \ge 0.$$

Now recall that $\phi(\lambda)\cdot\lambda = \|\phi(\lambda)\|.\|\lambda\|\cos\alpha$, where α is the angle between $\phi(\lambda)$ and λ; hence, the previous computation implies that, when $\|\lambda\| = R$, the vector field ϕ points outwards the ball $\overline{B_R(0)}$. Is this enough to conclude that ϕ vanishes in $B_R(0)$?

Figure 2

Suppose it does not: then, for each fixed $\lambda \in \overline{B_R(0)}$ the line $L\delta := \lambda + \delta\phi(\lambda)$ hits the circumference $\partial B_R(0)$ at a unique value of $\delta = \delta(\lambda) \geq 0$. More precisely, from the equality

$$\|\lambda + \delta\phi(\lambda)\|^2 = R^2$$

we obtain:

$$\delta(\lambda) = \frac{\phi(\lambda)\cdot\lambda + \sqrt{(\phi(\lambda)\cdot\lambda)^2 + R^2 - \|\lambda\|^2}}{\|\phi(\lambda)\|^2}.$$

Moreover, note that, since $\phi(\lambda)\cdot\lambda > 0$ for $\|\lambda\| = R$, the term inside the square root in the previous expression is strictly positive; thus, the function δ is as smooth as ϕ.

We conclude that the mapping $r : \overline{B_R}(0) \to \mathbb{R}^2$ defined by

$$r(\lambda) = \lambda + \delta(\lambda)\phi(\lambda)$$

is smooth and has two very special properties:

1. $r(\overline{B_R}(0)) \subset \partial B_R(0)$.
2. $\lambda \in \partial B_R(0) \Rightarrow r(\lambda) = \lambda$.

In other words, r is a so-called *retraction*, that maps a closed disk $\overline{D}$ onto its boundary and leaves the points of ∂D fixed. As we shall see, such a mapping cannot exist.

2.2.2 In the beginning was Green

In this section, we shall prove that there are no C^2 retractions from the closed unit disk onto its boundary. Later on we shall generalize this result and prove that there are no *continuous* retractions; for the moment, the smoothness assumption shall be very helpful in order to use just very elementary tools. Indeed, a remarkable aspect of the following proof is the fact that it employs only arguments from a basic course in Calculus.

Let us proceed by contradiction: suppose that $r = (u, v) : \overline{B}_1(0) \to \partial B_1(0)$ is a C^2 mapping such that $r|_{\partial B_1(0)} = id$ and define the quantity

$$D := \int_{\partial B_1(0)} (u\nabla v - v\nabla u)\cdot d\sigma.$$

On the one hand, we may set $\gamma(t) := (\cos t, \sin t)$; then, as $r \circ \gamma = \gamma$ we deduce, by the chain rule:

$$\nabla u \circ \gamma \cdot \gamma' = \gamma_1', \quad \nabla v \circ \gamma \cdot \gamma' = \gamma_2'.$$

Thus,

$$D=\int_0^{2\pi}\left[u(\gamma(t))\gamma_2'(t)-v(\gamma(t))\gamma_1'(t)\right]dt=\int_0^{2\pi}\left[\gamma_1(t)\gamma_2'(t)-\gamma_2(t)\gamma_1'(t)\right]dt$$

$$=\int_0^{2\pi}\left[\cos^2(t)+\sin^2(t)\right]dt=2\pi.$$

On the other hand, we may write

$$D=\int_{\partial B_1(0)}Pdx+Qdy$$

where

$$P:=u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x},\qquad Q:=u\frac{\partial v}{\partial y}-v\frac{\partial u}{\partial y}.$$

As $u^2+v^2\equiv 1$, it is seen that

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}\equiv 0,$$

and we conclude that ∇u and ∇v are linearly dependent, that is:

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}=\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

Hence,

$$\frac{\partial Q}{\partial x}=\frac{\partial\left(u\frac{\partial v}{\partial y}-v\frac{\partial u}{\partial y}\right)}{\partial x}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial y\partial x}-v\frac{\partial^2 u}{\partial y\partial x}$$

$$=\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+u\frac{\partial^2 v}{\partial x\partial y}-v\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial\left(u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x}\right)}{\partial y}=\frac{\partial P}{\partial y}$$

and by Green's Theorem we get $D=0$, a contradiction.

Remark 2.2. Note that the explicit computation of $D=2\pi$ can be performed without invoking any specific parameterization of $\partial B_1(0)$. Indeed, for an arbitrary parameterization γ we may apply Green's Theorem again to obtain

$$D=\int_0^{2\pi}\left[\gamma_1(t)\gamma_2'(t)-\gamma_2(t)\gamma_1'(t)\right]dt=\int_\gamma x\,dy-y\,dx=\iint_{B_1(0)}2\,dxdy=2\pi.$$

Thus we have proven:

Theorem 2.1. *There are no C^2 mappings $r:\overline{B}_1(0)\to\partial B_1(0)$ such that $r|_{\partial B_1(0)}=id$.*

As a consequence of this 'smooth' version of the no-retraction theorem we also deduce, from the computations in the previous section, that if ϕ is a C^2 mapping from the closed unit ball that points outwards over the boundary, namely

$$\phi(x)\cdot x > 0 \qquad \text{for } x\in \partial B_1(0) \tag{2.6}$$

then it necessarily vanishes in the interior:

Theorem 2.2. *Assume that* $\phi : \overline{B}_1(0) \to \mathbb{R}^2$ *is* C^2 *and satisfies* (2.6). *Then* ϕ *has a zero in* $B_1(0)$.

Furthermore, we can also obtain a C^2 version of a very famous result:

Theorem 2.3. *(Brouwer's Theorem,* C^2 *version) Let* $f : \overline{B}_1(0) \to \overline{B}_1(0)$ *be a* C^2 *mapping. Then* f *has at least one fixed point, that is: there exists* $x \in \overline{B}_1(0)$ *such that* $f(x) = x$.

Proof. If $f(x) = x$ for some $x \in \partial B_1(0)$, then we are done. Otherwise, define the C^2 mapping $\phi(x) := x - f(x)$; then, for $x \in \partial B_1(0)$,

$$\phi(x)\cdot x = \|x\|^2 - f(x)\cdot x = 1 - f(x)\cdot x > 0$$

and hence ϕ vanishes in $B_1(0)$.

Add more dimensions to your Bolzano: the Poincaré-Miranda Theorem

Few pages ago, we have complained about the fact that Bolzano's theorem could not be used for a 2-dimensional shooting problem. Now it's time to confess that we were not completely sincere; indeed, the following result can be regarded as a generalization of Bolzano's theorem for vector fields in $\mathbb{R}^2$:

Theorem 2.4. *(Poincaré-Miranda,* C^2 *version) Let* $f : [-1,1]\times[-1,1] \to \mathbb{R}^2$ *be a* C^2 *mapping such that*

$$f_1(-1,x_2) < 0 < f_1(1,x_2) \qquad \forall x_2 \in [-1,1] \tag{2.7}$$

and

$$f_2(x_1,-1) < 0 < f_2(x_1,1) \qquad \forall x_1 \in [-1,1]. \tag{2.8}$$

Then there exists $x \in (-1,1)\times(-1,1)$ *such that* $f(x) = (0,0)$.

There are several ways of proving this result, also known as *the generalized intermediate value theorem*. For example, one may consider the function $g(x) := x - \frac{f(x)}{M}$, where M is a constant. It is readily seen that, for M large enough, g maps the square $[-1,1]\times[-1,1]$ inside a smaller square $C \subset (-1,1)\times(-1,1)$. Next, take a C^2-diffeomorphism $d : \overline{B}_1(0) \to K$ for some K such that $C \subset K \subset [-1,1]\times[-1,1]$. In particular, $g(K) \subset K$, so by Theorem 2.3, applied to the mapping $d^{-1}\circ g\circ d$, we conclude that g has a fixed point, which corresponds to a zero of f.

Remark 2.3. It is clear that both the generalized intermediate value theorem and Theorem 2.2 are still valid if any of the inequalities (2.6), (2.7) and (2.8) is reversed. Furthermore, these inequalities do not need to be strict: indeed, assume for example that $\phi : \overline{B}_1(0) \to \mathbb{R}^2$ is a C^2 mapping such that $\phi(x) \cdot x \geq 0$ over $\partial B_1(0)$ and define $\phi_n(x) := \phi(x) + \frac{x}{n}$. Then $\phi_n(x) \cdot x > 0$ over $\partial B_1(0)$, so ϕ_n has a zero x_n. By compactness, there exists a subsequence of $\{x_n\}$ that converges to some $x \in \overline{B}_1(0)$, which is obviously a zero of ϕ. A similar argument allows to prove a non-strict version of Theorem 2.4.

2.2.3 Green light (to topology)*

As shown in the previous section, there are no C^2 retractions from the closed unit ball onto its boundary; in particular, that property was used to prove Theorems 2.2, 2.3 and 2.4.

This would suffice for our immediate purposes: if we are just dealing with a simple system like (2.1), it does not make any damage to assume that the involved functions are smooth so the mapping $\phi(\lambda) := u_\lambda(1)$ is also smooth (see Appendix A). However, it is not a great effort to prove that the preceding "topological" results are, indeed, topological: in other words, they still hold if we assume that the mappings are only continuous. To this end, we shall make use of the Stone-Weierstrass theorem; the reader who wants to keep using only arguments from basic calculus may skip this part and proceed further with the remaining sections in this chapter.

Let us start, for example, with Theorem 2.2: assume that $\phi : \overline{B}_1(0) \to \mathbb{R}^2$ is continuous and satisfies (2.6), and consider a sequence of C^2 functions $\phi_n : \overline{B}_1(0) \to \mathbb{R}^2$ such that $\phi_n \to \phi$ uniformly. Let $\varepsilon := \inf_{x \in \partial B_1(0)} \phi(x) \cdot x > 0$ and fix n_0 such that $\|\phi_n - \phi\|_\infty < \varepsilon$ for $n \geq n_0$, then

$$\phi_n(x) \cdot x = \phi(x) \cdot x - [\phi(x) - \phi_n(x)] \cdot x \geq \varepsilon - \|\phi_n - \phi\|_\infty > 0$$

for any $x \in \partial B_1(0)$ and $n \geq n_0$. Thus, ϕ_n satisfies (2.6) for n large enough, and hence it has a zero x_n. Next, take a subsequence of $\{x_n\}$ converging to some $x \in \overline{B}_1(0)$, then it is immediate that $\phi(x) = 0$.

From this point, the extension of the no-retraction theorem is straightforward: if $r : \overline{B}_1(0) \to \partial B_1(0)$ is a continuous retraction then it satisfies (2.6) and, in consequence, it should vanish, a contradiction. Summarizing:

Theorem 2.5. *There are no continuous mappings $r : \overline{B}_1(0) \to \partial B_1(0)$ such that $r|_{\partial B_1(0)} = id$.*

Theorem 2.6. *Assume that $\phi : \overline{B}_1(0) \to \mathbb{R}^2$ is continuous and satisfies* (2.6). *Then ϕ has a zero in $B_1(0)$.*

Any of both statements can be used for proving the standard version of Brouwer's Theorem in $\mathbb{R}^2$ (alternatively, one might use the Stone-Weierstrass theorem again and deduce it directly from Theorem 2.3):

Theorem 2.7. *(Brouwer) Any continuous mapping $f : \overline{B}_1(0) \to \overline{B}_1(0)$ has at least one fixed point.*

The same is true for the generalized intermediate value theorem. Again, the proof follows from the Stone-Weierstrass theorem combined with Theorem 2.4, or just repeating the argument of the preceding section and applying Theorem 2.7. Alternative proofs can be obtained using Theorems 2.5 or 2.6. Regarding this last one, it is worth observing that if a function f satisfies (2.7) and (2.8) then it is an outward pointing field with respect to the square $[-1,1] \times [-1,1]$.

Theorem 2.8. *(Poincaré-Miranda) Let $f : [-1,1] \times [-1,1] \to \mathbb{R}^2$ be a continuous mapping satisfying* (2.7) *and* (2.8)*. Then there exists $x \in (-1,1) \times (-1,1)$ such that $f(x) = (0,0)$.*

Remark 2.4. Although we've presented the four previous theorems in a specific order, they are all equivalent: that is, any of them can be used to prove any of the others. Furthermore, any of them implies the completeness axiom for the real numbers (see exercise 2.4). This is also true if we consider only the C^2 versions of the last section: thus, we may conclude that Green is *really* "in the beginning". This might seem a bit odd, since quite a lot of work is required just to understand the statement of Green's theorem: one needs to deal with oriented curves and line integrals, double integrals, partial derivatives and so on. But, formally, it is true that all the properties of the real numbers are valid if the completeness axiom is replaced by the statement of Green's theorem over the unit ball.

2.3 The Poincaré mapping

In this section we shall investigate the application of shooting type methods to a periodic problem. To begin, let us consider a first order system

$$\begin{cases} u' = f(t,u) \\ u(0) = u(1), \end{cases} \tag{2.9}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and locally Lipschitz in its second variable $u \in \mathbb{R}^2$.

As before, the idea of the method consists in finding a solution u_λ with initial data $\lambda \in \mathbb{R}^2$ and try to find an appropriate value of λ such that u_λ is a solution of (2.9). More precisely, u_λ is defined as the unique solution of

$$\begin{cases} u' = f(t,u) \\ u(0) = \lambda, \end{cases}$$

and we look for some λ such that $u_\lambda(1) = \lambda$. In other words, we search a fixed point of the so-called *Poincaré mapping* P, defined by $P(\lambda) = u_\lambda(1)$. Again, trajectories are not necessarily defined up to $t = 1$, so the domain of P may not be the whole plane; however, in some cases it is possible to obtain enough information that ensures the existence of a fixed point. This is the case in the following elementary example:

Proposition 2.1. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a C^2 mapping and assume that*

$$f(t,u) \cdot u < 0 \qquad \text{for } \|u\| = R,$$

where R is some positive constant. Then (2.9) has at least one solution u such that $\|u\|_\infty \le R$.

Proof. Take $\lambda \in \mathbb{R}^2$ such that $\|\lambda\| \le R$ and assume that u_λ is defined on $[0,T]$ for some T. We claim that $\|u_\lambda(t)\| \le R$ for all $t \in [0,T]$. Indeed, setting $\psi(t) := \|u(t)\|^2$ we obtain:

$$\psi'(t) = 2u(t) \cdot u'(t) = 2u(t) \cdot f(t,u(t)).$$

In particular, if $\|u(t)\| = R$ then $\psi'(t) < 0$. Assume firstly that $u(0) \in B_R(0)$, then $u(t)$ cannot reach the boundary $\partial B_R(0)$ for any t. On the other hand, if $\|u(0)\| = R$ then $\psi(t)$ is initially decreasing and, again, $u(t)$ must remain inside the ball $B_R(0)$ for all $t \in (0,T]$. This implies, in first place, that u_λ is defined in $[0,1]$ for $\|\lambda\| \le R$ and, consequently, the Poincaré mapping P is well defined and C^2 (see Appendix A). Furthermore, $P(\overline{B}_R(0)) \subset \overline{B}_R(0)$ so the existence of a fixed point of P follows from Theorem 2.3.

*2.3.1 Smoothing for shooting**

The reader might have noticed, in the last proposition, that the smoothness assumption on f was made with the precise purpose of using the C^2 version of Brouwer's Theorem. This looks quite fair, since a non-starred section should use results from non-starred sections only... although it is quite obvious that the result is still valid if f is only continuous and locally Lipschitz in u. Indeed, this is not a big deal since we know that the more general Theorem 2.7 holds.

But: what happens if f is only continuous? The study of this case seems to go against the essence of the shooting method, which is based in the existence and uniqueness theorem for the initial value problem. However, an extension of Proposition 2.1 can be easily obtained by means of a procedure which is, in fact, very general. In the same way as in section 2.2.3, we shall approximate f by smooth functions and get a convergent sequence of solutions of the approximated problems. In Theorem 2.6, this was almost trivial because the closed unit ball in $\mathbb{R}^2$ is compact; in the present problem, we are dealing with a sequence of *functions*, so we shall make use of a well-known and powerful compactness result: the Arzelá-Ascoli theorem. Before going into the details, it is worth observing that even a more general result may be obtained with almost the same effort: again, the inequality for f does not need to be strict.

Corollary 2.1. *(Proposition* 2.1 *revisited)*
Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}^2$ be continuous and assume that

$$f(t,u)\cdot u \le 0 \qquad \text{for } \|u\| = R,$$

where R is some positive constant. Then (2.9) *has at least one solution u such that $\|u\|_\infty \le R$.*

Proof. By the Stone-Weierstrass theorem, for each n we may set a smooth mapping $f_n : [0,1]\times\mathbb{R}^2 \to \mathbb{R}^2$ such that $\|f_n(t,u) - f(t,u)\| < \frac{R}{n}$ for all $t \in [0,1]$ and $u \in \overline{B}_R(0)$. Next, define $g_n(t,u) := f_n(t,u) - \frac{u}{n}$; thus, if $\|u\| = R$ then

$$g_n(t,u)\cdot u = [f_n(t,u) - f(t,u)]\cdot u + f(t,u)\cdot u - \frac{R^2}{n} < 0.$$

From Proposition 2.1, problem (2.9) with g_n instead of f has a solution u_n such that $\|u_n\|_\infty \le R$. Furthermore, for arbitrary t it holds that

$$\|u_n'(t)\| = \|g_n(t,u_n(t))\| \le \max_{t\in[0,1],\|u\|\le R} \|g_n(t,u)\| \le K$$

for some constant K independent of n. From Arzelá-Ascoli's theorem, there exists a subsequence $\{u_{n_j}\}$ that converges uniformly on $\overline{B}_R(0)$ to some function u, so if we write

$$u_{n_j}(t) = u_{n_j}(0) + \int_0^t g_{n_j}(s,u_{n_j}(s))\,ds,$$

then we deduce that

$$u(t) = u(0) + \int_0^t f(s,u(s))\,ds$$

for all t, and thus u is a solution of (2.9).

2.3.2 When Poincaré met Miranda

Let us consider again the second order problem (2.1) with $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ a C^2 function, but now under the periodic conditions

$$u(0) = u(1), \quad u'(0) = u'(1). \tag{2.10}$$

Note that the problem is scalar although, unlike the case of Dirichlet conditions, it cannot be reduced to a one-dimensional fixed point problem. Indeed, here the Poincaré mapping has two parameters, that correspond to the initial values of the function and its derivative.

As a simple example, let us consider again the Hartman condition (2.4). In this case, we may observe that the 'cut off' function of section 2.1 does not preserve the smoothness; however, it is not difficult to find a bounded C^2 function $\tilde{f}$ that coincides with f over $[0,1]\times[-R,R]$ and such that $\tilde{f}(t,-u)<0<\tilde{f}(t,u)$ for all $u\geq R$ and $t\in[0,1]$. As before, we claim that if u satisfies (2.10) and solves the modified problem $u''(t)=\tilde{f}(t,u(t))$ then $\|u\|_\infty\leq R$.

Indeed, suppose that $u(t)$ achieves a global maximum at some t_0 with $u(t_0)>R$. If $t_0\in(0,1)$, then $u''(t_0)=\tilde{f}(t_0,u(t_0))>0$, a contradiction; otherwise, it follows from (2.10) that $u(0)=u(1)=\max_{0\leq t\leq 1}u(t)$. Hence, $u'(0)\leq 0\leq u'(1)$ and using (2.10) again we deduce that $u'(0)=0=u'(1)$. Moreover, $u''(t)$ is positive near $t=0$, so $u'(t)$ is initially increasing. As $u'(0)=0$, it follows that $u(t)$ is also initially increasing, which contradicts the fact that a global maximum is achieved at $t=0$. In the same way, it is proven that $u(t)\geq -R$ for all t.

Thus, it suffices to verify that the modified problem has a solution. To this end, for fixed $x,y\in\mathbb{R}$ let $u_{x,y}$ be the unique solution of the initial value problem

$$u''(t)=\tilde{f}(t,u(t))$$

$$u(0)=x,\quad u'(0)=y.$$

For convenience, instead of looking for a fixed point of the Poincaré mapping $P(x,y):=u_{x,y}(1)$ we shall try to find a zero of the function

$$F(x,y):=(u'_{x,y}(1)-y,u_{x,y}(1)-x). \tag{2.11}$$

In first place, note that

$$u'_{x,y}(t)=y+\int_0^t\tilde{f}(s,u_{x,y}(s))\,ds$$

and

$$u_{x,y}(t)=x+ty+\int_0^t(t-s)\tilde{f}(s,u_{x,y}(s))\,ds.$$

Hence, fixing $M>\|\tilde{f}\|_\infty$ we deduce that

$$u_{x,M}(1)-x>0>u_{x,-M}(1)-x$$

for all x. Finally, take any $y\in[-M,M]$ and let $\tilde{R}:=R+2M$, then

$$u_{\tilde{R},y}(t)=\tilde{R}+yt+\int_0^t(t-s)\tilde{f}(s,u_{\tilde{R},y}(s))\,ds\geq\tilde{R}-2M>R$$

and we conclude from (2.4) that $\tilde{f}(t,u_{\tilde{R},y}(t))>0$ for all t. Thus, $u'_{\tilde{R},y}(1)-y=\int_0^1\tilde{f}(t,u_{\tilde{R},y}(t))\,dt>0$.

In the same way, it follows that $u'_{-\tilde{R},y}(1)-y<0$ and Poincaré-Miranda Theorem applies on the rectangle $[-\tilde{R},\tilde{R}]\times[-M,M]$.

Remark 2.5. In particular, for any constant $r>0$ and f bounded the problem

$$u''(t)-ru(t)=f(t,u(t))$$

has a solution satisfying (2.10).

An interesting result in the same direction has been proven by Lazer in [5]. For simplicity, let $g \in C^2(\mathbb{R},\mathbb{R})$ be bounded and assume there exist $R_0 > 0$ and $c \in \mathbb{R}$ such that

$$g(u) < c < g(-u) \qquad \text{for all } u \text{ such that } |u| > R_0 \tag{2.12}$$

or

$$g(u) > c > g(-u) \qquad \text{for all } u \text{ such that } |u| > R_0. \tag{2.13}$$

Then the periodic problem

$$u''(t) + g(u(t)) = p(t), \qquad u(0) = u(1), \quad u'(0) = u'(1) \tag{2.14}$$

has a solution for each C^2 function p satisfying $\int_0^1 p(t)\,dt = c$.

Note that, in this situation, the Hartman condition would read

$$g(R_0) < p(t) < g(-R_0) \qquad \text{for all } t,$$

so it is somehow comparable to (2.12). However, this is a *pointwise* inequality: the remarkable aspect of Lazer's theorem is the fact that the condition is given only in terms of the *average* of the function p.

Lazer's original proof used a well known result called *Schauder Theorem*, that shall be introduced in Chapter 4; the methods described in this chapter allow a much more elementary proof. Indeed, we may essentially reproduce the previous argument with $\tilde{f}(t,u) := p(t) - g(u)$. Everything works in the same way until the inequality $u_{R,y}(t) > R_0$ is deduced for all t and all $y \in [-M,M]$.

But now... attention! We cannot deduce, as before, that $\tilde{f}(t,u_{R,y}(t)) > 0$ for all t, although if for example (2.12) holds then we obtain:

$$u'_{R,y}(1) - y = \int_0^1 p(t) - g(u_{R,y}(t))\,dt$$

$$= c - \int_0^1 g(u_{R,y}(t))\,dt = \int_0^1 [c - g(u_{R,y}(t))]\,dt > 0.$$

In the same way, it is seen that $u'_{-R,y}(1) - y < 0$ so Poincaré-Miranda theorem can be applied to the same mapping F defined in (2.11). The proof is similar under condition (2.12).

Remark 2.6. * The general result by Lazer is established for g being only continuous and satisfying one of the non-strict conditions

$$g(u) \geq c \geq g(-u) \qquad \text{for all } u \text{ such that } |u| > R_0 \tag{2.15}$$

or

$$g(u) \leq c \leq g(-u) \qquad \text{for all } u \text{ such that } |u| > R_0. \tag{2.16}$$

Furthermore, the boundedness condition on g may be relaxed to assume that g is *sublinear*, namely:

$$\lim_{|u|\to\infty} \frac{g(u)}{u} = 0. \tag{2.17}$$

Finally, a friction term can be added to the equation, namely

$$u''(t) + au'(t) + g(u(t)) = p(t) \tag{2.18}$$

for some constant a. In this case, the integral expression for $u(t)$ and $u'(t)$ is a bit different, but it may be easily obtained by the method of variation of parameters. It is interesting to observe that, when $a \neq 0$, the sublinearity assumption on g can be dropped; the same is true when (2.16) holds.

This general result can be deduced from the preceding case using the ideas of sections 2.1.2 and 2.3.1; for instance, let us show how a priori bounds can be obtained when $a \neq 0$ under conditions (2.13) or (2.12) and leave the rest of the proof and the remaining cases as an exercise for the reader. Assume that u is a solution of (2.18) such that $u(0) = u(1)$ and $u'(0) = u'(1)$. Multiplying the equation by u' and integrating, we obtain:

$$\int_0^1 [u''(t)u'(t) + au'(t)^2 + g(u(t))u'(t)]\,dt = \int_0^1 p(t)u'(t)\,dt.$$

Next, observe that if $G(u) := \int_0^u g(s)\,ds$ then

$$\int_0^1 [u''(t)u'(t) + g(u(t))u'(t)]\,dt = \left[\frac{u'^2}{2} + G(u)\right]\Big|_0^1 = 0$$

so we deduce that

$$\int_0^1 u'(t)^2\,dt = \frac{1}{a}\int_0^1 p(t)u'(t)\,dt \le \frac{1}{|a|}\left(\int_0^1 p(t)^2\,dt\right)^{1/2}\left(\int_0^1 u'(t)^2\,dt\right)^{1/2}.$$

Hence,

$$\left(\int_0^1 u'(t)^2\,dt\right)^{1/2} \le \frac{1}{|a|}\left(\int_0^1 p(t)^2\,dt\right)^{1/2} := r.$$

On the other hand, writing $u(t) - u(0) = \int_0^t u'(s)\,ds$, we also deduce that

$$|u(t) - u(0)| \le \left(\int_0^1 u'(t)^2\,dt\right)^{1/2} \le r$$

for all t. Finally, integrate (2.18) and use the periodic conditions (2.10) to obtain:

$$\int_0^1 g(u(t))\,dt = \int_0^1 p(t)\,dt = c.$$

In particular, there exists t_0 such that $g(u(t_0)) = c$. We claim that $|u(0)| \le R_0 + r$: indeed, otherwise $|u(t)| \ge |u(0)| - |u(t) - u(0)| > R_0$, for all t, which implies that $g(u(t)) \neq c$ for all t, a contradiction. We conclude that $|u(t)| \leq |u(t)-u(0)| + |u(0)| \leq R_0 + 2r$ for all t.

2.4 Let's make it complex: the index of a curve *

Few pages ago we have mentioned the fact that, although all the Bolzano-like arguments of this chapter can be extended for higher dimensions we have considered only the two-dimensional case since it enables a very simple and intuitive treatment using only Green's theorem. In this section, we shall introduce an even simpler argument which requires just a little bit more: basic complex analysis. In fact, nothing of that is needed for the C^2 case, as the following elementary proof of the fixed point theorem shows:

Let $f : \overline{B} \to \overline{B}$ be a C^2 mapping and fix a C^2 function $\lambda : [0,2] \to [0,1]$ such that $\lambda \equiv 1$ on $[0,\frac{1}{2}]$ and $\lambda \equiv 0$ on $[1,2]$. Define $h : [0,2\pi] \times [0,1] \to \mathbb{C}$ by

$$h(t,s) = \lambda(s)e^{it} - (1-\lambda(2s))f(\lambda(s)e^{it}).$$

Suppose that f has no fixed points, then it follows that h does not vanish and we may define

$$I(s) := \int_0^{2\pi} \frac{\frac{\partial h}{\partial t}(t,s)}{h(t,s)}\,dt. \tag{2.19}$$

Then

$$I'(s) = \int_0^{2\pi} \frac{\partial}{\partial s}\left(\frac{\frac{\partial h}{\partial t}(t,s)}{h(t,s)}\right) dt = \int_0^{2\pi} \frac{\partial}{\partial t}\left(\frac{\frac{\partial h}{\partial s}(t,s)}{h(t,s)}\right) dt = \left.\frac{\frac{\partial h}{\partial s}(\cdot,s)}{h(\cdot,s)}\right|_0^{2\pi} = 0$$

since $h(0,s) = h(2\pi,s)$ for all s and hence I is constant. On the other hand, $I(0) = 2\pi i$ and $I(1) = 0$, a contradiction.

As mentioned, the previous proof has required no complex analysis at all. Complex notation was used for convenience; the only fact that one should need to check is that the differentiation of curves satisfies the quotient rule, which is hidden in the middle step:

$$\left(\frac{\gamma(t)}{\varphi(t)}\right)' = \frac{\gamma'(t)\varphi(t) - \gamma(t)\varphi'(t)}{\varphi(t)^2}.$$

Also, there was perhaps a "dubious" step that involved differentiation under an integral sign, but this is easily justified in this case since the integrand is a C^1 function.

However, the reader who is familiar with the techniques in complex analysis has surely identified the key idea behind the preceding proof, which is based on two concepts:

1. The index or *winding number* of a continuous closed curve $\gamma : [a,b] \to \mathbb{C}\backslash\{0\}$, defined by

$$I(\gamma,0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz \in \mathbb{Z}.$$

2. *Homotopy invariance*: if $\gamma,\delta : [a,b] \to \mathbb{C}\setminus\{0\}$ are continuous closed curves and $h : [a,b]\times \to \mathbb{C}\setminus\{0\}$ is continuous and satisfies $h(t,0) = \gamma(t)$, $h(t,1) = \delta(t)$ for all t and $h(a,s) = h(b,s)$ for all s, then

$$I(\gamma,0) = I(\delta,0).$$

Remark 2.7. The readers who hate paragraphs starting with "the reader who is familiar with..." may employ the previous ideas for a better understanding of the last two concepts. On the one hand, it is possible to obtain an alternative proof of the homotopy invariance of the index when the homotopy is C^2: indeed, observe that the integral $I(s)$ in (2.19) is nothing else but $2\pi i$ times the index of the curve $h(\cdot,s)$ around 0, so any pair of homotopic curves have necessarily the same winding number.

On the other hand, even the definition of index and the fact that it is an integer can be explained without invoking the theory of complex analysis: for instance, let $\gamma : [0,1] \to \mathbb{C}\setminus\{0\}$ be a closed C^1 curve and assume for simplicity that it intersects the y-axis only finitely many times. Writing $\gamma(t) = x(t) + iy(t)$, let us compute

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{\gamma'(t)\overline{\gamma(t)}}{|\gamma(t)|^2} = \frac{x'(t)x(t)+y'(t)y(t)}{x(t)^2+y(t)^2} + \frac{y'(t)x(t)-x'(t)y(t)}{x(t)^2+y(t)^2}i.$$

The real part of this expression is the derivative of the function $\frac{1}{2}\ln(x(t)^2+y(t)^2)$ which has the same value at $t=0$ and $t=1$, so the imaginary part of $I(\gamma,0)$ vanishes. Moreover, the imaginary part of $\frac{\gamma'}{\gamma}$ has also an easy-to-compute primitive, namely $\arctan\frac{y(t)}{x(t)}$. However, this function is not defined when $x(t)$ vanishes, so we need to split the integral as the sum of several integrals over smaller intervals. From our assumptions, the set $\{t \in [0,1] : x(t) = 0\}$ is finite; moreover, observe that if $t_0 < t_1$ are two consecutive zeros of x, then

$$\int_{t_0}^{t_1} \frac{y'(t)x(t)-x'(t)y(t)}{x(t)^2+y(t)^2}\,dt = \arctan\frac{y(t)}{x(t)}\bigg|_{t_0}^{t_1} = \lim_{t\to t_1^-}\arctan\frac{y(t)}{x(t)} - \lim_{t\to t_0^+}\arctan\frac{y(t)}{x(t)}.$$

As the sign of $x(t)$ between t_0 and t_1 is constant, we have three possible cases:

1. If $sgn(y(t_0)) = sgn(y(t_1))$, then $\int_{t_0}^{t_1}\frac{y'(t)x(t)-x'(t)y(t)}{x(t)^2+y(t)^2}\,dt = 0$.
2. If for example $y(t_0) < 0 < y(t_1)$ and $x(t) > 0$ on (t_0,t_1), then

$$\int_{t_0}^{t_1} \frac{y'(t)x(t)-x'(t)y(t)}{x(t)^2+y(t)^2}\,dt = \lim_{u\to+\infty}\arctan u - \lim_{u\to-\infty}\arctan u = \pi.$$

   The same result is obtained if $y(t_0) > 0 > y(t_1)$ and $x(t) < 0$ on (t_0,t_1).
3. An analogous calculation shows that the result is $-\pi$ in the remaining two situations:

- $y(t_0) < 0 < y(t_1)$ and $x(t) < 0$ on (t_0, t_1).
- $y(t_0) > 0 > y(t_1)$ and $x(t) > 0$ on (t_0, t_1).

Thus, every arc between two consecutive zeros of x contributes to the integral $I(\gamma,0) = \frac{1}{2\pi i}\int \frac{\gamma'(t)}{\gamma(t)}\,dt$ with 0, $\frac{1}{2}$ or $-\frac{1}{2}$ according to the three preceding cases; as γ is closed, it is clear that the total number of arcs where cases 2 or 3 occur is even, so $I(\gamma,0)$ is an integer.

Such a machinery allows a more comprehensive proof of Brouwer's theorem in two simple steps, for example proving firstly the following:

Lemma 2.1. *Let* $f : \overline{B}_1(0) \to \mathbb{C}\backslash\{0\}$ *be continuous and let* $\gamma(t) = e^{it}$ *for* $t \in [0,2\pi]$. *Then* $I(f\circ\gamma, 0) = 0$.

Proof. It suffices to consider the homotopy $h(t,s) = f(s\gamma(t))$.

Using this lemma, the proof of Brouwer's theorem (and some of its generalizations) is straightforward: suppose that $f : \overline{B}_1(0) \to \overline{B}_1(0)$ is continuous with $f(z) \neq z$ for all z and define $g(z) = z - f(z)$. On the one hand, consider the homotopy $h(t,s) = \gamma(t) - sf(\gamma(t))$. As f has no fixed points, it is easy to see that h does not vanish, and hence $I(g\circ\gamma,0) = I(\gamma,0) = 1$. On the other hand, the previous lemma says that $I(g\circ\gamma,0) = 0$, so a contradiction yields.

Beside a variety of topological applications, the winding number can be also used in differential equations: for example, to prove the existence of solutions to a rather general class of boundary value problems. For simplicity, let us consider a particular example,

$$u''(t) + u(t)^3 = p(t) \tag{2.20}$$

with $p : [0,1] \to \mathbb{R}$ continuous, under the Dirichlet conditions (2.2). As before, we define $u_\lambda(t)$ as the unique solution of the initial value problem (2.20)-(2.3). Multiplication by $u'_\lambda(t)$ and integration yields

$$u'_\lambda(t)^2 + \frac{u_\lambda(t)^4}{2} = \lambda^2 + 2\int_0^t p(s)u'_\lambda(s)\,ds, \tag{2.21}$$

so assuming that $|\lambda|$ is large we deduce that

$$u'_\lambda(t)^2 \leq r\lambda^2 \text{ and } u_\lambda(t)^4 \leq r\lambda^2. \tag{2.22}$$

for some constant $r > 1$: for example, if $|\lambda| \geq \int_0^1 |p(t)|\,dt$ then a simple computation shows that $u'_\lambda(t)^2 \leq (1+\sqrt{2})^2\lambda^2$ and, consequently, $u_\lambda(t)^4 \leq [2+4(1+\sqrt{2})]\lambda^2$.

In particular, it follows that u_λ is defined on $[0,1]$. Moreover, if t_0 is a critical point of u, then for $|\lambda|$ sufficiently large we obtain:

$$u_\lambda(t_0)^4 \geq 2\lambda^2 - 4\int_0^1 |p(t)u'_\lambda(t)|\,dt > \|p\|_\infty^{4/3}$$

and hence, using (2.20), we deduce that $u_\lambda''(t_0)$ and $u_\lambda(t_0)$ have opposite signs. In other words, when λ is large, all local maxima are positive and local minima are negative. Using Rolle's theorem, we conclude that zeros and critical points of u alternate: more precisely, define $\mathscr{C}_\lambda := \{t \in [0,1] : u_\lambda(t) = 0 \text{ or } u_\lambda'(t) = 0\}$, then

$$\mathscr{C}_\lambda = \{0 = t_0 < t_1 < \ldots < t_N\}$$

for some $N = N(\lambda)$ and, for $j = 1, \ldots, N$,

$$u_\lambda(t_j) = 0 \neq u_\lambda'(t_j) \;\text{ if } j \text{ is even}$$

$$u_\lambda(t_j) \neq 0 = u_\lambda'(t_j) \;\text{ if } j \text{ is odd.}$$

Furthermore, when $j \leq N-2$ is odd, $u_\lambda(t_j)$ and $u_\lambda(t_{j+2})$ have opposite signs.

Following Remark 2.7, let us consider now the curve $\gamma_\lambda : [0,1] \to \mathbb{C}\backslash\{0\}$ given by $\gamma_\lambda(t) := u_\lambda'(t) + iu_\lambda(t)$ and the integral

$$I(\lambda) := \frac{1}{2\pi}\int_0^1 \frac{u_\lambda'(t)^2 - u_\lambda''(t)u_\lambda(t)}{u_\lambda'(t)^2 + u_\lambda(t)^2}\,dt = \frac{1}{2\pi}\int_0^1 \frac{u_\lambda'(t)^2 + u_\lambda(t)^4 - p(t)u_\lambda(t)}{u_\lambda'(t)^2 + u_\lambda(t)^2}\,dt.$$

When $|\lambda|$ is large, the integrand is positive so γ_λ rotates counterclockwise and $I(\lambda)$ measures the exact fraction of turns that the curve performs around the origin. In that case, $I(\lambda) = \frac{n}{2}$ with $n \in \mathbb{N}$ if and only if $t_N = 1$, if and only if u_λ is a solution of (2.20)-(2.2) with *exactly* $n-1$ zeros in $(0,1)$.

Figure 3

We claim that the distance between any pair of consecutive points of $\mathscr{C}_\lambda$ tends to 0 as $|\lambda| \to \infty$; in particular, this implies that $N(\lambda) \to \infty$ and hence $I(\lambda) \to +\infty$ as $|\lambda| \to \infty$. As I is continuous, we deduce that from some $y_0 \geq 0$ it takes all the values $y \geq y_0$ at least two times. Hence, for any $n \in \mathbb{N}$ with $n \geq 2y_0$ there exist at least two solutions of (2.20)-(2.2) having exactly n zeros.

Indeed, let us verify for example that $t_1 - t_0 \to 0$ as $\lambda \to +\infty$; the other cases are similar. In $(0,t_1)$, both $u(t)$ and $u'(t)$ are positive and, from (2.21), it follows for example that $u'_\lambda(t)^2 \geq \frac{\lambda^2 - u_\lambda(t)^4}{2}$ for λ large enough. Thus, if we define the value $t_\lambda := \sup\{t \in [0,t_1] : u_\lambda(t)^4 \leq \lambda^2\}$ then

$$\frac{u'_\lambda(t)}{\sqrt{\lambda^2 - u_\lambda(t)^4}} \geq \frac{\sqrt{2}}{2} \text{ for } t < t_\lambda$$

and consequently

$$\int_0^{t_\lambda} \frac{u'_\lambda(t)}{\sqrt{\lambda^2 - u_\lambda(t)^4}}\, dt \geq \frac{\sqrt{2}}{2} t_\lambda.$$

As

$$\int_0^{t_\lambda} \frac{u'_\lambda(t)}{\sqrt{\lambda^2 - u_\lambda(t)^4}}\, dt = \int_0^{u_\lambda(t_\lambda)} \frac{du}{\sqrt{\lambda^2 - u^4}}\, dt \leq \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{dv}{\sqrt{1-v^4}}\, dt,$$

it follows that $t_\lambda \leq \frac{C}{\sqrt{\lambda}}$ for some constant C. Finally, for $t \in [t_\lambda, t_1]$ it holds that $u_\lambda(t) \geq \sqrt{\lambda}$ and thus

$$u'_\lambda(t) = -\int_t^{t_1} u''_\lambda(s)\, ds = \int_t^{t_1} [u_\lambda(s)^3 - p(s)]\, ds \geq \lambda^{3/2}(t_1 - t) - \int_0^1 |p(t)|\, dt$$

which, in turn, implies:

$$u_\lambda(t_1) - u_\lambda(t_\lambda) = \int_{t_\lambda}^{t_1} u'_\lambda(s)\, ds \geq \lambda^{3/2} \frac{(t_1 - t_\lambda)^2}{2} - (t_1 - t_\lambda) \int_0^1 |p(t)|\, dt.$$

From the second inequality of (2.22), we deduce:

$$(r^{1/4} - 1)\lambda^{1/2} \geq \lambda^{3/2} \frac{(t_1 - t_\lambda)^2}{2} - (t_1 - t_\lambda) \int_0^1 |p(t)|\, dt.$$

Hence, $t_1 - t_\lambda \leq \frac{K}{\sqrt{\lambda}}$ for some constant K and so completes the proof.

Appendix

Historical notes

[...]

(To be completed)
Shooting method: Severini 1905 xxxxxxxxxxxxxx
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