

1 **DIFFUSIVE LIMIT TO A SELECTION-MUTATION EQUATION**
2 **WITH SMALL MUTATION FORMULATED ON THE SPACE OF**
3 **MEASURES**

4 AZMY S. ACKLEH AND NICOLAS SAINTIER

ABSTRACT. In this paper we consider a selection-mutation model with an advection term formulated on the space of finite signed measures on \mathbb{R}^d . The selection-mutation kernel is described by a family of measures which allows to handle continuous and discrete kernels under the same setting. We rescale the selection-mutation kernel to obtain a diffusively rescaled selection-mutation model. We prove that if the rescaled selection-mutation kernel converges to a pure selection kernel then the solution of the diffusively rescaled model converges to a solution of an advection-diffusion equation.

Keywords: Selection-mutation equation, small mutation diffusive limit, nonlinear first-order hyperbolic equation

AMS Subject Classification: 35L60, 35R06, 35Q92

5 1. INTRODUCTION

6 Selection-mutation equations with continuous trait values have been extensively
7 studied in the literature (e.g. [1, 2, 7, 8, 9, 10, 11, 12, 16, 17]). Such equations have
8 been formulated on several state spaces including the space of integrable functions
9 (e.g., [9, 10, 11]) and the space of finite signed measures (e.g., [1, 12, 16]). For example,
10 in [9] and [10] the authors consider selection-mutation equations in L^1 describing the
11 density of individuals with respect to a continuous phenotypic evolutionary trait.
12 They establish the existence of steady states and study their asymptotic behavior as
13 the mutation rate goes to zero. Papers with selection-mutation models formulated
14 on the space of measures are fewer. We mention for instance [1] where the authors
15 consider a selection-mutation model in $\mathcal{M}_b(Q)$, the space of finite signed measures
16 on a compact metric space Q , and study the long term behavior of the solution.
17 They show that for small mutation when the trait space is discrete there exists an
18 equilibrium measure that attracts all solutions with nonnegative not identically zero

1 initial measure/condition. In [12] the authors consider a selection-mutation model
 2 formulated on the space of measures and study existence-uniqueness of nonnegative
 3 solutions to this model.

4 In this paper, we consider a selection-mutation equation on $\mathcal{M}_b(\mathbb{R}^d)$ (the space of
 5 finite signed measures on \mathbb{R}^d), modeling the evolution of a population structured by
 6 a trait $x \in \mathbb{R}^d$. Denoting by μ_t the trait distribution at time t , μ_t evolves according
 7 to

$$(1) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = \int_{\mathbb{R}^d} K(z, \cdot) d\mu_t(z) - b\mu_t,$$

8 with $K : \mathbb{R}^d \rightarrow \mathcal{M}_{b,+}(\mathbb{R}^d)$ (the nonnegative cone of $\mathcal{M}_b(\mathbb{R}^d)$) and $b(x) = \int_{\mathbb{R}^d} K(x, dy)$.
 9 Here, the kernel $K(x, dy)$ models the trait distribution of the offspring of an individual
 10 with trait x and $b(x)$ is the birth rate of an individual with trait x . Considering such
 11 a general $K(x, dy)$ is important as it allows us to treat in a unified way continuous
 12 and discrete selection-mutation kernels and offspring distributions. The function v
 13 is a density-dependent vector-field and the advection term $\nabla \cdot (v[\mu_t] \mu_t)$ models the
 14 individuals actively adapting to their environment seeking phenotype changes that
 15 makes them fitter. These changes can be seen as stress-induced epi-mutations (e.g.,
 16 [5]). Such advection terms are also referred to in the literature as fast evolution with
 17 respect to the structured variable [15, 28].

We are interested in studying the limiting behavior of the solution to equations of
 type (1) when mutation becomes small, that is, when K is suitably rescaled so as
 to converge to a pure selection kernel. This question has already been addressed in
 several selection-mutation models in different settings. For example in [28][Chapter
 9] the author considers the model

$$\partial_t n(t, x) = \int_{\mathbb{R}^d} \widehat{K}(z, x - z) n(t, z) dz - b(x) n(t, x), \quad b(x) = \int_{\mathbb{R}^d} \widehat{K}(x, y) dy$$

on the state space $C([0, T], L^1(\mathbb{R}^d))$ where $\widehat{K}(x, y) \geq 0$ models the mutations. He
 then rescales this equation using diffusive rescaling as follows:

$$\partial_t n_\varepsilon(t, x) = \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} \varepsilon^{-d} \widehat{K}_\varepsilon(z, (x - z)/\varepsilon) n_\varepsilon(t, z) dz - b(x) n_\varepsilon(t, x) \right).$$

The family of kernels \widehat{K}_ε allows for a slight asymmetry of the offspring distribution namely

$$\int_{\mathbb{R}^d} y \widehat{K}_\varepsilon(x, y) dy = \varepsilon U(x) \quad \text{where} \quad U \in L^\infty(\mathbb{R}^d).$$

Under some regularity assumptions on \widehat{K}_ε the solution n_ε is shown to converge, up to a subsequence, weakly in $L^2([0, T] \times \mathbb{R}^d)$ to n , the solution of the convection-diffusion equation

$$\partial_t n + \nabla \cdot (Un) = \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}(B_{ij}n), \quad B_{ij}(x) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} y_i y_j \widehat{K}_\varepsilon(x, y) dy.$$

A different rescaling, namely the hyperbolic rescaling

$$\partial_t n_\varepsilon(t, x) = \frac{1}{\varepsilon} \left(\int_{\mathbb{R}^d} \varepsilon^{-d} \widehat{K}(z, (x-z)/\varepsilon) n(t, z) dz - b(x) n_\varepsilon(t, x) \right)$$

leads (in the limit as $\varepsilon \rightarrow 0$) to the transport equation

$$\partial_t n + \nabla \cdot (Un) = 0.$$

Similar results (but from a different perspective) can be found in the book [20] where non-local equations of the form

$$\partial_t u(t, x) = (J * u)(t, x) - u(t, x) = \int_{\mathbb{R}^d} J(x-z)(u(t, z) - u(t, x)) dz$$

1 are considered. Here, J is a smooth radial nonnegative kernel with integral one. By
 2 considering the rescaled kernel $J_\varepsilon(x) = \varepsilon^{-d} J(x/\varepsilon)$, it is proved that the solution to
 3 the rescaled equation converges to the solution of the heat equation and an estimate
 4 of the rate of convergence is provided. Many other variants of this equation are also
 5 considered in [20].

In the aforementioned works, the state spaces are Lebesgue or Sobolev function spaces. A similar result in measure spaces can be found in [14] where the authors investigate the model

$$\begin{aligned} \partial_t \xi_t(x) &= [(1 - \eta(x))b(x, V * \xi_t(x)) - d(x, U * \xi_t(x))] \xi_t(x) \\ &\quad + \int_{\mathbb{R}^d} M(z, x-z) \eta(z) b(z, V * \xi_t(z)) \xi_t(z) dz \end{aligned}$$

with initial condition $\xi_{|t=0} = \xi_0 \in \mathcal{M}_b(\mathbb{R}^d)$. The solution ξ_t is obtained as the limit of the law of a rescaled probabilistic particles system where each particle represents an animal and the location of the particle is the phenotypic characterization (trait) of

the animal. Here, b and d are the birth rate and death rate, $*$ denotes the convolution operator, which means that the interaction kernels U and V give the weight of each individual when interacting with a focal individual as a function of how phenotypically different they are, η is the mutation probability and $M(x, y)$ is the mutation kernel. They rescale the mutation kernel $M(x, y)$ assuming it is the density of a random variable with mean zero and variance-covariance matrix $\Sigma(x)/\kappa^\eta$, e.g., a Gaussian variable. They also accelerate the birth and death processes at a rate proportional to κ^η considering $b_\kappa(x, \zeta) = \kappa^\eta r(x) + b(x, \zeta)$ and $d_\kappa(x, \zeta) = \kappa^\eta r(x) + d(x, \zeta)$. Assuming $\eta < 1$, they prove among other results that the law of the particle system converges as $\kappa \rightarrow +\infty$ to the unique solution of the diffusion equation

$$\partial_t \xi_t(x) = [b(x, V * \xi_t(x)) - d(x, U * \xi_t(x))] \xi_t(x) + \frac{1}{2} \Delta(\sigma^2 r \mu \xi_t)(x).$$

1 Following those results we rescale (1) considering

$$(2) \quad \partial_t \mu_t^\varepsilon + \nabla \cdot (v[\mu_t^\varepsilon] \mu_t^\varepsilon) = \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon(z, \cdot) d\mu_t^\varepsilon(z) - b\mu_t^\varepsilon \right).$$

2 We will explain in detail below the appropriate way to rescale our general measure
3 kernel $K(x, dy)$. The main result of this paper states that as $\varepsilon \rightarrow 0$ the solution μ_t^ε
4 of (2) converges in $C([0, T], \mathcal{M}_{b,+}(\mathbb{R}^d))$, $T > 0$, to a solution of the diffusion equation

$$(3) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = \frac{1}{2} \sum_{i,j=1}^d \partial_{ij} (d_{ij}(\cdot, \mu_t) \mu_t)$$

where the diffusion coefficients d_{ij} are given by

$$d_{ij}(x, \mu) = \int_{\mathbb{R}^d} (y_i - x_i)(y_j - x_j) K(x, \mu, dy) \quad 1 \leq i, j \leq d.$$

5 Our model and proofs are flexible enough to allow for several generalizations including
6 asymmetric offspring distributions, hyperbolic rescaling and systems of equations.

7 This paper is organized as follows: in Section 2 we formulate a selection-mutation
8 model on the space of finite signed measures similar to (1). We present the rescaled
9 equation of type (2) and the main result concerning the limit of the rescaled equation.
10 We provide various examples of admissible selection-mutation kernels in Section 3. In
11 Section 4 we state some preliminary results and in Section 5 we establish the existence
12 and uniqueness of the measure solution to our model. The proof of the main theorem
13 concerning the convergence of μ_t^ε to μ_t (the solution of the limiting equation of the

1 form (3)) is presented in Section 6. A final section is devoted to some extensions of
 2 our model (asymmetric distribution, hyperbolic rescaling and systems of equations).

3 2. THE MODEL AND THE MAIN RESULT

4 We begin by modeling the selection-mutation kernel. To this end, consider a kernel
 5 $K : \mathbb{R}^d \rightarrow \mathcal{M}_{b,+}(\mathbb{R}^d)$ where for any $x \in \mathbb{R}^d$, $K(x, dy)$ is a nonnegative finite measure
 6 over \mathbb{R}^d which models the distribution of offspring of an animal with trait value x .
 7 Let $b(x) = \int_{\mathbb{R}^d} K(x, dy)$ be the total offspring (birth rate) of an individual with trait
 8 x . Note that $K(x, dy)$ is a probability measure when $b(x) \equiv 1$.

9 We point out that modeling the distribution of offspring $K(x, dy)$ by a general mea-
 10 sure has the advantage of allowing the treatment of singular and absolutely continuous
 11 distributions simultaneously. For example, following [7], we can choose

$$(4) \quad K(x, dy) = b(x)(1 - m(x))\delta_x + b(x)m(x)\beta(y - x)dy,$$

12 where $\beta \in L^1(\mathbb{R}^d)$ is nonnegative and $\beta(y - x)dy$ is the probability density that an
 13 animal with trait x gives rise to an offspring with trait y , and $m(x) \in [0, 1]$ is the
 14 proportion of offsprings with mutations (i.e., with different trait values).

15 Consider the equation

$$(5) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t]\mu_t) = \int_{\mathbb{R}^d} K(z, \cdot) d\mu_t(z) - b\mu_t,$$

16 where $v : \mathcal{M}_b(\mathbb{R}^d) \rightarrow W^{1,\infty}(\mathbb{R}^d)$ is a density-dependent vector-field. For example, a
 17 common choice of v that depends on μ through some weighted mean of μ is of the
 18 form

$$(6) \quad v[\mu](x) = V \left(x, \int_{\mathbb{R}^d} \eta(y) d\mu(y) \right)$$

19 for given maps $V : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ and $\eta : \mathbb{R}^d \rightarrow \mathbb{R}$.

20 We are interested in rescaling the reproduction kernel K so that the size of the
 21 mutations goes to 0 as the rescaling parameter ε goes to 0. For instance if $K(x, dy) =$
 22 $\beta(y - x)dy$ we let $\beta_\varepsilon(z) = \varepsilon^{-d}\beta(z/\varepsilon)$ and then rescale K considering $K_\varepsilon(x, dy) =$
 23 $\beta_\varepsilon(y - x)dy$. To do the same for a general kernel, it is useful to think that the
 24 probability measure $\beta_\varepsilon(y - x)dy$ converges to the Dirac mass δ_x as $\varepsilon \rightarrow 0$ and, from
 25 a more abstract point of view, that the curve $\varepsilon \in [0, 1] \rightarrow \beta_\varepsilon(y - x)dy$ connects δ_x to

1 $\beta(y-x)dy$ in $P(\mathbb{R}^d)$, the space of probability measures. Optimal transport theory
 2 provides us with a simple way to implement this idea when replacing $\beta(y-x)dy$
 3 by an arbitrary transition kernel $K(x, dy)$. Consider the constant map $T_x(y) = x$,
 4 $y \in \mathbb{R}^d$. Notice that $T_x\#\mu$, the push-forward of the bounded measure μ by T_x , is
 5 simply $\mu(\mathbb{R}^d)\delta_x$. We then consider the new kernel $K_\varepsilon(x, dy)$, $\varepsilon \in [0, 1]$, defined by

$$(7) \quad K_\varepsilon(x, dy) = (T_x + \varepsilon(Id - T_x))\#K(x, dy)$$

6 (where Id is the identity map). This means that for any test-function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$(8) \quad (K_\varepsilon(x, dy), \phi) = \int_{\mathbb{R}^d} \phi(x + \varepsilon(y - x)) K(x, dy).$$

7 Taking $\phi \equiv 1$, we observe that the measure $K_\varepsilon(x, dy)$ has the same total mass as
 8 $K(x, dy)$, namely $b(x)$. Thus, for each x , the map $\varepsilon \in [0, 1] \rightarrow K_\varepsilon(x, dy)$ is a curve in
 9 $\mathcal{M}_{b,+}(\mathbb{R}^d)$ connecting $b(x)\delta_x$ to $K(x, dy)$. When $b \equiv 1$, so that $K(x, dy)$ is a proba-
 10 bility measure in $P(\mathbb{R}^d)$ for each x , it is known that this curve is indeed the shortest
 11 path (or McCann interpolant) connecting these two measures in the Wasserstein space
 12 $(P(\mathbb{R}^d), W_2)$, where W_2 is the Monge-Kantorovich distance (see e.g. [29]).

13 Notice that in the smooth case where $K(x, dy) = \beta(y-x)dy$ and $\beta \in L^1(\mathbb{R}^d)$, we
 14 have for any test-function ϕ that

$$\begin{aligned} (K_\varepsilon(x, dy), \phi) &= \int_{\mathbb{R}^d} \phi(x + \varepsilon(y - x)) K(x, dy) = \int_{\mathbb{R}^d} \phi(x + \varepsilon(y - x)) \beta(y - x) dy \\ &= \int_{\mathbb{R}^d} \phi(z) \varepsilon^{-d} \beta((z - x)/\varepsilon) dz. \end{aligned}$$

15 So $K_\varepsilon(x, dy) = \beta_\varepsilon(y-x) dy$ and we recover the smooth case rescaling kernel.

16 Rescaling the kernel K by considering K_ε defined in (8) and at the same time
 17 increasing the mutation rate leads to the equation

$$(9) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon(z, \cdot) d\mu_t(z) - b\mu_t \right).$$

18 Inspired by model (9), in this paper we study the following more general equation
 19 with a nonlinear source term $\bar{N}(t, \cdot, \mu)\mu$ and a nonlinear selection-mutation kernel
 20 $K(x, \mu, dy)$:

$$(10) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = \bar{N}(t, \cdot, \mu_t) \mu_t + \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon(z, \mu_t, \cdot) d\mu_t(z) - b(\cdot, \mu_t) \mu_t \right)$$

1 with a given nonnegative initial condition $\mu_0 \in \mathcal{M}_{b,+}(\mathbb{R}^d)$, and where $b(x, \mu) =$
 2 $\int_{\mathbb{R}^d} K(x, \mu, dy)$. Introducing a nonlinearity with respect to μ in the kernel K and
 3 hence in the birth rate b allows to model competition between individuals due to
 4 limiting resources (e.g., [1, 2]). Furthermore, the added term $\bar{N}(t, \cdot, \mu_t)\mu_t$ can be used
 5 to model a loss term due to mortality (e.g., [1, 2]) or a source term due to external
 6 forces (e.g., seeds blown by wind in a population of plants).

7 For $\varepsilon > 0$, denote by μ_t^ε the solution to equation (10). In Theorem 5.3 below we
 8 extend the results obtained in [3] to show that given a nonnegative initial condition
 9 μ_0^ε , then equation (10) has a unique *nonnegative* solution μ_t^ε defined for any $t \geq 0$.
 10 Our main goal here is to study the asymptotic behavior of μ_t^ε , the solution of (10),
 11 as $\varepsilon \rightarrow 0$.

12 The need to consider nonnegative initial conditions μ_0 , and thus nonnegative so-
 13 lutions μ_t^ε , is motivated both from the biological and the mathematical point of
 14 view. From a biological perspective μ_t^ε models the distribution of the population
 15 in the trait space and thus must be a nonnegative measure. From a mathematical
 16 point of view, the non-negativity of μ_t^ε implies that the total variation norm of μ_t^ε
 17 is $\|\mu_t^\varepsilon\|_{TV} = \int_{\mathbb{R}^d} d\mu_t^\varepsilon$ which is useful to obtain the bound (30) below. Moreover, it
 18 allows us to use the norm $\|\cdot\|_{BL,3}$ defined in (18) below which is more suitable to our
 19 problem than the usual Bounded Lipschitz (BL) norm and at the same time has the
 20 same convergence properties as the BL norm on the positive cone of the measures.

21 We will show that up to a subsequence, as $\varepsilon \rightarrow 0$, the solution μ_t^ε of (10) converges
 22 to μ_t a solution of the non-linear convection-diffusion equation

$$(11) \quad \begin{cases} \partial_t \mu_t + \nabla \cdot (v[\mu_t]\mu_t) = \bar{N}(t, \cdot, \mu_t)\mu_t + \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}(d_{ij}(\cdot, \mu_t)\mu_t) \\ \mu|_{t=0} = \mu_0 \end{cases}$$

23 where the diffusion coefficients d_{ij} are given by

$$(12) \quad d_{ij}(x, \mu) = \int_{\mathbb{R}^d} (y_i - x_i)(y_j - x_j) K(x, \mu, dy) \quad 1 \leq i, j \leq d.$$

24 To our knowledge, existence for a parabolic but possibly degenerate equations like
 25 (11) has not been established in the literature. Furthermore, this limiting equation
 26 provides a connection between two types of approaches that have been used to model
 27 *small* mutation including using an integral operator (e.g., [1]) and a diffusion operator
 28 (e.g., [26]).

The matrix $(d_{ij}(x, \mu))_{ij}$ can be seen as a covariance matrix. Indeed, let $\tilde{K}(x, \mu, dy) := \tau_{-x}\#K(x, \mu, dy)$ where $\tau_{-x}(y) = y - x$. Thus, $\int_{\mathbb{R}^d} \phi(y) \tilde{K}(x, \mu, dy) = \int_{\mathbb{R}^d} \phi(y-x) K(x, \mu, dy)$ for any ϕ . In particular, we can rewrite d_{ij} as

$$d_{ij}(x, \mu) = \int_{\mathbb{R}^d} y_i y_j \tilde{K}(x, \mu, dy) \quad 1 \leq i, j \leq d.$$

Recall that $K(x, \mu, dy)$ is a nonnegative finite measure with total mass $b(x, \mu)$. Therefore, so is $\tilde{K}(x, \mu, dy)$. Moreover, assumption (K4) below implies that $\int_{\mathbb{R}^d} y \tilde{K}(x, \mu, dy) = 0$. Thus, the matrix $(d_{ij}(x, \mu)/b(x, \mu))_{ij}$ is the covariance matrix of the probability measure $\tilde{K}(x, \mu, dy)/b(x, \mu)$. In particular, it follows that $(d_{ij}(x, \mu))_{ij}$ is a symmetric nonnegative matrix. The non-negativity can be seen directly by noticing that for any $z \in \mathbb{R}^d$,

$$\sum_{i,j=1}^d d_{ij}(x, \mu) z_i z_j = \int_{\mathbb{R}^d} \sum_{i,j=1}^d (y_i z_i)(y_j z_j) \tilde{K}(x, \mu, dy) = \int_{\mathbb{R}^d} \left(\sum_{k=1}^d y_k z_k \right)^2 \tilde{K}(x, \mu, dy) \geq 0.$$

Before stating the assumptions on the model parameters, we need to recall some known measure-theoretical facts. Unless specified otherwise the space $\mathcal{M}_b(\mathbb{R}^d)$ of bounded Borel measures on \mathbb{R}^d is always endowed with the BL norm defined by

$$\|\mu\|_{BL} = \sup \left\{ \int_{\mathbb{R}^d} \phi d\mu : \phi \in W^{1,\infty}(\mathbb{R}^d), \|\phi\|_{\infty} + Lip(\phi) \leq 1 \right\}.$$

Here, $W^{1,\infty}(\mathbb{R}^d)$ denotes the space of bounded Lipschitz continuous functions, i.e., $\phi \in W^{1,\infty}(\mathbb{R}^d)$ if ϕ is bounded and there exists $C > 0$ such that

$$|\phi(x) - \phi(y)| \leq C|x - y|, \text{ for any } x, y \in \mathbb{R}^d.$$

- 1 We denote by $Lip(\phi)$ the least admissible constant C .

Other topologies can be considered on $\mathcal{M}_b(\mathbb{R}^d)$ like the weak* convergence and the total-variation norm. However, the BL norm seems to be the most useful when dealing with transport equations. We recall a few facts concerning the relationship between these three topologies. Let $\mu_n, \mu \in \mathcal{M}_b(\mathbb{R}^d)$. If $\mu_n \rightarrow \mu$ weak* (in the sense that $(\mu_n, \phi) \rightarrow (\mu, \phi)$ for any $\phi \in C_b(\mathbb{R}^d)$) then $\|\mu_n - \mu\|_{BL} \rightarrow 0$ but in general the converse is false except when $\mu_n, \mu \geq 0$ (see [Thm 6 and 8 in \[18\]](#)). Also, notice that a converging sequence $(\mu_n)_n$ in the BL norm may not be tight nor bounded in total variation. As an example, consider $\mu_n = \sqrt{n}(\delta_{n+1/n} - \delta_n)$. Then $\|\mu_n\|_{BL} = \sqrt{n}/n \rightarrow 0$ but $(\mu_n)_n$ is neither tight nor TV-bounded since $\|\mu_n\|_{TV} = 2\sqrt{n}$. Notice that $\mathcal{M}_b(\mathbb{R}^d)$

is not complete under the BL norm (see Thm 4.8 in [22] for a complete treatment of the completeness under the BL norm) but that a subset of the form

$$\mathcal{M}_{b,R}(\mathbb{R}^d) := \{\mu \in \mathcal{M}_b(\mathbb{R}^d) : \|\mu\|_{TV} \leq R\},$$

1 i.e., the ball in $\mathcal{M}_b(\mathbb{R}^d)$ of total variation radius R , is complete (see e.g. Thm 2.7 in
2 [23]).

We denote by $C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ the space of continuous functions from $[0, T]$ to $(\mathcal{M}_b(\mathbb{R}^d), \|\cdot\|_{BL})$ endowed with the usual sup-norm. Next we state our assumptions on the model functions v , \bar{N} , K and b appearing in (10). We start with the vector-field v and assume that

$$v : \mu \in \mathcal{M}_b(\mathbb{R}^d) \rightarrow v[\mu] \in W^{1,\infty}(\mathbb{R}^d)$$

3 satisfies

(V1) $v : \mathcal{M}_b(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)$ is continuous and for any $R > 0$ there exists $L_{v,R} > 0$ such that

$$\|v[\mu] - v[\tilde{\mu}]\|_\infty \leq L_{v,R} \|\mu - \tilde{\mu}\|_{BL} \quad \text{for any } \mu, \tilde{\mu} \in \mathcal{M}_{b,R}(\mathbb{R}^d),$$

(V2) there exists $C_v > 0$ such that

$$\text{Lip}(v[\mu]) \leq C_v \quad \text{for any } \mu \in \mathcal{M}_b(\mathbb{R}^d).$$

Concerning the source term \bar{N} , we assume that

$$\bar{N} : (t, x, \mu) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow \bar{N}(t, x, \mu) \in \mathbb{R}$$

4 is continuous in (t, x, μ) and

(N1) for any $R > 0$, there exist $L_{\bar{N},R} > 0$ such that for any $t \in \mathbb{R}$, any $\mu, \tilde{\mu} \in \mathcal{M}_{b,R}(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$,

$$|\bar{N}(t, x, \mu) - \bar{N}(t, x, \tilde{\mu})| \leq L_{\bar{N},R} \|\mu - \tilde{\mu}\|_{BL},$$

(N2) there exists $C_{\bar{N}} > 0$ such that

$$\|\bar{N}(t, \cdot, \mu)\|_{W^{1,\infty}} \leq C_{\bar{N}} \quad \text{for any } \mu \in \mathcal{M}_b(\mathbb{R}^d) \text{ and any } t \geq 0.$$

Finally, concerning the reproduction kernel K we assume that

$$K : (x, \mu) \in \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow K(x, \mu, dy) \in \mathcal{M}_b(\mathbb{R}^d)$$

5 satisfies

- 1 (K0) $K(x, \mu, dy) \geq 0$ whenever $\mu \geq 0$,
- 2 (K1) for any $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ measurable and bounded and any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$, the
3 function $x \rightarrow \int_{\mathbb{R}^d} \phi(y) K(x, \mu, dy)$ is measurable,
- 4 (K2) the function $b : (x, \mu) \in \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow \int_{\mathbb{R}^d} K(x, \mu, dy) \in \mathbb{R}$ is continuous
5 and satisfies
- 6 (i) there exists $C_b > 0$ such that $\|b(\cdot, \mu)\|_{W^{1,\infty}} \leq C_b$ for any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$,
- (ii) for any $R > 0$ there exists $L_{b,R} > 0$ such that

$$|b(x, \mu) - b(x, \tilde{\mu})| \leq L_{b,R} \|\mu - \tilde{\mu}\|_{BL} \text{ for any } \mu, \tilde{\mu} \in \mathcal{M}_{b,R}(\mathbb{R}^d) \text{ and any } x \in \mathbb{R}^d,$$

- (K3) for any $\phi \in W^{1,\infty}(\mathbb{R}^d)$, the function $x \rightarrow (K(x, \mu, \cdot), \phi)$ is Lipschitz with

$$\sup_{\mu \in \mathcal{M}_b(\mathbb{R}^d), \|\phi\|_{W^{1,\infty}} \leq 1} Lip(x \rightarrow (K(x, \mu, \cdot), \phi)) < \infty,$$

- (K4) for any $x \in \mathbb{R}^d$ and any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (y - x) K(x, \mu, dy) = 0,$$

- (K5) for any $R > 0$ there exists $C_{K,R} > 0$ such that

$$\int_{\mathbb{R}^d} |y - x|^3 K(x, \mu, dy) \leq C_{K,R} \text{ for any } x \in \mathbb{R}^d \text{ and } \mu \in \mathcal{M}_{b,R}(\mathbb{R}^d),$$

- (K6) if $\|\mu_n - \mu\|_{BL} \rightarrow 0$ then $\|K(x, \mu_n, \cdot) - K(x, \mu, \cdot)\|_{BL} \rightarrow 0$ for any $x \in \mathbb{R}^d$, and
for any $R > 0$ there exists $L_{K,R} > 0$ such that

$$\|K(x, \tilde{\mu}, \cdot) - K(x, \mu, \cdot)\|_{BL} \leq L_{K,R} \|\tilde{\mu} - \mu\|_{BL}$$

- 7 for any $x \in \mathbb{R}^d$ and any $\mu, \tilde{\mu} \in \mathcal{M}_{b,R}(\mathbb{R}^d)$,
- 8 (K7) the function $d_{ij}(x, \mu) = \int_{\mathbb{R}^d} (y_i - x_i)(y_j - x_j) K(x, \mu, dy)$ satisfies
- (i) for any $R > 0$ there exists $L_{d,R} > 0$ such that

$$\|d_{ij}(\cdot, \mu)\|_{W^{1,\infty}} \leq L_{d,R} \text{ for any } \mu \in \mathcal{M}_{b,R}(\mathbb{R}^d).$$

- 9 (ii) for any $x \in \mathbb{R}^d$ and for any nonnegative TV-bounded sequence $(\mu_n)_n$
10 converging in the BL norm to some μ , we have $d_{ij}(x, \mu_n) \rightarrow d_{ij}(x, \mu)$.

- (K8) There exists $C_K > 0$ such that

$$\|K(x, \mu, \cdot)\|_{TV} \leq C_K \text{ for any } x \in \mathbb{R}^d \text{ and any } \mu \in \mathcal{M}_b(\mathbb{R}^d).$$

1 An example of admissible vector fields $v[\mu]$ can be obtained through weighted means
2 of μ such as

$$(13) \quad v[\mu](x) = V \left(x, \int_{\mathbb{R}^d} \eta(y) d\mu(y) \right)$$

for given maps $V : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ and $\eta : \mathbb{R}^d \rightarrow \mathbb{R}$. Indeed, if we assume that V and η are both bounded and globally Lipschitz then for any $\mu, \tilde{\mu} \in \mathcal{M}_b(\mathbb{R}^d)$,

$$Lip(v[\mu]) \leq Lip(V) \quad \text{and} \quad \|v[\mu] - v[\tilde{\mu}]\|_\infty \leq Lip(V) \|\eta\|_{W^{1,\infty}} \|\mu - \tilde{\mu}\|_{BL}$$

3 so that assumptions (V1) and (V2) hold. We can build examples of admissible source
4 terms \bar{N} in the same spirit considering

$$(14) \quad \bar{N}(t, x, \mu) = A \left(t, x, \int_{\mathbb{R}^d} \eta(y) d\mu(y) \right)$$

for given bounded and globally Lipschitz maps $A : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow \mathbb{R}$ and $\eta : \mathbb{R}^d \rightarrow \mathbb{R}$. Then \bar{N} is continuous and satisfies assumptions (N1) and (N2) since

$$|\bar{N}(t, x, \mu) - \bar{N}(t, x, \tilde{\mu})| \leq Lip(A) \|\eta\|_{W^{1,\infty}} \|\mu - \tilde{\mu}\|_{BL}$$

and

$$\|\bar{N}(t, \cdot, \mu)\|_{W^{1,\infty}} \leq \|A\|_{W^{1,\infty}}.$$

5 Since the general formulation of the selection-mutation kernel K (and its rescaled
6 version K_ϵ) that we consider in this paper is important and novel, we will devote
7 the next section to provide several examples of biologically relevant and admissible
8 kernels, K , that satisfy the assumptions above.

9 Now we are in a position to state our main result:

10 **Theorem 2.1.** *Assume that $v : \mathcal{M}_b(\mathbb{R}^d) \rightarrow W^{1,\infty}(\mathbb{R}^d)$ satisfies (V1)-(V2), that
11 $\bar{N} : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow \mathbb{R}$ is continuous and satisfies (N1)-(N2), and that
12 $K : \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow \mathcal{M}_b(\mathbb{R}^d)$ satisfies (K0)-(K8). Then, for any nonnegative ini-
13 tial condition $\mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x| d\mu_0(x) < \infty$, there exists a solution
14 $\mu \in C([0, +\infty), \mathcal{M}_{b,+}(\mathbb{R}^d))$ to*

$$\begin{cases} \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = \bar{N}(t, \cdot, \mu_t) \mu_t + \frac{1}{2} \sum_{i,j=1}^d \partial_{ij} (d_{ij}(\cdot, \mu_t) \mu_t) \\ \mu|_{t=0} = \mu_0 \end{cases}$$

where the diffusion coefficients d_{ij} are given by

$$d_{ij}(x, \mu) = \int_{\mathbb{R}^d} (y_i - x_i)(y_j - x_j) K(x, \mu, dy) \quad 1 \leq i, j \leq d.$$

This solution is obtained as the limit as $\varepsilon \rightarrow 0$ (up to a subsequence) of the unique solution μ_t^ε of equation (10), namely

$$\partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = \bar{N}(t, \cdot, \mu_t) \mu_t + \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon(z, \mu_t, \cdot) d\mu_t(z) - b(\cdot, \mu_t) \mu_t \right),$$

in the sense that $\mu^\varepsilon \rightarrow \mu$ in $C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ for any $T > 0$. Moreover for any $T > 0$ there exist $C_T > 0$ such that

$$\int_{\mathbb{R}^d} (1 + |x|) d\mu_t \leq C_T \quad 0 \leq t \leq T.$$

1 We point out that the existence a unique solution μ_t^ε of equation (10) stated in
2 Theorem 2.1 will be established in Theorem 5.3 below.

3. EXAMPLES OF ADMISSIBLE SELECTION-MUTATION KERNELS

4 To demonstrate the usefulness of the general formulation of the kernel $K(x, \mu, dy)$
5 that we consider here, we present several examples of selection-mutation kernels from
6 the literature that can be unified under this general formulation.

7 • *A pure selection kernel:* a simple example is $K_S(x, \mu, dy) = \delta_x$ which models
8 the case where mutation never occurs, i.e., the offspring inherits exactly the
9 trait of its parent. In this case $b = 1$, $d_{ij} = 0$ and the assumptions (K0)-(K8)
10 trivially hold. Such a mutation kernel has been considered in [2].

11 • *A continuous mutation distribution:* for this kernel the offspring's trait is
assumed to be distributed according to a continuous probability distribution
 β , namely,

$$K_{CM}(x, \mu, dy) = \beta(x, \mu, y - x) dy$$

where $\beta(x, \mu, \cdot)$ is a nonnegative integrable function (so that $K(x, \mu, dy) \in \mathcal{M}_{b,+}(\mathbb{R}^d)$). This type of kernel has been studied, for example, in [9]. Here,

$$d_{ij}(x, \mu) = \int_{\mathbb{R}^d} z_i z_j \beta(x, \mu, z) dz.$$

Notice in particular that assumption (K7) is satisfied if we assume that d_{ij} is continuous in μ for any $x \in \mathbb{R}^d$, and if for any $z \in \mathbb{R}^d$ and any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$, the function $x \in \mathbb{R}^d \rightarrow \beta(x, \mu, z)$ is Lipschitz and satisfies for any $R > 0$,

$$\sup_{\mu \in \mathcal{M}_{b,R}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |z|^2 \|\beta(\cdot, \mu, z)\|_{W^{1,\infty}} dz < \infty.$$

Indeed,

$$\sup_{\mu \in \mathcal{M}_{b,R}(\mathbb{R}^d)} \|d_{ij}(\cdot, \mu)\|_{W^{1,\infty}} \leq \sup_{\mu \in \mathcal{M}_{b,R}(\mathbb{R}^d)} \int_{\mathbb{R}^d} |z|^2 \|\beta(\cdot, \mu, z)\|_{W^{1,\infty}} dz < \infty.$$

1
2

- *A discrete mutation distribution:* for this kernel the offspring's trait is assumed to be distributed according to a discrete probability distribution, namely,

$$K_{DM}(x, \mu, dy) = \sum_{k=1}^N \alpha_k(x, \mu) \delta_{x+h_k(x,\mu)}$$

3
4
5
6
7

where $N \in \mathbb{N}$, $\alpha_1, \dots, \alpha_N \in [0, 1]$ with $\sum_{k=1}^N \alpha_k = 1$, and $h_1(x, \mu), \dots, h_N(x, \mu) \in \mathbb{R}^d$. This reproduction kernel models the situation where an offspring of a parent with trait x will have trait $x + h_k(x, \mu)$ with probability $\alpha_k(x, \mu) \in [0, 1]$. A similar discrete selection-mutation kernel has been considered, for example, in [4]

- *A mixed mutation distribution:* we can combine the previous cases to obtain a reproduction kernel of the form

$$K(x, \mu, dy) = (1 - a(x, \mu) - b(x, \mu))\delta_x + a(x, \mu)K_{CM}(x, \mu, dy) + b(x, \mu)K_{DM}(x, \mu, dy)$$

8
9
10
11

where $a, b \in [0, 1]$ are the probabilities of choosing K_{CM} and K_{DM} , respectively. Of course this is only an example, since our model allows a priori any kind of measure (for instance a measure supported on some sub-manifold like a sphere for example).

12
13
14
15
16

A simple particular case is worth mentioning which corresponds to the case where K is homogeneous in the trait space, i.e., K depends on x only through a translation at x of a given measure $K(\mu, dy)$. To treat this case, consider a nonnegative bounded measure $K(\mu, dy)$. Let $\tau_x : y \in \mathbb{R}^d \rightarrow y + x \in \mathbb{R}^d$. Define $K(x, \mu, dy) := \tau_x \# K(\mu, dy)$ so that $\int_{\mathbb{R}^d} \phi(y) K(x, \mu, dy) = \int_{\mathbb{R}^d} \phi(y + x) K(\mu, dy)$ for any ϕ . Here $K(\mu, dy)$ is the

1 trait distribution of an offspring with parent's trait 0 and $K(x, \mu, dy)$ is obtained
 2 centering $K(\mu, dy)$ at x .

Notice that $K_S(x, dy) = \delta_x$ is space homogeneous (take $K(\mu, dy) = \delta_0$). Space homogeneous version of K_{CM} and K_{DM} are

$$K_{CM}(x, \mu, dy) = \beta(\mu, y - x)dy$$

obtained from $\beta(\mu, y)dy$, and

$$K_{DM}(x, \mu, dy) = \sum_{k=1}^N \alpha_k(\mu) \delta_{x+h_k(\mu)}$$

3 obtained from $\sum_{k=1}^N \alpha_k(\mu) \delta_{h_k(\mu)}$.

4 In the case of a space homogeneous kernel, it is easily seen that (K1) and (K3) are
 5 satisfied. Towards (K1) note that if $\nu \in \mathcal{M}_{b,+}(\mathbb{R}^d)$ then for any $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ measur-
 6 able and bounded and any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$ we have $\int_{\mathbb{R}^d \times \mathbb{R}^d} |\phi(y+x)| K(\mu, dy) \nu(dx) < \infty$
 7 and a simple application of Fubini's Theorem results in $x \rightarrow \int_{\mathbb{R}^d} \phi(y+x) K(\mu, dy)$
 8 being measurable.

9 As for (K3), since for any $\phi \in W^{1,\infty}(\mathbb{R}^d)$, $\|\phi\|_{W^{1,\infty}} \leq 1$, we have $(K(x, \mu, \cdot), \phi) =$
 10 $\int \phi(y+x) K(\mu, dy)$ so that

$$\begin{aligned} |(K(x, \mu, \cdot), \phi) - (K(\tilde{x}, \mu, \cdot), \phi)| &\leq \int_{\mathbb{R}^d} |\phi(y+x) - \phi(y+\tilde{x})| K(\mu, dy) \\ &\leq |x - \tilde{x}| \|K(\mu, dy)\|_{TV} \leq C_K |x - \tilde{x}|, \end{aligned}$$

11 where we used (K8) in the last inequality. Thus, the function $x \rightarrow (K(x, \mu, \cdot), \phi)$
 12 is Lipschitz with Lipschitz constant less than C_K for any $\|\phi\|_{W^{1,\infty}} \leq 1$ and any
 13 $\mu \in \mathcal{M}_b(\mathbb{R}^d)$.

14 The remaining assumptions can be simplified as follows by dropping the dependence
 15 on x .

16 (K0') $K(\mu, dy) \geq 0$ whenever $\mu \geq 0$,

(K2') the function $b : \mu \in \mathcal{M}_b(\mathbb{R}^d) \rightarrow \int_{\mathbb{R}^d} K(\mu, dy) \in \mathbb{R}$ is continuous and for any
 $R > 0$ there exists $L_{b,R} > 0$ such that

$$|b(\mu) - b(\tilde{\mu})| \leq L_{b,R} \|\mu - \tilde{\mu}\|_{BL} \quad \text{for any } \mu, \tilde{\mu} \in M_{b,R}(\mathbb{R}^d),$$

17 (K4') $\int_{\mathbb{R}^d} y K(\mu, dy) = 0$

(K5') for any $R > 0$ there exists $C_{K,R} > 0$ such that

$$\int_{\mathbb{R}^d} |y|^3 K(\mu, dy) \leq C_{K,R} \quad \text{for any } \mu \in \mathcal{M}_{b,R}(\mathbb{R}^d),$$

(K6') if $\|\mu_n - \mu\|_{BL} \rightarrow 0$ then $\|K(\mu_n, \cdot) - K(\mu, \cdot)\|_{BL} \rightarrow 0$, and for any $R > 0$ there exists $L_{K,R} > 0$ such that

$$\|K(\tilde{\mu}, \cdot) - K(\mu, \cdot)\|_{BL} \leq L_{K,R} \|\tilde{\mu} - \mu\|_{BL} \quad \text{for any } \mu, \tilde{\mu} \in \mathcal{M}_{b,R}(\mathbb{R}^d),$$

1 (K7') the function $d_{ij}(\mu) = \int_{\mathbb{R}^d} y_i y_j K(\mu, dy)$ is such that $d_{ij}(\mu_n) \rightarrow d_{ij}(\mu)$ for any
 2 nonnegative TV-bounded sequence $(\mu_n)_n$ converging for the BL-norm to μ .

(K8') There exists $C_K > 0$ such that

$$\|K(\mu, \cdot)\|_{TV} \leq C_K \quad \text{for any } \mu \in \mathcal{M}_b(\mathbb{R}^d).$$

3 We can make a further simplification assuming that K is space homogeneous and
 4 also independent of μ i.e. $K(x, \mu, dy) = K(x, dy) = \tau_x \# K(dy)$ for some nonnegative
 5 $K(dy) \in \mathcal{M}_{b,+}(\mathbb{R}^d)$. It is easily verified that all the assumptions hold with the
 6 exception of (K4) and (K5) that can be simplified to

7 (K4'') $\int_{\mathbb{R}^d} y K(dy) = 0$

8 (K5'') $\int_{\mathbb{R}^d} |y|^3 K(dy) < +\infty$

9 For instance in the case of a continuous kernel $K_{CM}(dy) = \beta(y)dy$ with $\beta \in L^1(\mathbb{R}^d)$,
 10 $\beta \geq 0$, assumptions (K4'') and (K5'') reduce to

$$(15) \quad \int_{\mathbb{R}^d} y \beta(y) dy = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |y|^3 \beta(y) dy < \infty.$$

11 In the case of a discrete mutation kernel $K_{DM}(dy) = \sum_{k=1}^N \alpha_k \delta_{h_k}$ with $\alpha_1, \dots, \alpha_N \in$
 12 $[0, 1]$, $\alpha_1 + \dots + \alpha_N = 1$ and $h_1, \dots, h_N \in \mathbb{R}^d$, (K5'') is void and (K4'') is $\sum_{k=1}^N \alpha_k h_k = 0$.

13 Using space homogenous kernels, we can introduce easily some space inhomogeneity
 14 and measure-dependence by considering a convex combination like

$$(16) \quad \begin{aligned} K(x, \mu, dy) &= (1 - a_1(x, \mu) - a_2(x, \mu)) \delta_x + a_1(x, \mu) \beta(y - x) dy \\ &+ a_2(x, \mu) \sum_{k=1}^N \alpha_k \delta_{x+h_k}. \end{aligned}$$

15 We then have the following direct corollary of Theorem 2.1:

16 **Corollary 3.1.** *Assume that $a_1, a_2 : \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow [0, 1]$ satisfy $a_1 + a_2 \leq 1$, and*
 17 *for $k = 1, 2$,*

- 1 (i) there exists $C > 0$ such that $\|a_k(\cdot, \mu)\|_{W^{1,\infty}} \leq C$ for any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$,
 (ii) for any $R > 0$ there exists $L_{b,R} > 0$ such that

$$|a_k(x, \mu) - a_k(x, \tilde{\mu})| \leq L_{b,R} \|\mu - \tilde{\mu}\|_{BL} \quad \text{for any } \mu, \tilde{\mu} \in M_{b,R}(\mathbb{R}^d),$$

- 2 (iii) a_k is continuous in μ for any fixed $x \in \mathbb{R}^d$.

3 Assume also that $\beta \in L^1(\mathbb{R}^d)$ is nonnegative and satisfies (15) and that $\alpha_1, \dots, \alpha_N \geq 0$
 4 and $h_1, \dots, h_N \in \mathbb{R}^d$ satisfy $\sum_{k=1}^N \alpha_k h_k = 0$.

Then, for any nonnegative initial condition $\mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x| d\mu_0(x) < \infty$, there exists a solution $\mu \in C([0, +\infty), \mathcal{M}_{b,+}(\mathbb{R}^d))$ to (11) with diffusion coefficients

$$d_{ij}(x, \mu) = a_1(x, \mu) \int_{\mathbb{R}^d} z_i z_j \beta(z) dz + a_2(x, \mu) \sum_{k=1}^N h_k^i h_k^j$$

5 (where $h_k = (h_k^1, \dots, h_k^d)$). This solution is obtained as the limit as $\varepsilon \rightarrow 0$ (up to a
 6 subsequence) of the solutions μ_t^ε of (10) in the sense that $\mu^\varepsilon \rightarrow \mu$ in $C([0, T], \mathcal{M}_b(\mathbb{R}^d))$
 7 for any $T > 0$.

8 *Proof.* Notice that

$$\begin{aligned} b(x, \mu) &= \int_{\mathbb{R}^d} K(x, \mu, dy) \\ &= (1 - a_1(x, \mu) - a_2(x, \mu)) + a_1(x, \mu) \int_{\mathbb{R}^d} \beta(y - x) dy + a_2(x, \mu) \sum_{k=1}^N \alpha_k. \end{aligned}$$

9 In view of the assumptions on a_1 and a_2 , b satisfies (K2) and d_{ij} satisfies (K7). The
 10 assumptions (K3) and (K6) clearly hold, while (K4) and (K5) are satisfied by the
 11 assumptions made on β and $\alpha_1, \dots, \alpha_N$. \square

12 As we already saw in examples (13) and (14) of admissible vector fields $v[\mu]$ and
 13 source terms $\bar{N}(t, x, \mu)$, we can verify that admissible functions a_k , $k = 1, 2$, are given
 14 by

$$(17) \quad a_k(x, \mu) := A_k \left(x, \int_{\mathbb{R}^d} \eta_k(y) d\mu(y) \right)$$

15 for some bounded and globally Lipschitz functions $\eta_k : \mathbb{R}^d \rightarrow \mathbb{R}$ and $A_k : \mathbb{R}^d \times \mathbb{R} \rightarrow$
 16 $[0, +\infty)$ such $A_1(x, t) + A_2(x, t) \leq 1$ for any $(x, t) \in \mathbb{R}^d \times \mathbb{R}$.

17 Except for the last section which contains comments on extension of our results, the
 18 rest of the paper is devoted to the proof of Theorem 2.1. In particular, in Section 4 we

1 establish some technical results necessary for the proofs. Specifically, we introduce a
 2 variant of the BL norm denoted by $\|\cdot\|_{BL,N}$ and show that this norm is equivalent to
 3 the BL norm on the cone of nonnegative measures and we also establish the existence
 4 of $\int_{\mathbb{R}^d} K(x, \mu, dy) d\mu$ as a Bochner integral. We then proceed in section 5 with the
 5 existence and uniqueness of the solution μ_t^ε of equation (10). We finally present the
 6 proof of Theorem 2.1 in section 6.

7 4. PRELIMINARY TECHNICAL RESULTS.

8 We begin this section by introducing a variant of the BL norm and showing that
 9 it is equivalent to the BL norm for nonnegative measures.

10 **4.1. An equivalent norm on nonnegative measures.** We will need the following
 11 variant of the BL-norm: given an integer $N \geq 1$ we consider the norm

$$(18) \quad \|\mu\|_{BL,N} = \sup \left\{ \int_{\mathbb{R}^d} \phi d\mu : \phi \in C^N(\mathbb{R}^d), \|D^\alpha \phi\|_\infty \leq 1, 0 \leq |\alpha| \leq N \right\}.$$

12 This kind of norm has been considered in [21] but taking $|\alpha| \geq 1$.

13 **Proposition 4.1.** *Let $\mu_n, \mu \in \mathcal{M}_{b,+}(\mathbb{R}^d)$. Then*

$$(19) \quad \|\mu_n - \mu\|_{BL} \rightarrow 0 \quad \text{if and only if} \quad \|\mu_n - \mu\|_{BL,N} \rightarrow 0.$$

14 *As a consequence, a function from $[0, T]$ to $(\mathcal{M}_{b,+}(\mathbb{R}^d), \|\cdot\|_{BL})$ is continuous if and*
 15 *only if it is continuous from $[0, T]$ to $(\mathcal{M}_{b,+}(\mathbb{R}^d), \|\cdot\|_{BL,N})$. Moreover, consider $\mu^\varepsilon, \mu \in$
 16 $C([0, T], \mathcal{M}_{b,+}(\mathbb{R}^d))$. Then, convergence as $\varepsilon \rightarrow 0$ of μ^ε to μ in $C([0, T], (\mathcal{M}_{b,+}(\mathbb{R}^d), \|\cdot\|_{BL}))$
 17 is equivalent to convergence in $C([0, T], (\mathcal{M}_{b,+}(\mathbb{R}^d), \|\cdot\|_{BL,N}))$.*

18 The proof of (19) is inspired by [18][proof of Thm 8].

Proof. We first prove (19). Since $\|\mu\|_{BL,N} \leq \|\mu\|_{BL}$ the \implies implication is obvious. Let us assume that $\|\mu_n - \mu\|_{BL,N} \rightarrow 0$. We have to prove that $\mu_n \rightarrow \mu$ in the BL-norm. Since $\mu_n, \mu \geq 0$, this is equivalent to proving that $\mu_n \rightarrow \mu$ weak*. According to Portmanteau Theorem (see e.g. Thm 2.1. in [6]) this amount to showing that $\liminf \mu_n(U) \geq \mu(U)$ for any open subset $U \subset \mathbb{R}^d$. Let $F_m = \{x \in U : \text{dist}(x, \mathbb{R}^d \setminus U) \geq 1/m, |x| \leq m\}$, $m \in \mathbb{N}$. Notice that the sets F_m are compact and they form an increasing sequence with $\cup_m F_m = U$. Thus $\mu(F_m) \uparrow \mu(U)$ so that, given $\varepsilon > 0$, we can fix m such that $\mu(F_m) \geq \mu(U) - \varepsilon$. Let $\phi \in C_c^\infty(U)$ such that $0 \leq \phi \leq 1$

and $\phi = 1$ on F_m . (ϕ is obtained as the convolution of the characteristic function of a neighborhood of F_m contained in U and the standard mollifiers (see, e.g., [Thm 1.4.1 in \[25\]](#)). We have $\|D^\alpha \phi\|_\infty \leq c_k m^k$ for $|\alpha| \leq k$. Then, $C_N \phi$ is an admissible test function for $\|\cdot\|_{BL,N}$ for some $C_N > 0$. Thus,

$$\liminf_n \mu_n(U) \geq \liminf_n \int \phi d\mu_n = \int \phi d\mu \geq \mu(F_m) \geq \mu(U) - \varepsilon$$

1 It follows that $\liminf \mu_n(U) \geq \mu(U)$.

2 We now consider $\mu^\varepsilon, \mu \in C([0, T], \mathcal{M}_b(\mathbb{R}^d))$. Since $\|\cdot\|_{BL,N} \leq \|\cdot\|_{BL}$, convergence of
3 μ^ε to μ in $C([0, T], (\mathcal{M}_b(\mathbb{R}^d), \|\cdot\|_{BL}))$ implies convergence in $C([0, T], (\mathcal{M}_b(\mathbb{R}^d), \|\cdot\|_{BL,N}))$.

4 Let us now assume that the convergence holds in $C([0, T], (\mathcal{M}_b(\mathbb{R}^d), \|\cdot\|_{BL,N}))$ but not
5 in $C([0, T], (\mathcal{M}_b(\mathbb{R}^d), \|\cdot\|_{BL}))$. Then there exist $\delta > 0$, $(t_k)_k \subset [0, T]$, and $\varepsilon_k \rightarrow 0$ such
6 that $\|\mu_{t_k}^{\varepsilon_k} - \mu_{t_k}\|_{BL} \geq \delta$. We can assume without loss of generality that there exists
7 $t := \lim_k t_k$. Then $\delta \leq \|\mu_{t_k}^{\varepsilon_k} - \mu_t\|_{BL} + \|\mu_{t_k} - \mu_t\|_{BL}$. Since $\|\mu_{t_k} - \mu_t\|_{BL,N} \rightarrow 0$, we
8 have $\|\mu_{t_k} - \mu_t\|_{BL} \rightarrow 0$ by (19). Thus for large k ,

$$(20) \quad \|\mu_{t_k}^{\varepsilon_k} - \mu_t\|_{BL} \geq \delta/2.$$

9 On the other hand

$$\begin{aligned} \|\mu_{t_k}^{\varepsilon_k} - \mu_t\|_{BL,N} &\leq \|\mu_{t_k}^{\varepsilon_k} - \mu_{t_k}\|_{BL,N} + \|\mu_{t_k} - \mu_t\|_{BL,N} \\ &\leq \max_{0 \leq s \leq T} \|\mu_s^{\varepsilon_k} - \mu_s\|_{BL,N} + o(1) \rightarrow 0 \end{aligned}$$

10 since $\mu^\varepsilon \rightarrow \mu$ in $C([0, T], (\mathcal{M}_b(\mathbb{R}^d), \|\cdot\|_{BL,N}))$. Thus, $\|\mu_{t_k}^{\varepsilon_k} - \mu_t\|_{BL} \rightarrow 0$ by (19) which
11 contradicts (20). \square

4.2. Existence of $\int_{\mathbb{R}^d} K(x, \mu, dy) d\mu$ as a Bochner integral. Consider a mutation kernel $K : (x, \mu) \in \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow K(x, \mu, dy) \in \mathcal{M}_b(\mathbb{R}^d)$ and fix a nonnegative measure $\mu \in \mathcal{M}_b(\mathbb{R}^d)$. Using results from [19][Appendix C] we will argue that the integral $\int_{\mathbb{R}^d} K(x, \mu, dy) d\mu(x)$ exists as a Bochner integral in $\mathcal{M}_b(\mathbb{R}^d)$. Let $p(x) := K(x, \mu, dy)$ so that $p : \mathbb{R}^d \rightarrow \mathcal{M}_b(\mathbb{R}^d)$. From assumption (K1) it follows that the map $x \in \mathbb{R}^d \rightarrow (p(x), \phi)$ is measurable. Thus according to [19][Appendix C], p is Bochner measurable as a map with values in $\overline{\mathcal{M}}_b(\mathbb{R}^d)$ (the completion of $\mathcal{M}_b(\mathbb{R}^d)$ in the BL norm). Using (K8), the map $x \rightarrow \|p(x)\|_{TV}$ is in $L^1(\mathbb{R}^d, \mu)$. It follows from [19][Appendix C] that the integral $\int_{\mathbb{R}^d} K(x, \mu, dy) d\mu$ exists as a Bochner integral in $\overline{\mathcal{M}}_b(\mathbb{R}^d)$. Furthermore, its total variation is less or equal to $\int_{\mathbb{R}^d} \|K(x, \mu, dy)\|_{TV} d\mu(x)$, and thus

$\int_{\mathbb{R}^d} K(x, \mu, dy) d\mu$ is in fact a bounded measure. Finally, for any bounded measurable function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\left(\int_{\mathbb{R}^d} K(x, \mu, dy) d\mu, \phi \right) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi(y) K(x, \mu, dy) \right) d\mu.$$

1 Notice that the above still hold true for a signed finite measure μ by writing $\mu =$
 2 $\mu^+ - \mu^-$ and defining $\int_{\mathbb{R}^d} K(x, \mu, dy) d\mu := \int_{\mathbb{R}^d} K(x, \mu, dy) d\mu^+ - \int_{\mathbb{R}^d} K(x, \mu, dy) d\mu^-$.

3 5. GLOBAL NONNEGATIVE SOLUTION TO A NONLINEAR TRANSPORT EQUATION
 4 AND EXISTENCE OF μ^ε .

5 Consider the following nonlinear transport equation in \mathbb{R}^d

$$(21) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = N(t, \mu_t).$$

6 Assume that the vector-field $v : \mathcal{M}_b(\mathbb{R}^d) \rightarrow W^{1,\infty}(\mathbb{R}^d)$ satisfies (V1) and (V2)
 7 above. Furthermore, assume that the source term $N : \mathbb{R}_+ \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow \mathcal{M}_b(\mathbb{R}^d)$
 8 is continuous in (t, μ) (recall that we endow $\mathcal{M}_b(\mathbb{R}^d)$ with the BL norm). We also
 9 assume there exists $C^N > 0$, and for any $R > 0$, that there exist $L_{N,R} > 0$ such that
 10 the following is satisfied:

11 (N1') $\|N(t, \mu) - N(t, \tilde{\mu})\|_{BL} \leq L_{N,R} \|\mu - \tilde{\mu}\|_{BL}$ for any $t \in \mathbb{R}$ and any $\mu, \tilde{\mu} \in$
 12 $\mathcal{M}_{b,R}(\mathbb{R}^d)$,

13 (N2') $\|N(t, \mu)\|_{TV} \leq C^N (1 + \|\mu\|_{TV})$ for any $t \in \mathbb{R}$ and $\mu \in \mathcal{M}_b(\mathbb{R}^d)$.

14 The following result is from [3][Cor. 7.3]:

Theorem 5.1. *Assume that the vector-field $v : \mathcal{M}_b(\mathbb{R}^d) \rightarrow W^{1,\infty}(\mathbb{R}^d)$ satisfies (V1) and (V2) above, and that the source term $N : \mathbb{R}_+ \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow \mathcal{M}_b(\mathbb{R}^d)$ is continuous and satisfies (N1')-(N2') above. Then for any initial condition $\mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$ there exists a unique solution $\mu \in C([0, +\infty), \mathcal{M}_b(\mathbb{R}^d))$ to equation (21), namely*

$$\partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = N(t, \mu_t).$$

15 Moreover for any $T > 0$, there exists $R_T > 0$ such that $\|\mu_t\|_{TV} \leq R_T$ for $t \in [0, T]$.

16 In order to obtain a nonnegative solution when $\mu_0 \in \mathcal{M}_{b,+}(\mathbb{R}^d)$ we first consider an
 17 equation of the form

$$(22) \quad \partial_t \mu_t + \nabla \cdot (v(t, \cdot) \mu_t) = c(t, \cdot) \mu_t + B(\mu_t) \quad t \in [0, T]$$

1 with initial condition $\mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$, where $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $c : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$
 2 and $B : \mathcal{M}_b(\mathbb{R}^d) \rightarrow \mathcal{M}_b(\mathbb{R}^d)$. We assume that

3 (H1) v is continuous bounded in (t, x) and globally Lipschitz in x uniformly in t ,
 4 i.e., there exists $C > 0$ such that $|v(t, x) - v(t, x')| \leq C|x - x'|$ for any $t \in [0, T]$
 5 and any $x, x' \in \mathbb{R}^d$.

6 (H2) c is continuous and there exists $C > 0$ such that $\|c(t, \cdot)\|_{W^{1, \infty}} \leq C$ for any
 7 $t \in [0, T]$,

8 (H3) B is continuous, $B(\mu) \geq 0$ if $\mu \geq 0$, $\|B(\mu)\|_{TV} \leq C(1 + \|\mu\|_{TV})$ for any
 9 $\mu \in \mathcal{M}_b(\mathbb{R}^d)$, and for any $R > 0$ there exists $C_R > 0$ such that $\|B(\mu) -$
 10 $B(\tilde{\mu})\|_{BL} \leq C_R\|\mu - \tilde{\mu}\|_{BL}$ for any $\mu, \tilde{\mu} \in \mathcal{M}_{b, R}(\mathbb{R}^d)$.

11 **Proposition 5.1.** *Assume that $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $c : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and*
 12 *$B : \mathcal{M}_b(\mathbb{R}^d) \rightarrow \mathcal{M}_b(\mathbb{R}^d)$ satisfy assumptions (H1)-(H3) above. Then equation (22)*
 13 *has a unique solution $\mu \in C([0, T], \mathcal{M}_b(\mathbb{R}^d))$. Moreover, the solution μ_t is nonnegative*
 14 *if $\mu_0 \geq 0$.*

Proof. With the above assumptions it is easily seen from [3][Cor. 7.3] and [3][Prop. 5.1] that (22) has a unique solution $\tilde{\mu} \in C([0, T], \mathcal{M}_b(\mathbb{R}^d))$. Let us verify that this solution is nonnegative if $\mu_0 \geq 0$. To this end, we denote by $T_{s,t}$ the flow of v (which is well-defined for any $s, t \in [0, T]$) and consider

$$h(s, t, x) = \exp \left(\int_s^t c(\tau, T_{t,\tau}(x)) d\tau \right).$$

15 Notice that

$$(23) \quad \begin{aligned} h(s, t, x) &\leq e^{\|c\|_{\infty} T}, & |h(s, t', x) - h(s, t, x)| &\leq C|t - t'|, \\ |h(s, t, x') - h(s, t, x)| &\leq C|x - x'|. \end{aligned}$$

Then, for $\mu \in C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ we define $\Gamma(\mu) \in C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ by

$$\Gamma(\mu)_t := h(0, t, \cdot)(T_{0,t}\#\mu_0) + \int_0^t h(s, t, \cdot)(T_{s,t}\#B(\mu_s)) ds.$$

16 This means that for any $\phi \in C_b(\mathbb{R}^d)$,

$$\begin{aligned} (\Gamma(\mu)_t, \phi) &= \int_{\mathbb{R}^d} \phi(T_{0,t}(x)) h(0, t, T_{0,t}(x)) d\mu_0(x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \phi(T_{s,t}(x)) h(s, t, T_{s,t}(x)) d(B(\mu_s))(x) ds \end{aligned}$$

1 and thus

$$\begin{aligned}
 (\Gamma(\mu)_t, \phi) &= \int_{\mathbb{R}^d} \phi(T_{0,t}(x)) \exp\left(\int_0^t c(\tau, T_{0,\tau}(x)) d\tau\right) d\mu_0(x) \\
 &\quad + \int_0^t \int_{\mathbb{R}^d} \phi(T_{s,t}(x)) \exp\left(\int_s^t c(\tau, T_{s,\tau}(x)) d\tau\right) d(B(\mu_s))(x) ds.
 \end{aligned}$$

2 Notice that if $\mu \in C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ is a fixed-point of Γ in the sense that $\mu_t = \Gamma(\mu)_t$
 3 for any $t \in [0, T]$, then μ is a [solution of \(22\)](#) and thus is equal to $\tilde{\mu}$, the unique
 4 solution to (22).

We look for a nonnegative fixed-point of Γ in

$$X_S = \{\mu \in C([0, S], \mathcal{M}_b(\mathbb{R}^d)) : \mu|_{t=0} = \mu_0, \|\mu_t\|_{TV} \leq 2\|\mu_0\|_{TV}, \mu_t \geq 0, t \in [0, S]\}$$

5 for some S . Notice that X_S is complete as a [closed subset](#) of $C([0, S], \mathcal{M}_{b,R}(\mathbb{R}^d))$
 6 with $R = 2\|\mu_0\|_{TV}$. Moreover, since $B(\mu) \geq 0$ for $\mu \geq 0$ we have that $\Gamma(\mu)_t \geq 0$ if
 7 $\mu_t \geq 0$. We can then prove in a standard way that for S small enough, depending
 8 only on $\|\mu_0\|_{TV}$, $\Gamma(X) \subset X$ and that Γ is a strict contraction. Thus Γ has a unique
 9 fixed-point in X_S which is therefore $\tilde{\mu}$. We can then extend the fixed-point μ_t to
 10 a maximal time interval $[0, S^*)$. If $S^* < T$ then $\lim_{t \rightarrow S^*} \|\mu_t\|_{TV} = +\infty$. But since
 11 $\mu_t = \tilde{\mu}_t$ on $[0, T^*)$ and we know that $\|\tilde{\mu}_t\|_{TV} \leq C$ on $[0, T]$ we obtain a contradiction.
 12 Thus $\tilde{\mu}$, the unique solution to (22), is nonnegative. \square

13 Combining Theorem 5.1 and Proposition 5.1 we can obtain nonnegative global
 14 solution to the following equation:

$$(24) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = \bar{N}(t, \cdot, \mu_t) \mu_t + B(\mu_t) \quad t > 0$$

15 with initial condition $\mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$.

16 **Theorem 5.2.** *Assume that $v : \mathcal{M}_b(\mathbb{R}^d) \rightarrow W^{1,\infty}(\mathbb{R}^d)$ satisfies (V1)-(V2), $\bar{N} :$
 17 $[0, +\infty) \times \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow \mathbb{R}$ is continuous and satisfies (N1)-(N2), and $B : \mathcal{M}_b(\mathbb{R}^d) \rightarrow$
 18 $\mathcal{M}_b(\mathbb{R}^d)$ satisfies (H3). Then equation (24) has a unique solution $\mu \in C([0, +\infty), \mathcal{M}_b(\mathbb{R}^d))$
 19 for any initial condition $\mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$. Furthermore, if $\mu_0 \geq 0$ then $\mu_t \geq 0$ for any
 20 $t > 0$.*

21 *Proof.* The source term $N(t, \mu) := \bar{N}(t, \cdot, \mu) \mu + B(\mu)$ is continuous in (t, μ) and
 22 satisfies (N1') and (N2') from Theorem (5.1) above. Indeed, this is obvious for B

1 from assumption (H3). Concerning $\bar{N}(t, \cdot, \mu)\mu$, we clearly have (N2') in view of (N2).
 2 Concerning (N1'), for any $\mu, \tilde{\mu} \in \mathcal{M}_{b,R}(\mathbb{R}^d)$ and any $\phi \in W^{1,\infty}(\mathbb{R}^d)$ with $\|\phi\|_{W^{1,\infty}} \leq 1$
 3 we have

$$\begin{aligned} & (\bar{N}(t, \cdot, \mu)\mu - \bar{N}(t, \cdot, \tilde{\mu})\tilde{\mu}, \phi) \\ &= \int_{\mathbb{R}^d} \phi(x) [\bar{N}(t, x, \mu) - \bar{N}(t, x, \tilde{\mu})] d\mu(x) + \int_{\mathbb{R}^d} \phi(x) \bar{N}(t, x, \tilde{\mu}) d(\mu - \tilde{\mu})(x) \\ &\leq \|\bar{N}(t, \cdot, \mu) - \bar{N}(t, \cdot, \tilde{\mu})\|_{\infty} \|\mu\|_{TV} + \|\phi \bar{N}(t, \cdot, \tilde{\mu})\|_{W^{1,\infty}} \|\mu - \tilde{\mu}\|_{BL}. \end{aligned}$$

In view of (N1)-(N2) we get

$$(\bar{N}(t, \cdot, \mu)\mu - \bar{N}(t, \cdot, \tilde{\mu})\tilde{\mu}, \phi) \leq (L_{\bar{N},R}R + C_{\bar{N}}) \|\mu - \tilde{\mu}\|_{BL}$$

4 which yields (N1') after taking the supremum over all such ϕ . We thus obtain the
 5 existence of a unique solution $\mu \in C([0, +\infty), \mathcal{M}_b(\mathbb{R}^d))$ to equation (24).

6 We define $v(t, x) := v[\mu_t](x)$ and $c(t, x) := \bar{N}(t, x, \mu_t)$. For a given $T > 0$, we know
 7 from Theorem 5.1 that there exists $R_T > 0$ such that $\|\mu_t\|_{TV} \leq R_T$ for $t \in [0, T]$.
 8 Then it follows from (V1)-(V2) that v satisfies (H1) and from (N1)-(N2) that c satisfies
 9 (H2). Thus, if $\mu_0 \geq 0$ we deduce from Proposition 5.1 that $\mu_t \geq 0$ for $t \in [0, T]$. Since
 10 $T > 0$ is arbitrary we obtain that $\mu_t \geq 0$ for any $t \geq 0$. \square

11 We can now easily obtain the existence and uniqueness of μ^ε .

12 **Theorem 5.3.** *Let (V1)-(V2), (N1)-(N2) and (K0), (K1), (K2), (K3), (K6) and (K8)*
 13 *hold. For any $\varepsilon > 0$ and any initial condition $\mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$ equation (10) has a*
 14 *unique solution $\mu^\varepsilon \in C([0, +\infty), \mathcal{M}_b(\mathbb{R}^d))$ and for any $T > 0$ there exists $C_{T,\varepsilon} > 0$*
 15 *such that*

$$(25) \quad \|\mu_t^\varepsilon\|_{TV} \leq C_{T,\varepsilon} \quad \text{for any } t \in [0, T].$$

16 *Moreover, if $\mu_0 \geq 0$ then $\mu_t^\varepsilon \geq 0$ for any $t \geq 0$.*

17 *Proof.* Since v satisfies (V1)-(V2) and $\bar{N}(t, x, \mu)$ is continuous and satisfies (N1)-(N2),
 18 we only need to check in view of Theorem 5.2 that (i) $b(x, \mu)$ is continuous and satisfies
 19 (N1)-(N2), and that (ii) $B(\mu) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} K_\varepsilon(x, \mu, \cdot) d\mu(x)$ satisfies (H3).

20 Concerning b this is obvious in view of assumption (K2). As for B , first notice that
 21 $B(\mu) \geq 0$ if $\mu \geq 0$ since $K(x, \mu, dy) \geq 0$ according to (K0). Moreover, for any ϕ with

1 $\|\phi\|_\infty \leq 1$, by recalling definition (8) we obtain

$$\begin{aligned} (\varepsilon^2 B(\mu), \phi) &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi(x + \varepsilon(y - x)) K(x, \mu, dy) \right) d\mu(x) \\ &\leq \int_{\mathbb{R}^d} \|K(x, \mu, \cdot)\|_{TV} d\mu(x) \\ &\leq C_K \|\mu\|_{TV} \end{aligned}$$

2 where we used assumption (K8) in the last equality. Thus, $\|B(\mu)\|_{TV} \leq C_\varepsilon \|\mu\|_{TV}$.

3 It remains to check that B is continuous and locally Lipschitz. For any $\mu, \tilde{\mu} \in$
4 $\mathcal{M}_b(\mathbb{R}^d)$ and any test-function ϕ such that $\|\phi\|_\infty + Lip(\phi) \leq 1$ we have

$$\begin{aligned} \varepsilon^2 (B(\mu) - B(\tilde{\mu}), \phi) &= \left(\int_{\mathbb{R}^d} K_\varepsilon(x, \mu, dy) d\mu(x) - \int_{\mathbb{R}^d} K_\varepsilon(x, \tilde{\mu}, dy) d\tilde{\mu}(x), \phi \right) \\ &= \int_{\mathbb{R}^d} (K_\varepsilon(x, \mu, dy) - K_\varepsilon(x, \tilde{\mu}, dy), \phi) d\mu(x) \\ &\quad + \int_{\mathbb{R}^d} (K_\varepsilon(x, \tilde{\mu}, dy), \phi) d(\mu - \tilde{\mu})(x) \\ &\leq C_\varepsilon \int_{\mathbb{R}^d} \|K(x, \mu, \cdot) - K(x, \tilde{\mu}, \cdot)\|_{BL} d|\mu|(x) + C_\varepsilon \|\mu - \tilde{\mu}\|_{BL} \end{aligned}$$

5 where we used assumption (K3) in the last equality. Thus

$$(26) \quad \|B(\mu) - B(\tilde{\mu})\|_{BL} \leq C_\varepsilon \int_{\mathbb{R}^d} \|K(x, \mu, \cdot) - K(x, \tilde{\mu}, \cdot)\|_{BL} d|\mu|(x) + C_\varepsilon \|\mu - \tilde{\mu}\|_{BL}.$$

6 Notice that for any $x \in \mathbb{R}^d$, $\|K(x, \mu, \cdot) - K(x, \tilde{\mu}, \cdot)\|_{BL} \rightarrow 0$ when $\tilde{\mu} \rightarrow \mu$ by (K6)
7 and also that $\|K(x, \mu, \cdot) - K(x, \tilde{\mu}, \cdot)\|_{BL} \leq \|K(x, \mu, \cdot)\|_{TV} + \|K(x, \tilde{\mu}, \cdot)\|_{TV} \leq 2C_K$ by
8 (K8). Thus, the integral on the right-hand side goes to 0 as $\tilde{\mu} \rightarrow \mu$ by the Dominated
9 Convergence Theorem. It follows that B is continuous. Moreover, if $\mu, \tilde{\mu} \in \mathcal{M}_{b,R}(\mathbb{R}^d)$
10 for some $R > 0$ then from (26) and (K6) we get $\|B(\mu) - B(\tilde{\mu})\|_{BL} \leq C_{\varepsilon,R} \|\mu - \tilde{\mu}\|_{BL}$.
11 Thus, B satisfies (H3). \square

12 Hence, for any $\phi \in C_c^1(\mathbb{R}^d)$ and any $t \geq 0$, we have

$$(27) \quad \begin{aligned} \int_{\mathbb{R}^d} \phi d\mu_t^\varepsilon &= \int_{\mathbb{R}^d} \phi d\mu_0 + \int_0^t \int_{\mathbb{R}^d} v[\mu_s^\varepsilon] \nabla \phi d\mu_s^\varepsilon ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \tilde{N}(s, x, \mu_s^\varepsilon) \phi d\mu_s^\varepsilon ds + \int_0^t (\tilde{N}_\varepsilon(\mu_s^\varepsilon), \phi) ds \end{aligned}$$

13 where $\tilde{N}_\varepsilon(\mu) = \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon(x, \mu, dy) d\mu(x) - b(\cdot, \mu)\mu \right)$. A priori the test-function ϕ
14 must have compact support. However, as a direct consequence of (30) we can easily

1 show that the weak formulation (27) holds for bounded functions ϕ with bounded
2 first derivative without assuming that they have compact support:

3 **Proposition 5.2.** *The weak formulation (27) holds for any bounded $\phi \in C^1(\mathbb{R}^d)$ with
4 bounded first derivative.*

5 *Proof.* Take a bounded $\phi \in C^1(\mathbb{R}^d)$ with bounded first derivative. Given $R > 0$ we
6 truncate ϕ considering $\phi_R(x) = \phi(x)\rho(|x|/R)$ where $\rho : [0, +\infty) \rightarrow [0, 1]$ is smooth,
7 supported in $[0, 2]$ and $\rho = 1$ in $[0, 1]$. Since ϕ_R is C^1 with compact support we can
8 use it as a test-function in (27):

$$(28)$$

$$\begin{aligned} & \int_{\mathbb{R}^d} \phi(x)\rho(|x|/R) d\mu_t^\varepsilon - \int_{\mathbb{R}^d} \phi(x)\rho(|x|/R) d\mu_0 = \frac{1}{R} \int_0^t \int_{\mathbb{R}^d} \phi(x)\rho'(|x|/R) \frac{x}{|x|} v[\mu_s^\varepsilon] d\mu_s^\varepsilon ds \\ & + \int_0^t \int_{\mathbb{R}^d} \rho(|x|/R) v[\mu_s^\varepsilon] \nabla \phi d\mu_s^\varepsilon ds + \int_0^t \int_{\mathbb{R}^d} \bar{N}(s, x, \mu_s^\varepsilon) \phi(x)\rho(|x|/R) d\mu_s^\varepsilon ds \\ & + \int_0^t (\tilde{N}_\varepsilon(\mu_s^\varepsilon), \phi(\cdot)\rho(|\cdot|/R)) ds. \end{aligned}$$

We now pass to the limit $R \rightarrow +\infty$ in each integral to drop ρ . We pass to the limit
in the two integrals on the left-hand side by Monotone Convergence. Using (30) and
(31) we bound the first integral on the right-hand side by

$$\frac{1}{R} \|\phi\|_\infty \|\rho'\|_\infty \int_0^t \|v[\mu_s^\varepsilon]\|_\infty \mu_s^\varepsilon(\mathbb{R}^d) ds \leq C/R \rightarrow 0.$$

9 Concerning the second integral, notice that for any s we have $|\rho(|x|/R)v[\mu_s^\varepsilon]\nabla\phi| \leq C$.
10 Thus, the Dominated Convergence Theorem implies that $\int_{\mathbb{R}^d} \rho(|x|/R)v[\mu_s^\varepsilon]\nabla\phi d\mu_s^\varepsilon \rightarrow$
11 $\int_{\mathbb{R}^d} v[\mu_s^\varepsilon]\nabla\phi d\mu_s^\varepsilon$. Moreover, $|\int_{\mathbb{R}^d} \rho(|x|/R)v[\mu_s^\varepsilon]\nabla\phi d\mu_s^\varepsilon| \leq C\mu_s^\varepsilon(\mathbb{R}^d) \leq C$ so that we can
12 pass to the limit in the second integral once again applying the Dominated Conver-
13 gence Theorem. We treat the third and fourth integral terms on the right-hand side
14 in the same way recalling that $|\bar{N}(s, x, \mu_s^\varepsilon)| \leq C_{\bar{N}}$ by (N2) and $\|K(x, \mu_s^\varepsilon, \cdot)\|_{TV} \leq C_K$
15 for any $x \in \mathbb{R}^d$ and any $s \in [0, t]$ by (K8). We conclude that ϕ satisfies (27). \square

6. PROOF OF THEOREM 2.1

We denote by $\mu_t^\varepsilon \in C([0, +\infty), \mathcal{M}_b(\mathbb{R}^d))$ the unique solution to equation (10) as given by [Theorem 5.3](#). Thus, for any $\phi \in C_c^1(\mathbb{R}^d)$ and any $t \geq 0$ there holds

$$(29) \quad \begin{aligned} \int_{\mathbb{R}^d} \phi d\mu_t^\varepsilon &= \int_{\mathbb{R}^d} \phi d\mu_0 + \int_0^t \int_{\mathbb{R}^d} v[\mu_s^\varepsilon] \nabla \phi d\mu_s^\varepsilon ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \bar{N}(s, x, \mu_s^\varepsilon) \phi d\mu_s^\varepsilon ds + \int_0^t (\tilde{N}_\varepsilon(\mu_s^\varepsilon), \phi) ds \end{aligned}$$

where $\tilde{N}_\varepsilon(\mu) = \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon(x, \mu, dy) d\mu(x) - b(\cdot, \mu)\mu \right)$.

From now on we assume that the initial condition μ_0 is a nonnegative measure. In view of [Theorem 5.3](#) it follows that $\mu_t^\varepsilon \geq 0$ for any $t \geq 0$.

We divide the proof into several steps.

Step 6.1. *For any $T > 0$ there exists $C_T > 0$ independent of ε such that*

$$(30) \quad \|\mu_t^\varepsilon\|_{TV} \leq C_T \quad \text{for any } t \in [0, T].$$

Proof. Since $\mu_t^\varepsilon \geq 0$ we have $\|\mu_t^\varepsilon\|_{TV} = \int_{\mathbb{R}^d} d\mu_t^\varepsilon$. In view of Proposition 5.2 we can take $\phi \equiv 1$ in (29). Recalling that $(\tilde{N}_\varepsilon(\mu), 1) = 0$ by definition of b , and using assumption (N2), we obtain

$$\begin{aligned} \|\mu_t^\varepsilon\|_{TV} &= \int_{\mathbb{R}^d} d\mu_t^\varepsilon = \int_{\mathbb{R}^d} d\mu_0 + \int_0^t \int_{\mathbb{R}^d} \bar{N}(s, x, \mu_s^\varepsilon) d\mu_s^\varepsilon ds \\ &\leq \|\mu_0\|_{TV} + C_{\bar{N}} \int_0^t \|\mu_s^\varepsilon\|_{TV} ds. \end{aligned}$$

Using Gronwall lemma we deduce

$$\|\mu_t^\varepsilon\|_{TV} \leq \|\mu_0\|_{TV} e^{C_N t} \leq \|\mu_0\|_{TV} e^{C_N T} =: C_T.$$

12

□

Notice that in view of (30) and assumptions (V1) and (V2), the [vector fields](#) $v^\varepsilon(t, x) := v[\mu_t^\varepsilon](x)$ are bounded in $W^{1, \infty}(\mathbb{R}^d)$:

$$(31) \quad \|v^\varepsilon\|_{W^{1, \infty}} \leq C.$$

15 This fact will be used below.

- 1 **Step 6.2.** For any $T > 0$ there exists $C_T > 0$ such that for any bounded $\phi \in C^3(\mathbb{R}^d)$
 2 with all derivatives up to order 3 bounded and for any $t \in [0, T]$, there holds

(32)

$$\left| \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x) - b(\cdot, \mu_t^\varepsilon) \mu_t^\varepsilon, \phi \right) - \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,j=1}^d \partial_{ij} \phi(x) d_{ij}(x, \mu_t^\varepsilon) d\mu_t^\varepsilon(x) \right|$$

$$\leq C_T \varepsilon \|D^3 \phi\|_\infty$$

where

$$d_{ij}(x, \mu) = \int_{\mathbb{R}^d} (y_i - x_i)(y_j - x_j) K(x, \mu, dy).$$

- 3 *Proof.* Recalling the definition of b and of K_ε given in (8) we can write

$$\begin{aligned} & \left(\int_{\mathbb{R}^d} K_\varepsilon(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x) - b(\cdot, \mu_t^\varepsilon) \mu_t^\varepsilon, \phi \right) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\phi(y) - \phi(x)) K_\varepsilon(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x) \\ & = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} [\phi(x + \varepsilon(y - x)) - \phi(x)] K(x, \mu_t^\varepsilon, dy) \right) d\mu_t^\varepsilon(x). \end{aligned}$$

- 4 Performing a Taylor expansion we then obtain

(33)

$$\begin{aligned} & \left(\int_{\mathbb{R}^d} K_\varepsilon(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x) - b(\cdot, \mu_t^\varepsilon) \mu_t^\varepsilon, \phi \right) \\ & = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (\varepsilon \nabla \phi(x)(y - x) + \frac{\varepsilon^2}{2} (y - x)^T D^2 \phi(x + \theta \varepsilon(y - x))(y - x)) K(x, \mu_t^\varepsilon, dy) \right) d\mu_t^\varepsilon(x) \end{aligned}$$

- 5 where $\theta = \theta(x, y, \varepsilon) \in (0, 1)$. From (K4) we have $\int_{\mathbb{R}^d} (y - x) K(x, \mu_t^\varepsilon, dy) = 0$ for any
 6 $x \in \mathbb{R}^d$, thus the first integral term on the right-hand side of (33) of order ε equals
 7 to zero. Hence, we get

$$\begin{aligned} & \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x) - b(\cdot, \mu_t^\varepsilon) \mu_t^\varepsilon, \phi \right) \\ & = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (y - x)^T D^2 \phi(x + \theta \varepsilon(y - x))(y - x) K(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x). \end{aligned}$$

- 8 Thus, the left-hand side of (32) is bounded by

$$\begin{aligned} & \leq \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |D^2 \phi(x + \theta \varepsilon(y - x)) - D^2 \phi(x)| |y - x|^2 K(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x) \\ & \leq \frac{\varepsilon}{2} \|D^3 \phi\|_\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y - x|^3 K(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x). \end{aligned}$$

- 9 Hence, we obtain the result by invoking (30) and assumption (K5). □

1 **Step 6.3.** For any $T > 0$ there exists a positive constant C_T independent of ε such
 2 that

$$(34) \quad \int_{\mathbb{R}^d} |x| d\mu_t^\varepsilon \leq C_T \quad \text{for any } t \in [0, T].$$

3 *Proof.* We denote by $|\cdot|_\delta$ the function $|\cdot|$ smoothed near 0 in such a way that
 4 $|\cdot| \leq |\cdot|_\delta$. Given $R > 0$ we then truncate $|\cdot|_\delta$ considering $\psi_{\delta,R}(x) = |x|_\delta \rho(|x|/R)$
 5 where $\rho : [0, +\infty) \rightarrow [0, 1]$ is smooth, supported in $[0, 2]$ and $\rho = 1$ in $[0, 1]$. Notice
 6 that for any $x \in \mathbb{R}^d$ the following is satisfied: $0 \leq \psi_{\delta,R}(x) \leq |x|_\delta$ and $|D^\alpha \psi_{\delta,R}(x)| \leq$
 7 $C_\delta 1_{\{|x| \leq 2R\}}$, $1 \leq |\alpha| \leq 3$.

8 Since $\psi_{\delta,R}$ is smooth and compactly supported we can use it as a test-function in
 9 (29):

$$(35) \quad \begin{aligned} \int_{\mathbb{R}^d} \psi_{\delta,R} d\mu_t^\varepsilon &= \int_{\mathbb{R}^d} \psi_{\delta,R} d\mu_0 + \int_0^t \int_{\mathbb{R}^d} \bar{N}(s, x, \mu_s^\varepsilon) \psi_{\delta,R} d\mu_s^\varepsilon ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} v[\mu_s^\varepsilon] \nabla \psi_{\delta,R} d\mu_s^\varepsilon ds + \int_0^t (\tilde{N}_\varepsilon(\mu_s^\varepsilon), \psi_{\delta,R}) ds. \end{aligned}$$

Using assumption (N2), $0 \leq \psi_{\delta,R}(x) \leq |x|_\delta$ and $\int_{\mathbb{R}^d} |x| d\mu_0(x) < \infty$ we can bound the
 first two integrals on the right-hand side by

$$\int_{\mathbb{R}^d} |x|_\delta d\mu_0 + C_{\bar{N}} \int_0^t \left(\int_{\mathbb{R}^d} \psi_{\delta,R} d\mu_s^\varepsilon \right) ds \leq C + C_{\bar{N}} \int_0^t \left(\int_{\mathbb{R}^d} \psi_{\delta,R} d\mu_s^\varepsilon \right) ds.$$

10 Recalling that from (31) $|v[\mu_s^\varepsilon](x)| \leq C$ (because of (V1) and (30)), $\|\mu_t^\varepsilon\|_{TV} \leq C$ by
 11 (30) and $|\nabla \psi_{\delta,R}(x)| \leq C_\delta$, we can bound the integral involving $\nabla \psi_{\delta,R}$ by $C_\delta T$. It
 12 remains to bound the last integral on the right-hand side of (35). Using (32) we have

$$|(\tilde{N}_\varepsilon(\mu_s^\varepsilon), \psi_{\delta,R})| \leq \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,j=1}^d |\partial_{ij} \psi_{\delta,R}(x)| |d_{ij}(x, \mu_s^\varepsilon)| d\mu_t^\varepsilon(x) + C_T \varepsilon \|D^3 \psi_{\delta,R}\|_\infty.$$

13 It follows from assumptions (K7) and (30) that

$$(36) \quad |d_{ij}(x, \mu_t^\varepsilon)| \leq C \quad \text{for any } t \in [0, T] \text{ and any } \varepsilon.$$

We then obtain

$$|(\tilde{N}_\varepsilon(\mu_s^\varepsilon), \psi_R)| \leq C_{\delta,T}.$$

Thus,

$$\int_{\mathbb{R}^d} \psi_{\delta,R} d\mu_t^\varepsilon \leq C_{\delta,T} + C_{\bar{N}} \int_0^t \left(\int_{\mathbb{R}^d} \psi_{\delta,R} d\mu_s^\varepsilon \right) ds.$$

Gronwall lemma gives

$$\int_{\mathbb{R}^d} \psi_{\delta,R} d\mu_t^\varepsilon \leq C_{\delta,T} e^{C_N T}.$$

Letting $R \rightarrow +\infty$ and using the Monotone Convergence theorem we obtain

$$\int_{\mathbb{R}^d} |x|_\delta d\mu_t^\varepsilon \leq C_{\delta,T}.$$

1 Since $|x| \leq |x|_\delta$ we deduce the claim. □

2 **Step 6.4.** *The sequence $\mu^\varepsilon : [0, T] \rightarrow \mathcal{M}_b(\mathbb{R}^d)$ is uniformly equicontinuous, i.e., there*
 3 *exists $C_T > 0$ such that for any $\varepsilon \in (0, 1]$ and any $t, t' \in [0, T]$,*

$$(37) \quad \|\mu_t^\varepsilon - \mu_{t'}^\varepsilon\|_{BL,3} \leq C_T |t - t'|,$$

4 *where the norm $\|\cdot\|_{BL,3}$ is defined in (18) with $N = 3$.*

5 *Proof.* Without loss of generality, we assume that $t' > t$ and take a bounded test-
 6 function $\phi \in C^3(\mathbb{R}^d)$ whose derivatives up to order three are bounded. Then according
 7 to Proposition 5.2,

$$\begin{aligned} (\mu_{t'}^\varepsilon - \mu_t^\varepsilon, \phi) &= \int_t^{t'} \int_{\mathbb{R}^d} \bar{N}(s, x, \mu_s^\varepsilon) \phi(x) d\mu_s^\varepsilon(x) ds + \int_t^{t'} \int_{\mathbb{R}^d} v[\mu_s^\varepsilon] \nabla \phi d\mu_s^\varepsilon(x) ds \\ &\quad + \int_t^{t'} (\tilde{N}_\varepsilon(\mu_s^\varepsilon), \phi) ds. \end{aligned}$$

8 Recalling that $\|v[\mu_s^\varepsilon]\|_\infty \leq C$ from (31), the TV-bound (30) and using (32) and (36)
 9 we have

$$\begin{aligned} |(\mu_{t'}^\varepsilon - \mu_t^\varepsilon, \phi)| &\leq C_{N,T} \left(\|\phi\|_\infty + \|\nabla \phi\|_\infty + \varepsilon \|D^3 \phi\|_\infty \right) |t - t'| \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_t^{t'} \int_{\mathbb{R}^d} |\partial_{ij} \phi(x)| |d_{ij}(x, \mu_s^\varepsilon)| d\mu_s^\varepsilon(x) \\ &\leq C_T |t - t'| \sum_{0 \leq |\alpha| \leq 3} \|D^\alpha \phi\|_\infty. \end{aligned}$$

10 Now, we obtain the result [by taking the supremum](#) over all such ϕ and recalling the
 11 definition (18) of the norm $\|\cdot\|_{BL,3}$. □

Notice that a sequence of measures $(\mu_n)_n$ satisfying $\int_{\mathbb{R}^d} |x| d|\mu_n|(x) \leq C$ for some constant C is tight since $\int_{\{|x| \geq R\}} d|\mu_n| \leq C/R$. It follows that a set of the form

$$\mathcal{K} = \left\{ \mu \in \mathcal{M}_{b,+}(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |x|) d\mu \leq C \right\}$$

1 for some constant $C > 0$, is compact under the **BL norm** (see [18]) and thus also
 2 compact under the norm $\|\cdot\|_{BL,3}$ in view of Proposition 4.1. According to (30) and
 3 (34), and recalling that $\mu_t^\varepsilon \geq 0$, we know that the sequence μ^ε belongs to $C([0, T], \mathcal{K})$.
 4 Moreover, it is uniformly equicontinuous in view of (37). It then follows from Arzela-
 5 Ascoli theorem that up to a subsequence there exists $\mu := \lim_{\varepsilon \rightarrow 0} \mu^\varepsilon$ in $C([0, T], \mathcal{K})$.
 6 Notice that the convergence holds not only in $C([0, T], (\mathcal{M}_b(\mathbb{R}^d), \|\cdot\|_{BL,3}))$ but also in
 7 $C([0, T], (\mathcal{M}_b(\mathbb{R}^d), \|\cdot\|_{BL}))$ by Proposition 4.1.

8 The last Step of the proof is

9 **Step 6.5.** μ is a solution of

$$(38) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = \bar{N}(t, \cdot, \mu) \mu + \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}(d_{ij}(\cdot, \mu_t) \mu_t).$$

10 *Proof.* According to (29) and (32) we have for any $T > 0$, any $t \in [0, T]$, any $\varepsilon > 0$
 11 and any $\phi \in C_c^\infty(\mathbb{R}^d)$, that

$$(39) \quad \begin{aligned} \int_{\mathbb{R}^d} \phi d\mu_t^\varepsilon &= \int_{\mathbb{R}^d} \phi d\mu_0 + \int_0^t \int_{\mathbb{R}^d} \bar{N}(s, x, \mu_s^\varepsilon) \phi(x) d\mu_s^\varepsilon ds + \int_0^t \int_{\mathbb{R}^d} \nabla \phi \cdot v[\mu_s^\varepsilon] d\mu_s^\varepsilon ds \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \int_{\mathbb{R}^d} d_{ij}(x, \mu_s^\varepsilon) \partial_{ij} \phi(x) d\mu_s^\varepsilon(x) ds + O(\varepsilon) \|\phi^{(3)}\|_\infty. \end{aligned}$$

12 Let us first check that for any $s \in [0, T]$,

$$(40) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} d_{ij}(x, \mu_s^\varepsilon) \partial_{ij} \phi(x) d\mu_s^\varepsilon(x) = \int_{\mathbb{R}^d} d_{ij}(x, \mu_s) \partial_{ij} \phi(x) d\mu_s(x).$$

13 Indeed

$$\begin{aligned} &\int_{\mathbb{R}^d} d_{ij}(x, \mu_s^\varepsilon) \partial_{ij} \phi(x) d\mu_s^\varepsilon(x) - \int_{\mathbb{R}^d} d_{ij}(x, \mu_s) \partial_{ij} \phi(x) d\mu_s(x) \\ &= \int_{\mathbb{R}^d} d_{ij}(x, \mu_s^\varepsilon) \partial_{ij} \phi(x) d(\mu_s^\varepsilon - \mu_s)(x) + \int_{\mathbb{R}^d} (d_{ij}(x, \mu_s^\varepsilon) - d_{ij}(x, \mu_s)) \partial_{ij} \phi(x) d\mu_s(x) \\ &=: A + B. \end{aligned}$$

Using assumption (K7) with TV-bound (30) we have

$$|A| \leq \|\mu_s^\varepsilon - \mu_s\|_{BL} \|d_{ij}(\cdot, \mu_s^\varepsilon) \partial_{ij} \phi\|_{W^{1,\infty}} \leq C \|\mu_s^\varepsilon - \mu_s\|_{BL} \rightarrow 0.$$

14 Moreover since $\|d_{ij}(\cdot, \mu_s^\varepsilon)\|_\infty, \|d_{ij}(\cdot, \mu_s)\|_\infty \leq C$ by (K7) and (30), and $d_{ij}(x, \mu_s^\varepsilon) \rightarrow$
 15 $d_{ij}(x, \mu_s)$ as $\varepsilon \rightarrow 0$ for any $x \in \mathbb{R}^d$, we have $B \rightarrow 0$ by the Dominated Convergence
 16 Theorem. This proves (40).

Now using (40) and the bound $|\int_{\mathbb{R}^d} d_{ij}(x, \mu_s^\varepsilon) \partial_{ij} \phi(x) d\mu_s^\varepsilon(x)| \leq C_T$ for any $s \in [0, T]$, we can pass to the limit in the [fourth integral](#) on the right-hand side using the Dominated Convergence Theorem to obtain

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathbb{R}^d} d_{ij}(x, \mu_s^\varepsilon) \partial_{ij} \phi(x) d\mu_s^\varepsilon(x) ds = \int_0^t \int_{\mathbb{R}^d} d_{ij}(x, \mu_s) \partial_{ij} \phi(x) d\mu_s(x) ds.$$

We prove in the same way that

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathbb{R}^d} \bar{N}(s, x, \mu_s^\varepsilon) \phi(x) d\mu_s^\varepsilon ds = \int_0^t \int_{\mathbb{R}^d} \bar{N}(s, x, \mu_s) \phi(x) d\mu_s ds.$$

We now pass to the limit in the [third integral](#) term on the right-hand side of (39).

We first write it as

$$\int_0^t \int_{\mathbb{R}^d} \nabla \phi(v[\mu_s^\varepsilon] - v[\mu_s]) d\mu_s^\varepsilon ds + \int_0^t \int_{\mathbb{R}^d} \nabla \phi.v[\mu_s] d\mu_s^\varepsilon ds =: I + II.$$

1 We bound I using assumption (V1) by

$$\begin{aligned} \|\nabla \phi\|_\infty L_R^1 \int_0^t \int_{\mathbb{R}^d} \|\mu_s^\varepsilon - \mu_s\|_{BL} d\mu_s^\varepsilon ds &\leq \|\nabla \phi\|_\infty L_R^1 \max_{0 \leq s \leq T} \|\mu_s^\varepsilon - \mu_s\|_{BL} \|\mu_s^\varepsilon\|_{TV} T \\ &\leq C \max_{0 \leq s \leq T} \|\mu_s^\varepsilon - \mu_s\|_{BL} \end{aligned}$$

which goes to 0 since $\mu^\varepsilon \rightarrow \mu$ in $C([0, T], \mathcal{M}_b(\mathbb{R}^d))$. Moreover, II goes to $\int_0^t \int_{\mathbb{R}^d} \nabla \phi.v[\mu_s] d\mu_s ds$ by the Dominated Convergence Theorem since $|\int_{\mathbb{R}^d} \nabla \phi.v[\mu_s] d\mu_s^\varepsilon| \leq C$. Thus

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \int_{\mathbb{R}^d} \nabla \phi.v[\mu_s^\varepsilon] d\mu_s^\varepsilon ds = \int_0^t \int_{\mathbb{R}^d} \nabla \phi.v[\mu_s] d\mu_s ds.$$

Hence, we can pass to the limit in (39) to obtain

$$\int_{\mathbb{R}^d} \phi d\mu_t = \int_{\mathbb{R}^d} \phi d\mu_0 + \int_0^t \int_{\mathbb{R}^d} \nabla \phi.v[\mu_s] d\mu_s ds + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \int_{\mathbb{R}^d} d_{ij} \partial_{ij} \phi d\mu_s ds$$

2 which is the weak form of (38). □

3

7. REMARKS AND EXTENSIONS

4 In this paper we formulated a selection-mutation model on the space of finite signed
 5 measures and we modeled the selection-mutation kernel as a family of finite nonneg-
 6 ative measures in the trait parameter x . This allows to simultaneously treat discrete
 7 and continuous kernels under the same theory. We established the convergence of
 8 a nonlinear diffusively rescaled selection-mutation model with a nonlinear transport

1 and source terms to a nonlinear reaction-advection-diffusion equation. To end this
 2 paper, below we provide remarks on possible extensions to the theory we presented
 3 here.

4 **7.1. Asymmetric distribution.** We can slightly generalize the results above by
 5 considering instead of K a sequence of kernels $K^\varepsilon(x, \mu, \cdot) \in \mathcal{M}_b(\mathbb{R}^d)$, $\varepsilon > 0$, satisfying
 6 basically the same assumptions as K with estimates to be uniform in ε but relaxing
 7 the assumption (K4) allowing for slightly **asymmetric** distribution:

8 (K0') $K^\varepsilon(x, \mu, dy) \geq 0$ whenever $\mu \geq 0$,

9 (K1') For any $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ measurable and bounded and any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$ and $\varepsilon > 0$,
 10 the function $x \rightarrow \int_{\mathbb{R}^d} \phi(y) K^\varepsilon(x, \mu, dy)$ is measurable.

11 (K2') the function $b^\varepsilon : (x, \mu) \in \mathbb{R}^d \times \mathcal{M}_b(\mathbb{R}^d) \rightarrow \int_{\mathbb{R}^d} K^\varepsilon(x, \mu, dy) \in \mathbb{R}$ is continuous
 12 and satisfies

13 (i) there exists $C_b > 0$ such that $\|b^\varepsilon(\cdot, \mu)\|_{W^{1,\infty}} \leq C_b$ for any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$
 14 and any $\varepsilon > 0$,

(ii) for any $R > 0$ there exists $L_{b,R} > 0$ such that

$$|b^\varepsilon(x, \mu) - b^\varepsilon(x, \tilde{\mu})| \leq L_{b,R} \|\mu - \tilde{\mu}\|_{BL} \quad \text{for any } \mu, \tilde{\mu} \in \mathcal{M}_{b,R}(\mathbb{R}^d) \text{ and any } \varepsilon > 0,$$

(K3') For any $\phi \in W^{1,\infty}(\mathbb{R}^d)$, the function $x \rightarrow (K^\varepsilon(x, \mu, \cdot), \phi)$ is Lipschitz with

$$\sup_{\mu \in \mathcal{M}_b(\mathbb{R}^d), \|\phi\|_{W^{1,\infty}} \leq 1} \text{Lip}\left(x \rightarrow (K^\varepsilon(x, \mu, \cdot), \phi)\right) \leq C \quad \text{where } C \text{ is independent of } \varepsilon.$$

15 (K4') the functions $V^\varepsilon(x, \mu) := \frac{1}{\varepsilon} \int_{\mathbb{R}^d} (y - x) K^\varepsilon(x, \mu, dy)$ satisfy

– for any $R > 0$ there exists $C_{V,R} > 0$ such that

$$\|V^\varepsilon(\cdot, \mu)\|_{W^{1,\infty}} \leq C_{V,R} \quad \text{for any } \mu \in \mathcal{M}_{b,R}(\mathbb{R}^d) \text{ and any } \varepsilon > 0,$$

16 – there exists $V(x, \mu)$ such that for any $x \in \mathbb{R}^d$ and any nonnegative TV-
 17 bounded sequence $(\mu_\varepsilon)_\varepsilon$ converging for the BL-norm to some μ , we have

18 $V^\varepsilon(x, \mu_\varepsilon) \rightarrow V(x, \mu)$.

(K5') for any $R > 0$ there exists $C_{K,R} > 0$ such that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |y - x|^3 K^\varepsilon(x, \mu, dy) \leq C_{K,R} \quad \text{for any } \mu \in \mathcal{M}_{b,R}(\mathbb{R}^d) \text{ and any } \varepsilon > 0.$$

(K6') for any $R > 0$ there exists $L_{K,R} > 0$ such that

$$\|K^\varepsilon(x, \tilde{\mu}, \cdot) - K^\varepsilon(x, \mu, \cdot)\|_{BL} \leq L_{K,R} \|\tilde{\mu} - \mu\|_{BL}$$

1 for any $x \in \mathbb{R}^d$, any $\mu, \tilde{\mu} \in \mathcal{M}_{b,R}(\mathbb{R}^d)$ and any $\varepsilon > 0$.

2 (K7') the functions $d_{ij}^\varepsilon(x, \mu) = \int_{\mathbb{R}^d} (y_i - x_i)(y_j - x_j) K^\varepsilon(x, \mu, dy)$ satisfy
 – for any $R > 0$ there exists $L_{d,R} > 0$ such that

$$\|d_{ij}^\varepsilon(\cdot, \mu)\|_{W^{1,\infty}} \leq L_{d,R} \quad \text{for any } \mu \in \mathcal{M}_{b,R}(\mathbb{R}^d) \text{ and any } \varepsilon > 0.$$

3 – there exists $d_{ij}(x, \mu)$ such that for any $x \in \mathbb{R}^d$ and any nonnegative TV-
 4 bounded sequence $(\mu_\varepsilon)_\varepsilon$ converging for the BL-norm to some μ , we have

$$5 \quad d_{ij}^\varepsilon(x, \mu_\varepsilon) \rightarrow d_{ij}(x, \mu).$$

(K8') There exists $C_K > 0$ such that

$$\|K^\varepsilon(x, \mu, \cdot)\|_{TV} \leq C_K \quad \text{for any } x \in \mathbb{R}^d, \text{ any } \mu \in \mathcal{M}_b(\mathbb{R}^d) \text{ and any } \varepsilon > 0.$$

6 The main difference with the setting of the previous section lies in assumption
 7 (K4) where we do not assume anymore that $\int_{\mathbb{R}^d} (y-x)K(x, dy) = 0$, i.e., the offspring
 8 is distributed symmetrically around x , but instead that this distribution could be
 9 slightly asymmetric with an asymmetry of order ε .

We rescale as before the kernels $K^\varepsilon(x, \mu, dy)$ defining $K_\varepsilon^\varepsilon(x, \mu, dy)$ by

$$(K_\varepsilon^\varepsilon(x, \mu, dy), \phi) = \int_{\mathbb{R}^d} \phi(x + \varepsilon(y-x)) K^\varepsilon(x, \mu, dy).$$

10 We then obtain as before a unique nonnegative global solution μ_t^ε to

$$(41) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) = \bar{N}(t, \cdot, \mu_t) \mu_t + \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon^\varepsilon(z, \cdot) d\mu_t(z) - b^\varepsilon \mu_t \right).$$

11

12 We still assume that the vector-field v satisfies (V1)-(V2) and that the source term
 13 \bar{N} satisfies (N1)-(N2).

14 The asymmetry of the reproduction kernel K^ε quantified by V^ε in assumption (K4')
 15 is reflected by an additional transport term $\nabla \cdot (V \mu_t)$ in the limit equation:

16 **Theorem 7.1.** *For any nonnegative initial condition $\mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x| d\mu_0 <$
 17 ∞ , the μ_t^ε converge up to a subsequence as $\varepsilon \rightarrow 0$ in $C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ for any $T > 0$
 18 to some $\mu \in C([0, +\infty), \mathcal{M}_{b,+}(\mathbb{R}^d))$ which solves*

$$(42) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) + \nabla \cdot (V(\cdot, \mu_t) \mu_t) = \bar{N}(t, \cdot, \mu_t) \mu_t + \frac{1}{2} \partial_{ij} (d_{ij}(\cdot, \mu_t) \mu_t)$$

19 with initial condition $\mu|_{t=0} = \mu_0$ and where V and d_{ij} are defined in (K4') and (K7')
 20 respectively.

1 *Proof.* The proof is an adaptation of the proof of Theorem 2.1. We only highlight
 2 the main changes. Step 6.1 remains unchanged. Step 6.2 must be slightly modified
 3 replacing (32) by

$$(43) \quad \left| \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon^\varepsilon(x, \mu, dy) d\mu_t^\varepsilon(x) - b^\varepsilon(\cdot, \mu_t^\varepsilon) \mu_t^\varepsilon, \phi \right) \right. \\ \left. - \int_{\mathbb{R}^d} V^\varepsilon(x, \mu_t^\varepsilon) \nabla \phi d\mu_t^\varepsilon(x) - \frac{1}{2} \int_{\mathbb{R}^d} \partial_{ij} \phi(x) d_{ij}^\varepsilon(x, \mu_t^\varepsilon) d\mu_t^\varepsilon(x) \right| \\ \leq C_T \varepsilon \|D^3 \phi\|_\infty.$$

4 Indeed, the Taylor expansion (33) can be written now as

$$\frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon^\varepsilon(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x) - b^\varepsilon(\cdot, \mu_t^\varepsilon) \mu_t^\varepsilon, \phi \right) \\ = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left(\frac{1}{\varepsilon} \nabla \phi(x)(y-x) + \frac{1}{2} (D^2 \phi(x + \theta \varepsilon(y-x)))(y-x), y-x \right) K^\varepsilon(x, \mu_t^\varepsilon, dy) \right) d\mu_t^\varepsilon(x) \\ = \int_{\mathbb{R}^d} \nabla \phi(x) V^\varepsilon(x, \mu_t^\varepsilon) d\mu_t^\varepsilon(x) \\ + \frac{1}{2} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (D^2 \phi(x + \theta \varepsilon(y-x)))(y-x), y-x \right) K^\varepsilon(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x).$$

5 Step 6.3 and Step 6.4 remain unchanged since $|d_{ij}^\varepsilon(x, \mu_t^\varepsilon)|$ is bounded uniformly in
 6 ε by assumption (K7').

7 The other change is in the last Step 6.5 of the proof when proving that $\mu_t :=$
 8 $\lim_{\varepsilon \rightarrow 0} \mu_t^\varepsilon$ solves equation (42). Indeed we have to pass to the limit in the additional
 9 term and prove that

$$(44) \quad \int_0^t \int_{\mathbb{R}^d} \nabla \phi(x) V^\varepsilon(x, \mu_s^\varepsilon) d\mu_s^\varepsilon(x) ds \rightarrow \int_0^t \int_{\mathbb{R}^d} \nabla \phi(x) V(x, \mu_s) d\mu_s(x) ds.$$

10 To this end, we write

$$\int_0^t \int_{\mathbb{R}^d} \nabla \phi(x) V^\varepsilon(x, \mu_s^\varepsilon) d\mu_s^\varepsilon(x) ds - \int_0^t \int_{\mathbb{R}^d} \nabla \phi(x) V(x, \mu_s) d\mu_s(x) ds \\ = \int_0^t \int_{\mathbb{R}^d} \nabla \phi(x) V^\varepsilon(x, \mu_s^\varepsilon) d(\mu_s^\varepsilon - \mu_s)(x) ds + \int_0^t \int_{\mathbb{R}^d} \nabla \phi(x) (V^\varepsilon(x, \mu_s^\varepsilon) - V(x, \mu_s)) d\mu_s(x) ds \\ = A + B$$

We have for any $s \in [0, T]$ that

$$\left| \int_{\mathbb{R}^d} \nabla \phi(x) V^\varepsilon(x, \mu_s^\varepsilon) d(\mu_s^\varepsilon - \mu_s)(x) \right| \leq \|\mu_s^\varepsilon - \mu_s\|_{BL} \|\nabla \phi V^\varepsilon(\cdot, \mu_s^\varepsilon)\|_{W^{1,\infty}} \leq C \|\mu_s^\varepsilon - \mu_s\|_{BL} \rightarrow 0$$

in view of assumption (K4'). We deduce that $A \rightarrow 0$ by the Dominated Convergence Theorem. Moreover,

$$\int_{\mathbb{R}^d} \nabla \phi(x) (V^\varepsilon(x, \mu_s^\varepsilon) - V(x, \mu_s)) d\mu_s(x) \rightarrow 0$$

1 by the Dominated Convergence Theorem in view of assumption (K4'). We then
2 deduce that $B \rightarrow 0$ again by the Dominated Convergence Theorem.

3 We also have to pass to the limit in the diffusion term and prove that for any
4 $s \in [0, T]$,

$$(45) \quad \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} d_{ij}^\varepsilon(x, \mu_s^\varepsilon) \partial_{ij} \phi(x) d\mu_s^\varepsilon(x) = \int_{\mathbb{R}^d} d_{ij}(x, \mu_s) \partial_{ij} \phi(x) d\mu_s(x).$$

5 This can be done exactly in the same way as (44) using (K7'). □

As an example of admissible kernels K^ε consider

$$K^\varepsilon(x, \mu, dy) := K(x, \mu, dy) + \varepsilon \tilde{K}(x, \mu, dy)$$

where K satisfies (K0)-(K8) and \tilde{K} satisfies (K0)-(K8) except (K4). Then K^ε satisfy (K0')-(K8') except (K4'). Concerning (K4') we have

$$V^\varepsilon(x, \mu) = V(x, \mu) = \int_{\mathbb{R}^d} (y - x) \tilde{K}(x, \mu, dy)$$

6 and we must add the assumption on \tilde{K} that V satisfies (K4') namely

(K4')(i) for any $R > 0$ there exists $C_{V,R} > 0$ such that

$$\|V(\cdot, \mu)\|_{W^{1,\infty}} \leq C_{V,R} \quad \text{for any } \mu \in \mathcal{M}_{b,R}(\mathbb{R}^d),$$

7 (K4')(ii) for any $x \in \mathbb{R}^d$ and any nonnegative TV-bounded sequence $(\mu_\varepsilon)_\varepsilon$ converging
8 for the BL-norm to some μ , we have $V(x, \mu_\varepsilon) \rightarrow V(x, \mu)$.

Concerning (K7') notice that

$$d_{ij}^\varepsilon(x, \mu) = d_{ij}(x, \mu) + \varepsilon \int_{\mathbb{R}^d} (y_i - x_i)(y_j - x_j) \tilde{K}(x, \mu, dy),$$

where

$$d_{ij}(x, \mu) = \int_{\mathbb{R}^d} (y_i - x_i)(y_j - x_j) K(x, \mu, dy).$$

1 Then (K7') is satisfied. Indeed if $(\mu_\varepsilon)_\varepsilon$ is a nonnegative TV-bounded sequence con-
 2 verging for the BL-norm to some μ then for any $x \in \mathbb{R}^d$,

$$\begin{aligned} |d_{ij}^\varepsilon(x, \mu_\varepsilon) - d_{ij}(x, \mu)| &\leq |d_{ij}(x, \mu_\varepsilon) - d_{ij}(x, \mu)| + \varepsilon \int_{\mathbb{R}^d} |y - x|^2 \tilde{K}(x, \mu_\varepsilon, dy) \\ &\leq o(1) + C\varepsilon \int_{\mathbb{R}^d} (1 + |y - x|^3) \tilde{K}(x, \mu_\varepsilon, dy) \\ &\leq o(1) + C\varepsilon \end{aligned}$$

3 since K satisfies (K7) and \tilde{K} satisfies (K5) and (K8).

7.2. Slower mutation rate: hyperbolic rescaling. We still consider variable ker-
 nels K^ε as in subsection 7.1. Instead of accelerating the mutation rate by $1/\varepsilon^2$, thus
 obtaining a diffusion equation in the limit, we only accelerate it by $1/\varepsilon$ (hyperbolic
 rescaling), i.e., we consider instead the equation

$$\partial_t \mu_t^\varepsilon + \nabla \cdot (v[\mu_t^\varepsilon] \mu_t^\varepsilon) = \bar{N}(t, \cdot, \mu_t) \mu_t + \frac{1}{\varepsilon} \left(\int K_\varepsilon^\varepsilon(z, \cdot) d\mu_t^\varepsilon(z) - b^\varepsilon \mu_t^\varepsilon \right).$$

4

5 We still assume that the vector-field v satisfies (V1)-(V2) and that the source term
 6 \bar{N} satisfies (N1)-(N2). Concerning K^ε , we assume (K0')-(K6') (we do not need (K7')
 7 anymore) of subsection 7.1 but we replace (K4') and (K5') by

8 (K4'') the functions $V^\varepsilon(x, \mu) := \int_{\mathbb{R}^d} (y - x) K^\varepsilon(x, \mu, dy)$ satisfy

(i) for any $R > 0$ there exists $C_{V,R} > 0$ such that

$$\|V^\varepsilon(\cdot, \mu)\|_{W^{1,\infty}} \leq C_{V,R} \quad \text{for any } \mu \in \mathcal{M}_{b,R}(\mathbb{R}^d) \text{ and any } \varepsilon > 0,$$

9 (ii) there exists $V(x, \mu)$ such that for any nonnegative TV-bounded sequence

10 $\mu_\varepsilon \rightarrow \mu$ for the BL-norm and any $x \in \mathbb{R}^d$, $V^\varepsilon(x, \mu_\varepsilon) \rightarrow V(x, \mu)$.

(K5'') for any $R > 0$ there exists $C_{K,R} > 0$ such that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |y - x|^2 K^\varepsilon(x, \mu, dy) \leq C_{K,R} \quad \text{for any } \mu \in \mathcal{M}_{b,R}(\mathbb{R}^d) \text{ and any } \varepsilon > 0.$$

The slower mutation rate implies that in the Taylor expansion (33) only the first
 order term matters. We thus have to replace (32) in Step 6.2 by

$$\left| \frac{1}{\varepsilon} \left(\int_{\mathbb{R}^d} K_\varepsilon^\varepsilon(x, \mu_t^\varepsilon, dy) d\mu_t^\varepsilon(x) - b^\varepsilon(\cdot, \mu_t^\varepsilon) \mu_t^\varepsilon, \phi \right) - \int_{\mathbb{R}^d} V^\varepsilon \nabla \phi d\mu_t^\varepsilon(x) \right| \leq C_T \varepsilon \|D^2 \phi\|_\infty.$$

11 **The other steps** of the proof remain unchanged. We then obtain

1 **Theorem 7.2.** For any nonnegative initial condition $\mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x| d\mu_0 <$
 2 ∞ , and for any $T > 0$, μ_t^ε converges as $\varepsilon \rightarrow 0$ in $C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ to the unique so-
 3 lution $\mu \in C([0, +\infty), \mathcal{M}_{b,+}(\mathbb{R}^d))$ of the differential equation

$$(46) \quad \partial_t \mu_t + \nabla \cdot (v[\mu_t] \mu_t) + \nabla \cdot (V \mu_t) = \bar{N}(t, \cdot, \mu_t) \mu_t.$$

4 with initial condition $\mu|_{t=0} = \mu_0$.

5 Notice that the whole sequence μ_t^ε converges since equation (46) has a unique solution
 6 (see [13]).

7 **7.3. Case of a system.** We can adapt the result and the proof in a straightforward
 8 way to the case of a system like

$$(47) \quad \begin{cases} \partial_t \mu_t^{\varepsilon,1} + \nabla \cdot (v^1[\mu_t^\varepsilon] \mu_t^{\varepsilon,1}) = \bar{N}^1(t, \cdot, \mu_t^\varepsilon) \mu_t^{\varepsilon,1} + \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon^1(z, \mu_t^\varepsilon, \cdot) d\mu_t^{\varepsilon,1}(z) - b^1(\cdot, \mu_t^\varepsilon) \mu_t^{\varepsilon,1} \right) \\ \partial_t \mu_t^{\varepsilon,2} + \nabla \cdot (v^2[\mu_t^\varepsilon] \mu_t^{\varepsilon,2}) = \bar{N}^2(t, \cdot, \mu_t^\varepsilon) \mu_t^{\varepsilon,2} + \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}^d} K_\varepsilon^2(z, \mu_t^\varepsilon, \cdot) d\mu_t^{\varepsilon,2}(z) - b^2(\cdot, \mu_t^\varepsilon) \mu_t^{\varepsilon,2} \right) \\ \mu_{|t=0}^{\varepsilon,1} = \mu_0^1, \mu_{|t=0}^{\varepsilon,2} = \mu_0^2 \end{cases}$$

9 where $\mu_t^\varepsilon = (\mu_t^{\varepsilon,1}, \mu_t^{\varepsilon,2})$ and $b^k(x, \mu) = \int_{\mathbb{R}^d} K^k(x, \mu, dy)$, $k = 1, 2$.

10 Assume that for any $k = 1, 2$, the vector fields v^k satisfy (V1)-(V2), the source
 11 term \bar{N}^k satisfies (N1)-(N2), and the kernel K^k satisfies (K0) through (K8), where
 12 $\mu = (\mu^1, \mu^2) \in \mathcal{M}_b(\mathbb{R}^d) \times \mathcal{M}_b(\mathbb{R}^d)$.

13 First the existence and uniqueness result stated in Theorem 5.1 holds since the
 14 result in [3][Cor. 7.3] deals with systems. We then deduce existence and uniqueness
 15 of a global nonnegative solution $\mu_t^\varepsilon = (\mu_t^{\varepsilon,1}, \mu_t^{\varepsilon,2})$ to the system (47) as in Theorem
 16 5.3. Eventually Step 6.1 to Step 6.5 hold with exactly the same proof. We thus
 17 obtain Theorem 2.1 for the system (47) namely

18 **Theorem 7.3.** For any nonnegative initial condition $\mu_0^1, \mu_0^2 \in \mathcal{M}_b(\mathbb{R}^d)$ such that
 19 $\int_{\mathbb{R}^d} |x| d\mu_0^k(x) < \infty$, $k = 1, 2$, there exists a solution $\mu \in C([0, +\infty), \mathcal{M}_{b,+}(\mathbb{R}^d) \times$
 20 $\mathcal{M}_{b,+}(\mathbb{R}^d))$ to the system

$$(48) \quad \begin{cases} \partial_t \mu_t^1 + \nabla \cdot (v^1[\mu_t] \mu_t^1) = \bar{N}^1(t, \cdot, \mu_t) \mu_t^1 + \frac{1}{2} \sum_{i,j=1}^d \partial_{ij} (d_{ij}^1(\cdot, \mu_t) \mu_t^1) \\ \partial_t \mu_t^2 + \nabla \cdot (v^2[\mu_t] \mu_t^2) = \bar{N}^2(t, \cdot, \mu_t) \mu_t^2 + \frac{1}{2} \sum_{i,j=1}^d \partial_{ij} (d_{ij}^2(\cdot, \mu_t) \mu_t^2) \\ \mu_{|t=0}^1 = \mu_0^1, \mu_{|t=0}^2 = \mu_0^2, \end{cases}$$

with

$$d_{ij}^k(x, \mu) = \int_{\mathbb{R}^d} (y_i - x_i)(y_j - x_j) K^k(x, \mu, dy), \quad 1 \leq i, j \leq d, k = 1, 2$$

1 for any $x \in \mathbb{R}^d$ and any $\mu = (\mu^1, \mu^2) \in \mathcal{M}_b(\mathbb{R}^d) \times \mathcal{M}_b(\mathbb{R}^d)$.

This solution is obtained as the limit as $\varepsilon \rightarrow 0$ (up to a subsequence) of the unique solution μ_t^ε of the system (47) in the sense that $\mu^{\varepsilon, k} \rightarrow \mu^k$ in $C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ for any $T > 0$ and $k = 1, 2$. Moreover for any $T > 0$ there exist $C_T > 0$ such that

$$\int_{\mathbb{R}^d} (1 + |x|) d\mu_t^k \leq C_T \quad 0 \leq t \leq T, k = 1, 2.$$

2 **Acknowledgement:** The authors would like to thank two anonymous referees for
3 useful suggestions that led to an improved version of the manuscript.

4 REFERENCES

- 5 [1] A.S. Ackleh, J. Cleveland, H. Thieme, *Population dynamics under selection and mutation:*
6 *long-time behavior of differential equations in measure spaces*, Journal of Differential Equations,
7 261(2016), 1472-1505.
- 8 [2] A.S. Ackleh, B.G. Fitzpatrick and H.R. Thieme, *Rate distributions and survival of the fittest:*
9 *A formulation on the space of measures*, Discrete and Continuous Dynamical Systems Series B,
10 5(2005), 917-928.
- 11 [3] A.S. Ackleh, N. Saintier, *Well-posedness for a system of transport and diffusion equations in*
12 *measure spaces*, submitted.
- 13 [4] A.S. Ackleh, B. Ma, P.L. Salceanu, *Persistence and global stability in a selection-mutation size-*
14 *structured model*, Journal of Biological Dynamics, 5(2011), 436-453.
- 15 [5] L. Almeida, R.H. Chisholm, J. Clairambault, T. Lorenzi, A. Lorz, C Poucho, *Why Is Evolu-*
16 *tion Important in Cancer and What Mathematics Should Be Used to Treat Cancer? Focus on*
17 *Drug Resistance*, In: Trends in Biomathematics: Modeling, Optimization and Computational
18 Problems (Mondaini R. eds), Springer, Cham (2018), 107-120.
- 19 [6] P. Billingsley, *Convergence of probability measures*, Wiley series in probability and statistics,
20 Wiley-interscience Publication, 1999.
- 21 [7] R. Burger and I.M. Bomze, *Stationary distributions under mutation-selection balance: structure*
22 *and properties*, Advances in Applied Probability, 28(1996), 227-251.
- 23 [8] A. Calsina and S. Cuadrado, *Small mutation rate and evolutionarily stable strategies in infinite*
24 *dimensional adaptive dynamics*, Journal of Mathematical Biology, 48(2004), 135-159.
- 25 [9] A. Calsina, S. Cuadrado, L. Desvillettes, G. Raoul, *Asymptotics of steady states of a selection*
26 *mutation equation for small mutation rate*, Proceedings of the Royal Society of Edinburgh
27 143((2013), 1123-1146.
- 28 [10] A. Calsina, S. Cuadrado, L. Desvillettes, G. Raoul, *Asymptotic profile in selection-mutation*
29 *equations: Gauss versus Cauchy distributions*, Journal of Mathematical Analysis and Applica-
30 tions 444(2016), 1515-1541.
- 31 [11] A. Calsina and J. Palmada, *Steady states of a selection-mutation model for an age structured*
32 *population*, 400(2013), 386-395.

- 1 [12] J.A. Cañizo, J.A. Carrillo, S. Cuadrado, *Measure solutions for some models in population dy-*
2 *namics*, Acta Applicandae Mathematicae, 123 (2013), 141-156 .
- 3 [13] J.A. Cañizo, J.A. Carrillo, J. Rosado, *A well-posedness theory in measures for some kinetic*
4 *models of collective motion*, Mathematical Models and Methods in Applied Sciences, 21 (2011),
5 515-539.
- 6 [14] N. Champagnat, R. Ferrière, S. Méléard, *Unifying evolutionary dynamics: from individual*
7 *stochastic processes to macroscopic models*, Theoretical population biology, 69 (2006), 297-321.
- 8 [15] R.H. Chisholma, T. Lorenzi, J. Clairambault, *Cell population heterogeneity and evolution to-*
9 *wards drug resistance in cancer: Biological and mathematical assessment, theoretical, treatment*
10 *optimisation*, Biochimica et Biophysica Acta, 1860 (2016), 2627-264.
- 11 [16] J. Cleveland and A.S. Ackleh, *Evolutionary game theory on measure spaces: Well-Posedness*,
12 *Nonlinear Analysis: Real World Applications*, 14(2013), 785-797.
- 13 [17] R. Cressman and J. Hofbauer, *Measure dynamics on a one-dimensional continuous trait space:*
14 *theoretical foundations for adaptive dynamics*, Theoretical Population Biology, 67(2005), 47-59.
- 15 [18] R.M. Dudley, *Convergence of Baire measures*, *Studia Mathematica*, T. XXVII (1966), 251-268.
- 16 [19] J.H.M. Evers, S. C. Hille, A. Muntean, *Mild solutions to a measure-valued mass evolution*
17 *problem with flux boundary conditions*, Journal of Differential Equations, 259 (2015), 1068-1097.
- 18 [20] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J.J. Toledo-Melero, *Nonlocal Diffusion Problems*.
19 *American Mathematical Society. Mathematical Surveys and Monographs 2010. Vol. 165. ISBN-*
20 *10: 0-8218-5230-2. ISBN-13: 978-0-8218-5230-9.*
- 21 [21] G. Gabetta, G. Toscani, B. Wennberg, *Metrics for probability distributions and the trend to*
22 *equilibrium for solutions of the Boltzmann equation*, Journal of Statistical Physics, 81 (1995),
23 901-934.
- 24 [22] P. Gwiazda, A. Marciniak Czochra, *Measures under the flat norm as ordered normed vector*
25 *space*, Positivity, 22 (2018), 105?138.
- 26 [23] P. Gwiazda, T. Lorenz, A. Marciniak Czochra, *A nonlinear structured population model: Lip-*
27 *schitz continuity of measure-valued solutions with respect to model ingredients*, Journal of Dif-
28 *ferential Equations*, 248 (2010), 2703-2735.
- 29 [24] S.C. Hille, T. Szarek, D.T.H. Worm, M.A. Ziemplanska, *On a Schur-like property for spaces of*
30 *measures*, arxiv.org/pdf/1703.00677.pdf
- 31 [25] L. Hormander, *The Analysis of Linear Partial Differential Operators. I, Distribution Theory*
32 *and Fourier Analysis*, Grundlehren Der Mathematischen Wissenschaften, Springer, 1990.
- 33 [26] P. Magal and G.F. Webb, *Mutation, selection, and recombination in a model of phenotype*
34 *evolution*, Discrete and Continuous Dynamical Systems, 6 (2000), 221-236.
- 35 [27] H. Othmer, T. Hillen, *The diffusion limit of transport equations II: chemotaxis equations*, SIAM
36 *Journal on Applied Mathematics*, 62 (2002), 1222-1250
- 37 [28] B. Perthame, *Parabolic equations in biology: Growth, Reaction, Movement and Diffusion*,
38 Springer, New York, 2015.
- 39 [29] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics 58, American
40 *Mathematical Society*, (2003).

1 DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA AT LAFAYETTE, LAFAYETTE,
2 LOUISIANA 70504, U.S.A.

3 *Email address:* `ackleh@louisiana.edu`

4 DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNI-
5 VERSIDAD DE BUENOS AIRES (1428) PABELLÓN I - CIUDAD UNIVERSITARIA - BUENOS AIRES -
6 ARGENTINA

7 *Email address:* `nsaintie@dm.uba.ar`