CHANGING SIGN SOLUTIONS OF A CONFORMALLY INVARIANT FOURTH ORDER EQUATION IN THE EUCLIDEAN SPACE

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Fourth order equations of critical Sobolev growth have been an intensive target of investigations in the last years. In particular because of the applications of the fourth order Paneitz operator to conformal geometry. Also because of the parallel that exists between fourth order equations of critical growth and their second order analogues. References for the Paneitz operator are Branson [2] and Paneitz [7]. We consider in this paper the following fourth order equation

$$\Delta^2 u = |u|^{2^\sharp - 2} u \tag{1}$$

on $\mathbb{R}^n$, $n \geq 5$, where $2^\sharp = \frac{2n}{n-4}$ is the critical exponent for the Sobolev embedding of H_2^2-spaces (consisting of functions in L^2 with two derivatives in L^2) into L^p-spaces, and $\Delta^2 = \Delta_\xi^2$ is the bilaplacian operator with respect to the euclidean metric ξ. In [6], Lin proved that the only smooth positive solutions of (1) are the functions given by

$$u_{\lambda,a}(x) = \alpha_n \left(\frac{\lambda}{1 + \lambda^2 |x-a|^2} \right)^{\frac{n-4}{2}} \tag{2}$$

where $\alpha_n = \big(n(n-4)(n^2-4)\big)^{\frac{n-4}{8}}$, $\lambda > 0$ and $a \in \mathbb{R}^n$. The result extends to nontrivial nonnegative solutions of (1) when they belong to the Beppo-Levi space $\mathcal{D}_2^2(\mathbb{R}^n)$. Following standard terminology, we say that two solutions u and v of an equation like (1) are equivalent if they are related by an equation like

$$v(x) = \lambda^{-\frac{n-4}{2}} u \left(\frac{x-a}{\lambda} \right) \tag{3}$$

for some $\lambda > 0$ and $a \in \mathbb{R}^n$. Thanks to the above mentioned result of Lin [6], two smooth positive solutions of (1) are always equivalent. Indeed,

$$u_{\lambda,a}(x) = \lambda^{\frac{n-4}{2}} u_{1,0}\left(\lambda(x-a)\right)$$

Moreover, it is easily checked that equivalent solutions have the same energy in the sense that

$$\int_{\mathbb{R}^n} (\Delta v)^2 dx = \int_{\mathbb{R}^n} (\Delta u)^2 dx$$

if u and v are related by (3). The energy of the $u_{\lambda,a}$'s in (2) is precisely the quantum of energy of a bubble in the blow-up study of positive solutions of Paneitz type equations. We refer to Hebey-Robert [4] for more details.

The purpose of this paper is to prove the following theorem. Such a theorem is the analogue of Ding's result [3] when passing from the second order critical equation $\Delta u = |u|^{4/(n-2)} u$ to the fourth order critical equation (1) we consider in this

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paper.

Theorem: *There exists a sequence* $(u_k)_{k=1}^{\infty}$ *of solutions of (1) whose energy tend to* $+\infty$ *as* $k \to +\infty$. *Namely such that*

$$\int_{\mathbb{R}^n} (\Delta u_k)^2 dx \to +\infty$$

as $k \to +\infty$. *In particular, there exist infinitely many non-equivalent solutions of equation (1). These solutions* u_k *necessarily change sign when* k *is large.*

We prove the theorem in the rest of the paper, following Ding's approach [3] when proving the existence of infinitely many non-equivalent solutions of the second order critical equation $\Delta u = |u|^{4/(n-2)} u$. Specific technical difficulties are attached to the fourth order case.

Proof of the theorem

The Paneitz operator P_h^n on the unit n-sphere (S^n, h) reads as

$$P_h^n u = \Delta_h^2 u + c_n \Delta_h u + d_n u$$

where $c_n = \frac{n^2-2n-4}{2}$ and $d_n = \frac{n(n-4)(n^2-4)}{16}$ (see Paneitz [7] and Branson [2] for the definition of P_h^n). We let $\Phi : S^n \backslash \{N\} \to \mathbb{R}^n$ be the stereographic projection of north pole N in S^n. Then, as is well known,

$$(\Phi^{-1})^* h = \phi^{\frac{4}{n-4}} \xi \tag{4}$$

where

$$\phi(x) = 4^{\frac{n-4}{4}} \left(1 + |x|^2\right)^{-\frac{n-4}{2}} \tag{5}$$

We let $u \in C^2(\mathbb{R}^n)$ be a solution of (1), and let $\hat{u} : S^n \to \mathbb{R}$ be given by

$$\hat{u} = (u\phi^{-1}) \circ \Phi \tag{6}$$

By the conformal properties of P_h^n

$$\phi^{2^\sharp - 1} (P_h^n \hat{u}) \circ \Phi^{-1} = P_\xi^n u = \Delta_\xi^2 u = |u|^{2^\sharp - 2} u = \phi^{2^\sharp - 1} (|\hat{u}|^{2^\sharp - 2} \hat{u}) \circ \Phi^{-1}$$

Therefore $\hat{u}$ is a solution of

$$P_h^n \hat{u} = |\hat{u}|^{2^\sharp - 2} \hat{u} \tag{7}$$

Moreover, it is easily checked that

$$\int_{\mathbb{R}^n} |u|^{2^\sharp} dx = \int_{S^n} |\hat{u}|^{2^\sharp} dv_h \tag{8}$$

Conversely, if $\hat{u}$ is a solution of (7) then $u : \mathbb{R}^n \to \mathbb{R}$ given by $u = (\hat{u} \circ \Phi^{-1})\phi$ is a solution of (1) satisfying (8). As a remark, if $\hat{u} \in H_2^2(S^n)$ is a solution of (7), then $\hat{u} \in L^p(S^n)$ for all p, and $\hat{u}$ is actually in $C^4(S^n)$. We claim now that

$$\int_{\mathbb{R}^n} (\Delta_\xi u)^2 dx < +\infty \tag{9}$$

In order to prove (9), we let $\tilde{\xi}$ be the Riemannian metric on $\mathbb{R}^n$ given by $\tilde{\xi} = \phi^{\frac{4}{n-4}} \xi$. Then, if g is a Riemannian metric on $\mathbb{R}^n$, we let L_g be the conformal Laplacian with respect to g given by

$$L_g u = \Delta_g u + \frac{n-2}{4(n-1)} S_g u$$

where S_g is the scalar curvature of g. By the conformal properties of L_g,

$$\begin{aligned}
\Delta_\xi u &= L_\xi u \\
&= \phi^{\frac{n+2}{n-4}} L_{\tilde{\xi}} \left(u\phi^{-\frac{n-2}{n-4}} \right) \\
&= \phi^{\frac{n+2}{n-4}} \left(\Delta_{\tilde{\xi}} (u\phi^{-\frac{n-2}{n-4}}) + \frac{n(n-2)}{4} u\phi^{-\frac{n-2}{n-4}} \right)
\end{aligned}$$

Therefore, we have that

$$\int_{\mathbb{R}^n} (\Delta_\xi u)^2 \, dv_\xi = \int_{\mathbb{R}^n} \phi^{\frac{4}{n-4}} \left(\Delta_{\tilde{\xi}} (u\phi^{-\frac{n-2}{n-4}}) + \frac{n(n-2)}{4} u\phi^{-\frac{n-2}{n-4}} \right)^2 dv_{\tilde{\xi}}$$

and we can also write that

$$\begin{aligned}
\Delta_{\tilde{\xi}} (u\phi^{-\frac{n-2}{n-4}}) &= \Delta_{\tilde{\xi}} \left((\hat{u} \circ \Phi^{-1}) \phi^{-\frac{2}{n-4}} \right) \\
&= \Delta_{\tilde{\xi}} (\hat{u} \circ \Phi^{-1}) \phi^{-\frac{2}{n-4}} + \Delta_{\tilde{\xi}} (\phi^{-\frac{2}{n-4}}) (\hat{u} \circ \Phi^{-1}) \\
&\quad -2 \langle \nabla (\hat{u} \circ \Phi^{-1}); \nabla \phi^{-\frac{2}{n-4}} \rangle_{\tilde{\xi}}
\end{aligned}$$

where $\langle \cdot ; \cdot \rangle_{\tilde{\xi}}$ is the scalar product with respect to $\tilde{\xi}$. It follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} (\Delta_\xi u)^2 \, dv_\xi &\leq 4 \int_{S^n} (\Delta_h \hat{u})^2 \, dv_h + C_1 (I_1 + I_2 + I_3) \\
&\leq C_2 + C_1 (I_1 + I_2 + I_3)
\end{aligned}$$

where $C_1, C_2 > 0$ are positive constants, and

$$\begin{aligned}
I_1 &= \int_{S^n} \left(\Delta_h (\phi^{-\frac{2}{n-4}} \circ \Phi) \right)^2 (\phi^{\frac{4}{n-4}} \circ \Phi) dv_h \\
I_2 &= \int_{S^n} \left(\phi^{\frac{4}{n-4}} \circ \Phi \right) \left| \nabla \left(\phi^{-\frac{2}{n-4}} \circ \Phi \right) \right|_h^2 dv_h \\
I_3 &= \int_{S^n} \left(\phi^{-2} \circ \Phi \right) (u \circ \Phi)^2 dv_h
\end{aligned}$$

Thanks once again to the conformal invariance of the conformal Laplacian, we can write that

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} \left(\Delta_{\tilde{\xi}} (\phi^{-\frac{2}{n-4}}) \right)^2 \phi^{\frac{4}{n-4}} dv_{\tilde{\xi}} \\
&= \int_{\mathbb{R}^n} \phi^{\frac{2n+4}{n-4}} \left(\phi^{-\frac{n+2}{n-4}} \Delta_\xi \phi - \frac{n(n-2)}{4} \phi^{-\frac{2}{n-4}} \right)^2 dx \\
&\leq C_3 \int_{\mathbb{R}^n} (\Delta_\xi \phi)^2 \, dx + C_4 \int_{\mathbb{R}^n} \phi^{2^\sharp} dx < +\infty
\end{aligned}$$

where $C_3, C_4 > 0$ are positive constants. In a similar way, we can write that

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^n} \phi^{\frac{4}{n-4}} \left| \nabla \phi^{-\frac{2}{n-4}} \right|_{\tilde{\xi}}^2 dv_{\tilde{\xi}} \\
&= \int_{\mathbb{R}^n} \phi^{2^\sharp} \left| \nabla \phi^{-\frac{2}{n-4}} \right|_\xi^2 dx < +\infty
\end{aligned}$$

At last, by (6), we also have that

$$\begin{aligned} |I_3| &\leq C_5 \int_{\mathbb{R}^n} dv_{\tilde{\xi}} \\ &= C_5 \int_{\mathbb{R}^n} \phi^{2^\sharp} dx < +\infty \end{aligned}$$

where $C_5 > 0$ is a positive constant. Hence, (9) is true. In a similar way, we claim that we also have that

$$\int_{\mathbb{R}^n} |\nabla u|^{2^\star} dx < +\infty \tag{10}$$

where $2^\star = 2n/(n-2)$ is the critical Sobolev exponent for the embedding of H_1^2-spaces (consisting of functions in L^2 with one derivative in L^2) into L^p-spaces. Another possible equation for 2^* is $2^* = 2 \times 1^\sharp$. In order to prove (10), we note that, by (6),

$$\left|\nabla(\hat{u} \circ \Phi^{-1})\right|_\xi = \phi^{\frac{2}{n-4}} \left|\nabla \hat{u}\right|_h$$

Then we write that

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla u|^{2^\star} dx &\leq C_6 \int_{\mathbb{R}^n} \left|\nabla(\hat{u} \circ \Phi^{-1})\right|_\xi^{2^\star} \phi^{2^\star} dx + C_7 \int_{\mathbb{R}^n} |\nabla \phi|_\xi^{2^\star} dx \\ &\leq C_8 \int_{\mathbb{R}^n} \phi^{2^\sharp} dx + C_6 \int_{\mathbb{R}^n} |\nabla \phi|_\xi^{2^\star} dx < +\infty \end{aligned}$$

where $C_6, C_7, C_8 > 0$ are positive constants. This proves (10).

Now we consider $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_0(1)$ and $\eta \equiv 0$ in $\mathbb{R}^n \backslash B_0(2)$ where $B_0(r)$ stands for the open Euclidean ball of center 0 and radius r in $\mathbb{R}^n$. For $R > 0$, we set

$$\eta_R(x) = \eta\left(\frac{x}{R}\right)$$

and let u be a solution of (1). Multiplying (1) by $\eta_r u$ and integrating by parts over $\mathbb{R}^n$, we get that

$$\int_{\mathbb{R}^n} \eta_R |u|^{2^\sharp} dx = \int_{\mathbb{R}^n} \Delta_\xi(\eta_R u) \Delta u dx = I_1(R) + I_2(R) - 2I_3(R) \tag{11}$$

where

$$\begin{aligned} I_1(R) &= \int_{B_0(2R)} \eta_R (\Delta_\xi u)^2 dx \\ I_2(R) &= \int_{A_R)} (\Delta_\xi \eta_R) u (\Delta_\xi u) dx \\ I_3(R) &= \int_{A_R} \langle \nabla \eta_R ; \nabla u \rangle_\xi (\Delta_\xi u) dx \end{aligned}$$

and where A_R is the annulus $A_R = B_0(2R) \backslash B_0(R)$. Clearly, thanks to (9), we have that

$$I_1(R) \to \int_{\mathbb{R}^n} (\Delta_\xi u)^2 dx$$

as $R \to +\infty$. We also have that

$$\int_{\mathbb{R}^n} \eta_R |u|^{2^\sharp} dx \to \int_{\mathbb{R}^n} |u|^{2^\sharp} dx$$

as $R \to +\infty$. Independently, letting $V(R) = \mathrm{Vol}_\xi(A_R)$, by help of Hölder'inequality, and noting that $V(R) \le CR^n$, we can write that

$$\begin{aligned} |I_2(R)| &= \left| \int_{A_R)} (\Delta_\xi \eta_R) u (\Delta_\xi u) dx \right| \\ &\le CR^{-2} \|u\|_{2^\sharp} \left(\int_{A_R} (\Delta_\xi u)^{\frac{2n}{n+4}} dx \right)^{\frac{n+4}{2n}} \\ &\le CR^{-2} \|u\|_{2^\sharp} \left(\int_{A_R} (\Delta_\xi u)^2 dx \right)^{1/2} V(R)^{\frac{2}{n}} \\ &\le C \|u\|_{2^\sharp} \left(\int_{A_R} (\Delta_\xi u)^2 dx \right)^{1/2} \end{aligned}$$

Hence, $I_2(R) \to 0$ as $R \to +\infty$. In a similar way, by (10), we can write that

$$\begin{aligned} |I_3(R)| &\le CR^{-1} \|\nabla u\|_{2^\star} \left(\int_{A_R} |\Delta_\xi u|^{\frac{2n}{n+2}} dx \right)^{\frac{n+2}{2n}} \\ &\le CR^{-1} \|\nabla u\|_{2^\star} \left(\int_{A_R} (\Delta_\xi u)^2 dx \right)^{\frac{1}{2}} V(R)^{\frac{1}{n}} \\ &\le C \|\nabla u\|_{2^\star} \left(\int_{A_R} (\Delta u)^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

Hence, we also have that $I_3(R) \to 0$ as $R \to +\infty$. Passing to the limit as $R \to +\infty$ in (11) we get that if $\hat{u}$ is a solution of (7), then $u : \mathbb{R}^n \to \mathbb{R}$ given by $u = (\hat{u} \circ \Phi^{-1})\phi$ is a solution of (1) such that

$$\begin{aligned} \int_{\mathbb{R}^n} (\Delta_\xi u)^2 dx &= \int_{\mathbb{R}^n} |u|^{2^\sharp} dx \\ &= \int_{S^n} |\hat{u}|^{2^\sharp} dv_h < +\infty \end{aligned}$$

In view of this result, and in order to prove the theorem, it suffices to prove that there exists a sequence $(\hat{u}_k)_k$ of solution of (7) such that

$$\int_{S^n} |\hat{u}_k|^{2^\sharp} dv_h \to +\infty$$

as $k \to +\infty$. Let J be the functional associated to (7) given by

$$J(u) = \frac{1}{2} \int_{S^n} \left((\Delta_h u)^2 + c_n |\nabla u|_h^2 + d_n u^2 \right) dv_h - \frac{1}{2^\sharp} \int_{S^n} |u|^{2^\sharp} dv_h$$

Let also G be a closed subgroup of the isometry group $Isom_h(S^n)$ of (S^n, h). For $q = 1, 2$, and $p > 1$, we let

$$H^p_{q,G}(S^n) = \left\{ u \in H^p_q(S^n) \text{ s.t } u(g.x) = u(x) \text{ for all } g \in G \text{ and a.a } x \in S^n \right\}$$

where $H^p_q(S^n)$ is the Sobolev space of functions in L^p with q derivatives in L^p. We denote by $O^x_G = \{g.x / g \in G\}$ the orbit of x under G and let

$$k = \min_{x \in S^n} dim\ O^x_G$$

The composition of a continuous embedding and of a compact embedding is compact. Moreover, we know from the general result in Hebey-Vaugon [5] that if $k \ge 1$, then the embedding $H^p_{1,G}(S^n) \subset L^q(S^n)$ is continuous for all $1 < q \le p^\star_G$, and compact for all $1 < q < p^\star_G$, where $p^\star_G = +\infty$ if $n-k \le p$, and $p^\star_G = (n-k)p/(n-k-p)$ if $n-k > p$. Noting that $(2^\star)^\star_G > 2^\sharp$ when $k \ge 1$, the sequence

$$H^2_{2,G}(S^n) \subset H^{2^\star}_{1,G}(S^n) \subset L^{(2^\star)^\star_G}(S^n)$$

then gives that the embedding $H^2_{2,G}(S^n) \subset L^{2^\sharp}(S^n)$ is compact when $k \ge 1$. In what follows, we let G be such that $k \ge 1$ and such that $H^2_{2,G}(S^n)$ is infinite-dimensional. For instance, as in Ding [3], we can let $G = O(n_1) \times O(n_2)$ where n_1, n_2 are such that $n_1 + n_2 = n + 1$ and $n_1, n_2 \ge 2$. In this example, $k = \min(n_1, n_2) - 1$. We claim now that there exists a sequence $(\hat{u}_m)_m$ of critical points of J restricted to $H^2_{2,G}(S^n)$ such that

$$\int_{S^n} \hat{u}_m^{2^\sharp} dv_h \to +\infty \tag{12}$$

as $m \to +\infty$. In order to prove this claim, we first let $\|\cdot\|$ be the norm on $H^2_2(S^n)$ be given by

$$\|u\|^2 = \int_{S^n} \left((\Delta_h u)^2 + c_n |\nabla u|^2_h + d_n u^2\right) dv_h$$

For J as above, it is easily seen that J is even, that $J(0) = 0$, and that

(A1) there exist $\rho, \alpha > 0$ such that $J > 0$ in $B_0(\rho)\backslash\{0\}$ and $J \ge \alpha$ on $S_0(\rho)$, and

(A2) J satisfies the Palais-Smale condition

where $B_0(\rho)$ is the ball of center 0 and radius ρ in $H^2_2(S^n)$, and $S_0(\rho)$ is the sphere of center 0 and radius ρ in $H^2_2(S^n)$. We can also prove that for any finite dimensional subspace $E \subset H^2_{2,G}(S^n)$,

(A3) $E \bigcap \{J \ge 0\}$ is bounded.

Indeed, since E is finte dimensional, there exists $C > 0$ such that for any $u \in E$, $\|u\| \le C\|u\|_{2^\sharp}$. Let $E = \text{span}\{f_1, \ldots, f_N\}$, where the u_i's are an orthonormal basis for E, and $u = \sum_{i=1}^N \alpha_i f_i$ be such that $\|u\| = 1$. Then, for $R > 0$,

$$\begin{aligned} J(Ru) &= \frac{R^2}{2} - \frac{R^{2^\sharp}}{2^\sharp}\|u\|^{2^\sharp}_{2^\sharp} \\ &\le \frac{R^2}{2}\left(1 - \frac{2R^{2^\sharp-2}}{2^\sharp C^{2^\sharp}}\right) \end{aligned}$$

and (A3) follows. Now, by (A1)–(A3) we can apply Theorem 2.13 of Ambrosetti-Rabinowitz [1] and we get the existence of an increasing sequence $(\alpha_m)_m$ of critical values for J given by

$$\alpha_m = \sup_{h \in \Gamma^\star} \inf_{u \in S \cap E^\perp_{m-1}} J\big(h(u)\big) \tag{13}$$

where $S = S_0(1)$, $E_m = \text{span}\{f_1, \ldots, f_m\}$, $E^\perp_m$ is the orthogonal complement of E_m, $(f_i)_{i\ge1}$ is an orthonormal basis of $H^2_{2,G}(S^n)$, and $\Gamma^\star$ is the space of odd homeomorphisms of $H^2_{2,G}(S^n)$ onto $H^2_{2,G}(S^n)$ such that $J\big(h(B)\big) \ge 0$ where B is the ball of center 0 and radius 1 in $H^2_{2,G}(S^n)$. Then, in order to prove that there exists a sequence $(\hat{u}_m)_m$ of critical points of J restricted to $H^2_{2,G}(S^n)$ such that (12) is true, it suffices to prove that

$$\alpha_m \to +\infty \tag{14}$$

as $m \to +\infty$. We define

$$T = \left\{u \in H^2_{2,G}(S^n) \text{ s.t. } 2^\sharp\|u\|^2 = 2\|u\|^{2^\sharp}_{2^\sharp}\right\}$$

and let

$$\beta_m = \inf_{u \in T \cap E_m^\perp} \|u\|$$

Then

$$\beta_m \to +\infty \tag{15}$$

as $m \to +\infty$. Indeed, if it is not the case, there exists $(u_m)_m$ such that $u_m \in E_m^\perp$ for all m, $u_m \in T$ for all m, the u_m's are bounded in $H_2^2(S^n)$, and $u_m \rightharpoonup 0$ in $H_{2,G}^2(S^n)$ since $u_m \in E_m^\perp$. The compactness of the embedding $H_{2,G}^2(S^n) \subset L^{2^\sharp}(S^n)$ then implies that (up to a subsequence) $u_m \to 0$ in $L^{2^\sharp}(S^n)$. It follows that $u_m \to 0$ in $H_2^2(S^n)$ since $u_m \in T$ for all m. On the other hand, by the Sobolev inequality corresponding to the embedding $H_2^2(S^n) \subset L^{2^\sharp}(S^n)$, and still since $u_m \in T$ for all m, there exists $C > 0$ such that $\|u_m\| \geq C$ for all m. A contradiction, and (15) is proved. For $u \in E_m^\perp$, we let

$$h_m(u) = \frac{1}{2}\beta_m u$$

Following Ambrosetti-Rabinowitz [1], it is easily seen that h_m extends to $\tilde{h}_m \in \Gamma^\star$. Given $u \in H_{2,G}^2(S^n)\backslash\{0\}$, we let $\beta(u) \in \mathbb{R}$ be such that $\beta(u)u \in T$. Then, if $u \in S \cap E_m^\perp$,

$$\begin{aligned} J\big(h_m(u)\big) &= \frac{1}{2}\left(\frac{\beta_m}{2}\right)^2 \left(1 - \left(\frac{\beta_m}{2\beta(u)}\right)^{2^\sharp - 2}\right) \\ &\geq \frac{1}{2}\left(\frac{\beta_m}{2}\right)^2 \left(1 - \left(\frac{1}{2}\right)^{2^\sharp - 2}\right) \end{aligned}$$

and we get with (13) and (15) that (14) holds. In particular, there exists a sequence $(\hat{u}_m)_m$ of critical points of J restricted to $H_{2,G}^2(S^n)$ such that (12) holds. The $\hat{u}_m$'s are solutions of (7) when restricted to $H_{2,G}^2(S^n)$ in the sense that for any m and any $\varphi \in H_{2,G}^2(S^n)$,

$$\int_{S^n} \left((\Delta_h \hat{u}_m)(\Delta_h \varphi) + c_n \langle \nabla \hat{u}_m, \nabla \varphi \rangle_h + d_n \hat{u}_m \varphi\right) dv_h = \int_{S^n} |\hat{u}_m|^{2^\sharp - 2} \hat{u}_m \varphi dv_h$$

Let φ be any smooth function on S^n, or any function in $H_2^2(S^n)$. Let also φ_G be given by the equation

$$\varphi_G(x) = \int_G \varphi\left(\sigma(x)\right) d\mu(\sigma)$$

where $d\mu$ is the Haar measure on G. Clearly, φ_G is smooth and G-invariant if φ is smooth, or $\varphi_G \in H_{2,G}^2(S^n)$ if $\varphi \in H_2^2(S^n)$. Then we can write that

$$\begin{aligned} &\int_{S^n} \left((\Delta_h \hat{u}_m)(\Delta_h \varphi_G) + c_n \langle \nabla \hat{u}_m, \nabla \varphi_G \rangle_h + d_n \hat{u}_m \varphi_G\right) dv_h \\ &= \int_G \Bigg(\int_{S^n} \Big((\Delta_h \hat{u}_m)\left(\Delta_h (\varphi \circ \sigma)\right) \\ &\qquad + c_n \langle \nabla \hat{u}_m, \nabla(\varphi \circ \sigma) \rangle_h + d_n \hat{u}_m (\varphi \circ \sigma) \Big) dv_h \Bigg) d\mu(\sigma) \\ &= |G| \int_{S^n} \left((\Delta_h \hat{u}_m)(\Delta_h \varphi) + c_n \langle \nabla \hat{u}_m, \nabla \varphi \rangle_h + d_n \hat{u}_m \varphi\right) dv_h \end{aligned}$$

where $|G|$ is the volume of G with respect to $d\mu$, and that

$$\begin{aligned}&\int_{S^n}|\hat{u}_m|^{2^\sharp-2}\hat{u}_m\varphi_G dv_h\\&=\int_G\left(\int_{S^n}|\hat{u}_m|^{2^\sharp-2}\hat{u}_m(\varphi\circ\sigma)dv_h\right)d\mu(\sigma)\\&=|G|\int_{S^n}|\hat{u}_m|^{2^\sharp-2}\hat{u}_m\varphi dv_h\end{aligned}$$

It follows that

$$\int_{S^n}\left((\Delta_h\hat{u}_m)(\Delta_h\varphi)+c_n\langle\nabla\hat{u}_m,\nabla\varphi\rangle_h+d_n\hat{u}_m\varphi\right)dv_h=\int_{S^n}|\hat{u}_m|^{2^\sharp-2}\hat{u}_m\varphi dv_h$$

for all $\varphi\in H_2^2(S^n)$ and all m. In particular, for any m, $\hat{u}_m$ is a solution of (7). The u_m's associated to the $\hat{u}_m$'s have to change sign for $m>>1$ according to the remark on equivalent solutions in the introduction and the fact that

$$\int_{\mathbb{R}}(\Delta u_m)^2dx=\int_{S^n}|\hat{u}_m|^{2^\sharp}dv_h\to+\infty$$

This ends the proof of the theorem.

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