ANALYSIS, PSYCHOANALYSIS, AND THE ART OF COIN-TOSSING[1]

Young man, in mathematics you don't understand things; you just get used to them.
John von Neumann

1. With a little help from my dice

The main subject of this article can be summarized in a very precise question: can we conceive the possibility of doing Analysis from the tosses of a coin? One may think, for example, about the illuminating case of Bridlegoose, the well-known judge of *Gargantua and Pantagruel*, who decided his sentences with the inestimable help of his dice:

> *For having well and exactly seen surveyed, overlooked, reviewed, recognized, read, and read over again, turned and tossed over, seriously perused and examined the bills of complaint, accusations, impeachments, indictments, warnings, citations, summonings, comparitions, appearances, mandates, commissions, delegations, instructions, informations, inquests, preparatories, productions, evidences, proofs, allegations, depositions, cross speeches, contradictions, supplications, requests, petitions, inquiries, instruments of the deposition of witnesses, rejoinders, replies, confirmations of former assertions, duplies, triplies, answers to rejoinders, writings, deeds, reproaches, disabling of exceptions taken, grievances, salvation bills, re-examination of witnesses, confronting of them together, declarations, denunciations, libels, certificates, royal missives, letters of appeal, letters of attorney, instruments of compulsion, delineatories, anticipatories, evocations, messages, dimissions, issues, exceptions, dilatory pleas, demurs, compositions, injunctions, reliefs, reports, returns, confessions, acknowledgments, exploits, executions, and other such-like confects and spiceries, both at the one and the other side, as a good judge ought to do [...], I posit on the end of a table in my closet all the pokes and bags of the defendant, and then allow unto him the first hazard of the dice, according to the usual manner of your other worships [...]. That being done, I thereafter lay down upon the other end of the same table the bags and satchels of the plaintiff, as your other worships are accustomed to do [...]. Then do I likewise and semblably throw the dice for him, and forthwith livre him his chance.*[2]

Even for Psychoanalysis, coin tossing is not a new affair; a good evidence to support this assertion could be the manner in which S. Freud describes the argument of the enemies of his new technique:

Heads I win, tails you lose.

In other words, what detractors said about psychoanalytic interpretation is:

> *If the patient agrees with us, then the interpretation is right, but if he contradicts us, that is only a sign of his resistance, which again shows that we are right.*[3]

1 Published in Almanac of Psychoanalysis 4 (2004).

2 F.Rabelais, *Gargantua and Pantagruel*, Third Book, chapter XXXIX: How Pantagruel was present at the trial of Judge Bridlegoose, who decided causes and controversies in law by the chance and fortune of the dice.

3 S.Freud, *Constructions in Psychoanalysis*.

Few decades later, Lacan introduced two remarkable topics that are closely related to the caprice of chance: the sequences of the *Seminar on 'The Purloined Letter'* and the fascinating theme of *Tyche and Authomathon*.

As far as Mathematics are concerned, we have to declare that the tosses of a coin place us in the context of the *binary*, as Lacan announced in the fifties; however, this simple scene is far away from that which could be called *absolute randomness*[4]. Still, binary has a lot to offer. It is a known fact that zeroes and ones alone allow writing any other natural number; for example, the binary version of 25 reads

$$11001$$

This "encoding" must be interpreted as follows:

$$25 = \mathbf{1}.2^4 + \mathbf{1}.2^3 + \mathbf{0}.2^2 + \mathbf{0}.2^1 + \mathbf{1}.2^0$$

Hence, any natural number can be regarded as a result of the successive tosses of a coin; it suffices to translate *heads* and *tails* respectively into zeroes and ones. Every finite sequence of tosses (which in *Seminar on 'The Purloined Letter'* represents the *speech*) determines a unique natural number and, conversely, for every natural number n there exists a unique sequence of tosses that determines n. This combinatorial picture of the set of positive integers shows that they constitute, as Borges would say, an *untouched and secret treasure*[5].

In this simple setting, a fundamental question arises: how is it possible to bring forth the *continuum* from a single, two-sided coin?

The problem of continuum is one of the most antique topics in Philosophy, and takes us back to the times of Zeno, who enunciated his celebrated *aporias* in order to refuse the pythagorean conception of Time and Space. In Mathematics, continuum has been extensively discussed for centuries until the set of *real numbers* has been formally constructed after the works of Bolzano, Weierstrass and Dedekind, among other authors. The infinite line is instituted just as the set of reals, each of whose elements can be written in a very precise fashion: decimal expansions. As we have learned at school, a real number is nothing else than a sequence of digits that *does not stop not being written*, e.g.

$$\pi = 3.141592653...$$

We have to admit that some of these expansions *do* stop: such a situation corresponds to those real numbers that are *rationals* (quotients of integers):

$$1/5 = 0.2$$

or

$$2/7 = 0.285714285714285714...$$

In this last example we should remark that, despite its infinite number of ciphers, it is accurate to say that the sequence stops, since we know exactly how it goes on after the first occurrence of "285714". Irrationals, on the other side, correspond to those expansions that are non-periodic and, as Lacan proclaims in *Encore*, they are only thinkable as a *limit*. This concept might sound a bit frightening, if we take into account its relation with the mysterious idea of

4 Indeed, if we think about heads and tails as the only two possible events, we are arbitrarily introducing a bound on the universe: it might happen, for instance, that a disrespectful and naughty bird captures the coin when it is still *in suspense* (from this point of view, the argument described in Freud, *op.cit.* is destined to fail). We have to accept, once more, that the absolute cannot be seized by Mathematics.

5 J.L.Borges, *The Library of Babel.* It is worth to mention that this conception of the Universe as "the variation of the 23 letters" was taken from the *Kabalah*, which captivated the Argentinean writer.

the *infinitesimal;* however, limit is well defined in mathematical Analysis, giving Calculus an appropriate axiomatic foundation.

Now let us go back to our coin. In the same way that we have described the numbers just as decimal expansions filling up the continuous line, we may replace *digits* by *bits* in order to obtain infinite sequences of zeroes and ones. It is an easy task to prove that every real number can be represented by a binary expansion, with an *integer part* and a pure decimal expression (the *mantissa*), e.g.

$$110.10011001110... = 110 + 0.10011001110...$$

That is to say:

integer part: $\mathbf{1}.2^2 + \mathbf{1}.2^1 + \mathbf{0}.2^0 = 6$

mantissa: $\mathbf{1}.1/2 + \mathbf{0}.1/2^2 + \mathbf{0}.1/2^3 + \mathbf{1}.1/2^4 + \mathbf{1}.1/2^5 ... = 0.59375...$

Thus, also the set of real numbers can be thought as a "treasure": a very singular treasure, comprised of a unique coin.

2. My unfair coin

In the previous section we have shown that every real number can be simply manufactured by the tosses of a coin: a hand-made (or, better: finger-made) representation of reals[6]. But coin-cidentally, coin tossing is related to *chance*; hence, it is natural to ask ourselves about the probability of obtaining some particular results. More precisely, we may restrict our examination to numbers between 0 and 1, i.e. those numbers having integer part equal to 0. If we make an infinity of tosses, each one of them defining an element of the mantissa, what is the probability of obtaining, say, a rational number?

Unlike other questions of considerable difficulty that can be formulated in this field, there is a very precise answer to this one: the probability is zero. This might sound astonishing, especially if we think that most of the numbers we "know" are rational. However, it is easy to prove that there are "much more" irrationals than rationals[7]; in fact, we may give some intuitive evidence of this just by observing that no periodicity should be expected in a random sequence. What would we think, for instance, about a "random" sequence with this appearance?

0.001001001001001...[8]

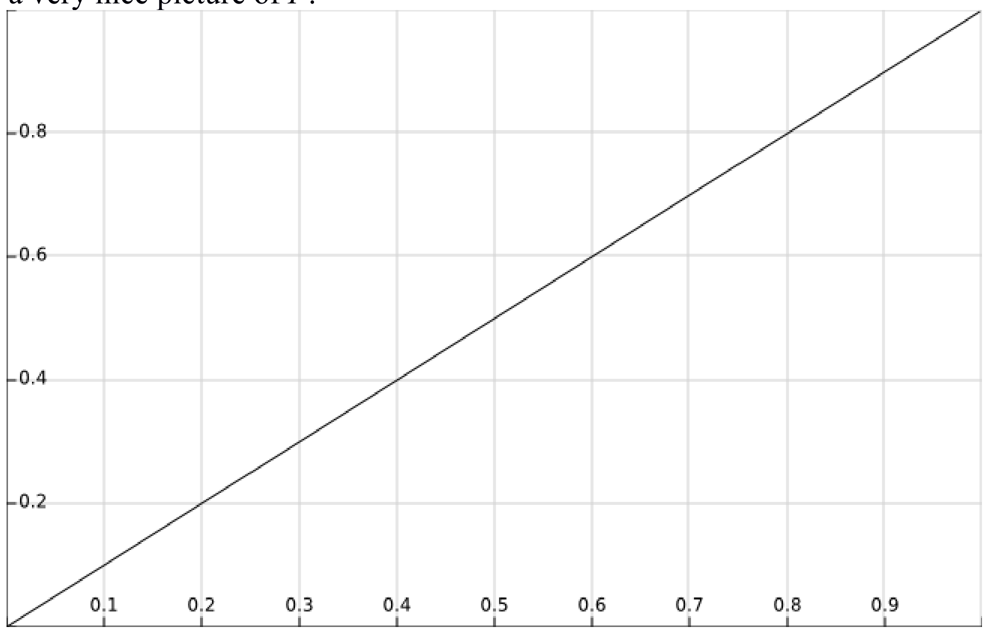
On the other hand, it is immediate to ask about the *distribution* related to this experiment, which can be expressed in the following terms: if we generate by coin-tossing a number a between 0 and 1, what is the probability that a is less or equal than a certain value x? This probability is usually denoted by $F(x)$, where F is the so-called *distribution function*. Since

6 This way of obtaining the reals may seem quite laborious; it is opportune then to remind the sharp way in which Sherlock Holmes characterizes his profession: *They say that genius is an infinite capacity for taking pains," he remarked with a smile. "It's a very bad definition, but it does apply to detective work. (A study in scarlet, chapter 3)*

7 This fact would have produced a quite strong impression to pythagoreans, who believed that only rational numbers existed (see e.g. P.Amster, *La matemática en la enseñanza de Lacan*. Ed.Lectour 2001).

8 This idea may provide a *definition* of what "randomness" means: random sequences must behave in a chaotic fashion, which certainly excludes periodicity and some other regularities. For example, this non-periodic sequence could never be considered "random": 0.101001000100001...

every sequence produces a non-negative number, it is clear that $F(0) = 0$; in the same way we deduce that $F(1) = 1$. In fact, it can be proved that $F(x) = x$ for every x between 0 and 1, which provides a very nice picture of F:

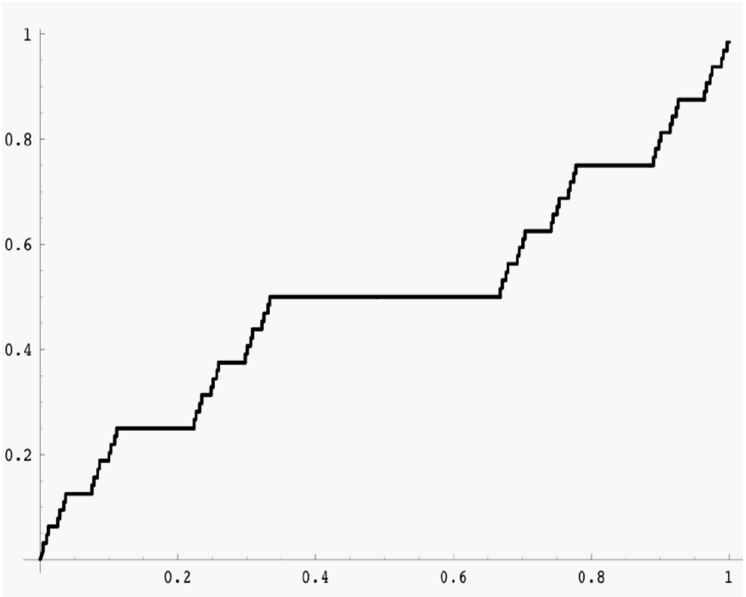


Graph of F

In such a way we have obtained the so-called *uniform distribution*. Certainly, this might not be a great surprise, since it seems reasonable that the probability of getting a result has to be "the same" at every sector of the segment. Why *reasonable*? Because we have employed that coin which is preferred by anyone who loves chance: the (impossible) *fair coin*, also known as the *coin of Tyche*. Here "fairness" means some behavior that is not exactly related to *decency*, but allows us to assume that the probability of getting heads or tails is equal in both cases to ½. If we are casting lots, it is fair to use a fair coin; otherwise we are cheating. But... don't we cheat when we speak?

This naive question opens a new field for our research: what happens if we use an *unfair* coin? For example, if the probability of getting heads is only ¼ then, in a sequence of a thousand tosses, we would expect to obtain "more or less" 250 heads and 750 tails; in this case, a sequence formed by 500 heads and 500 tails would be suspicious. For infinite tosses, both heads and tails should occur infinite times, with *relative frequencies* respectively equal to ¼ and ¾. But, what happens with the distribution function? It is clear that uniformity does not hold anymore, although it is not easy to explain exactly how this function looks like. It is worth to try, anyway: to begin with, note as before that $F(0) = 0$, $F(1) = 1$, and that $F(x)$ grows as x does. Namely, F joins the points (0,0) and (1,1) in a non-decreasing fashion, a fact that remains true if the probability p of getting heads is assumed to be any number between 0 and 1.

As mentioned, for $p = ½$ then F is very "right": is there anything that is more fair than a straight line, always realizing the shortest way between two points? For $p \neq ½$, the route of F draws necessarily more meanders; the striking fact is that, in a certain way, the graph of F will be *the largest* way between (0,0) and (1,1)[9]. Furthermore, it is proved that F behaves in a very strange manner: roughly speaking, the tangent of its graph is, at essentially any point, either horizontal or vertical. More precisely, the tangent will be horizontal *almost everywhere*, which says that F remains very quiet most of the time, although at some certain points it presents sudden "jumps", in which the tangent becomes vertical:



(Inconceivable) graph of F

As a final remark, let us mention that in the previous paragraph we have used an apparently imprecise expression: *almost everywhere*. It has, nevertheless, an exact meaning, which would be difficult to explain in this short note; however, in the present context we can replace it by a less obscure sentence: "with probability 1". Therefore, jumps are really infrequent, since they occur with probability 0. But they *do* exist, and this is the reason that permits F to achieve the end point (1,1). This amazing result holds for every $p \neq ½$: that is to say, fairness is a very delicate matter.

To conclude this paper, one more comment may be added. Perhaps the speech is comparable to the distribution of an unfair coin, almost everywhere well behaved, although producing from time to time some unexpected infinite jumps...

Pablo Amster

9 This assertion needs to be clarified. As Lacan mentions in his discussion on *Meno* by Plato, the length of the diagonal described by F when the coin is fair is the square root of 2: that is certainly a *minimum*. On the other hand, the length of the graph of a non-decreasing function cannot exceed 2. It can be proved that, in fact, the "length" of the graph of the distribution function F for p ½ is *exactly* 2 (see e.g. D.Stroock, *Doing analysis by tossing a coin*, The Mathematical Intelligencer 22, 2 (2000), *pp*. 66-73).

Post Scriptum:

It is a safe rule to apply that, when a mathematician or philosophical author writes with misty profundity, he is talking nonsense. A.N. Whitehead (1911).