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Measure valued opinion dynamics

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In this work we propose a method of opinion pooling based on pairwise interactions. We assume that each agent has a probability measure on the possible outcomes of some situation, and they try to find a single measure aggregating their estimates. This is a classical problem in Decision Theory, where expert opinions contain some degree of uncertainty, and a Decision Taker needs to pool these estimates.

We study this problem using a kinetic theory approach, obtaining a Boltzmann type equation for opinions which are symmetric probability measures defined on the real line. We obtain a non local, first order, mean field equation as its grazing limit when the parameter in the interaction goes to zero. Also, we prove the convergence to quasi-consensus with explicit estimates on the convergence time depending on the variance of these measures.

Let us remark that this model can be interpreted as a noisy model of opinion dynamics. In many models, the opinion of each agent is a point in the real line, the agents interact and observe other agents opinions. We can consider that observed opinions are perturbed or deformed by some noise in the transmission channel or in the interpretation of the agents, so we can think of agents opinions directly as random variables instead of a single point.

Keywords: Opinion formation models; active particles; Boltzmann equation; Grazing limit.

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1. Introduction

Let us consider a group of agents facing a situation where each one has its own opinion on the correct procedure to follow. Usually, this opinion is represented by binary -or more generally, finite- options, such as political candidates in an election, support or not some project, or choosing the meal, among many other cases.[25,44,48] In other problems, opinions can be represented by real numbers, as the degree of adhesion to some idea, or the rating of a product, a service or a movie.[1,12,27] Also, higher dimensional opinions, represented as vectors in $\mathbb{R}^d$, are used as continuous version of the Axelrod model,[4] and also as the position and velocity of a bird in a flock or a robot in a de-centralized search.[18]

However, there are many situations where the agents opinions are inherently uncertain, and only represent some degree of belief. This happens when they try to predict the future value of an asset, or forecast the weather, or they are playing some complex game with no obvious Nash equilibria. In those cases, it is customary to model the opinion of each agent as a probability distribution, and the problem now is to extract from them a single probability distribution as the experts advice. This is a classical topic in the areas of Artificial Intelligence, Decision Theory, and Expert Systems, where some Decision Taker aggregates the expert opinions, a procedure known as opinion pooling, and we refer the interested reader to Ref. 16, 29, 31 and the comprehensive work of Cooke[17] on the subject.

The study of opinion pooling given a group of experts goes back to Savage,[41] and several methods were proposed, mainly through weighted means of the probability distributions, or using behavioral procedures where each agent reviews their own probability distribution combining it with the distributions of the others experts, with important contributions of Stone,[45] Winkler,[52] and specially DeGroot,[21] who proposed the first dynamical model of this type.

It is known that each method has their own drawbacks, and cannot be applied in every situation. Weighted means, for instance, depend on the arbitrary weights assigned to the experts, while internal discussion among the experts is heavily influenced by psychological factors.[47] For example, DeGroot proposed in Ref. 21 a mix between both methods, where each agent updates its opinion by performing a weighted mean of the different probability distributions, assigning its own weight to each expert. His algorithm is a Markov process on the set of probability measures and hence the formation of consensus is guaranteed assuming the process is irreducible.

Let us observe that in behavioral methods, like DeGroot's method mentioned before, a full knowledge of the expert distributions is needed, and sometimes this is not possible. For instance, in a game, only the pure strategies played in a round are observed, instead of the mixed strategies used to select them. Moreover, it is difficult to learn the mixed strategy of the opponent if each player updates it after the game, and it is well known that the players cannot learn the optimal way of playing in several games using only this information.[28]

We propose here a simple method of opinion pooling based on pairwise interactions. We assume that each agent has a probability density on the real numbers, assumed to be the set of possible opinions, and they are willing to reach a consensus. This attitude toward a compromise is natural from the perspective of a Decision Taker, since she/he needs to pool the different estimates.

Let us mention also that the whole population can affect the individual opinion of a single agent through the publications of polls and surveys, and the results of elections or referendums. A mean field model covering this kind of interactions is out of the scope of this work and it would be interesting to consider a background -depending on the population- interacting with the individuals.

Now, agents are randomly matched in pairs, and each one generates a number with her/his distribution which is observed by the other agent. Then, they update their distributions, and different rules for the dynamic are available at this stage:

(a) Only the mean is moved toward the observed value.
(b) The mean is moved if the observed value belongs to some interval close to the mean, a model with *bounded confidence*.
(c) The full probability distribution is moved as a weighted mean with the Dirac's Delta at the single value observed.

We mainly study (a) in this work, and we deal with (b) and (c) in a separate paper. We prove for (a) that agents mean opinion converges to some value in each realization, although, due to the random nature of the process, it is not always the same. We derive the microscopic equations of the dynamics, and its continuous approximation by a Boltzmann type equation. We perform the grazing limit obtaining a first order, nonlocal, mean field equation, and characterize the expected value of consensus, or, more precisely, *quasi-consensus*, see below.

Rule (a) projects the dynamics on a one dimensional space, and we proceed as in continuous models of opinion formation, which were thoroughly studied in the last years using the kinetic theory of granular flows and gases. However, an important difference arise whenever agents have the possibility to take decisions, and the so-called kinetic theory of active particles introduced by Bellomo and its collaborators,[8,35] which is a technique well suited for study this kind of problems. There many important references in opinion modeling and related socio-economics problems, both by mathematicians,[9,15,36,37] and physicists.[10,11,25,42,43]

Observe that we can re-interpret the classical opinion dynamics models as a case of measure valued opinion dynamics, identifying agents opinions with Dirac Delta masses. However, the main difference is the lack of certainty in the observation of the other agents opinions, so the binary (or collective) interactions will depend on the particular realizations of the opinions, since they are random variables.

This will forces us to abandon the notion of *consensus*, and we talk instead of *quasi-consensus*. The uncertainty in the opinions implies that the agents never reach a consensus state, since new interactions could change this state. So, we will say that the agents reach a quasi-consensus state when all the means are concentrated in a small interval.

On the other hand, our model has another interpretation. Let us suppose that each agent i has a well-defined opinion, represented by a real number x_i without uncertainty. Now, when agent i interacts with agent j, and observes a noisy signal $m_j = x_j + \varepsilon_j$, where x_j is the true opinion of agent j plus a random noise ε_j. So, our results can be thought as an analysis of the consensus formation under noise, and the effect of the noise in the time to consensus.

Usually, noise appears in opinion dynamics as a random variable representing free thinking, and agent opinions perform independent random walks in the space of opinions, and adds a diffusion term in the continuous approximations of different dynamics.[1,20,22,27,38,49,50] However, this one is a different class of noise, depending on the communication channel, and was studied mainly trough simulations.[26,30,40,46] Here, very interesting phenomena appear, since the noise intensity can destroy the consensus both in discrete or continuous problems, or it could help to reach a quasi-consensus state destroying the clusters in Hegselmann-Krause models, or flip the agents between different equilibria. Here, we provide a rigourous theoretical analysis of the convergence to a continuous equation, and the time of convergence to a quasiconsensus.

1.1. *Organization of the paper and main results*

In order to describe the main results, let us assume that agents opinions belong to the same family of probability distributions with different means x_i. For instance every agent could have as an opinion a normal distribution $N(x_i, \sigma^2)$ where σ^2 is given (the same for every agent) and the mean opinion x_i will change during interactions. We fix a small positive parameter h, and if agent i observes a signal m_j from agent j, greater than (respectively, smaller than) its own mean x_i, then her/his new mean will be $x_i + h$ (resp., $x_i - h$). This rule of discrete updates was used recently in several works.[5–7,34,39]

Briefly, the opinion of agent i is the random variable $\mathbb{X}_i = \sigma\mathbb{Y} + x_i$, where $\mathbb{Y}$ is a fixed symmetric random variable with a continuous distribution and variance $Var[\mathbb{Y}] = 1$. Notice that $\mathbb{X}_i$ has expected value x_i and variance σ^2. We call F the cumulative distribution of $\mathbb{Y}$, $F_\sigma(\cdot) = F(\cdot/\sigma)$ the cumulative distribution of $\sigma\mathbb{Y}$, and we define

$$\Psi_\sigma = 1 - 2F_\sigma.$$

In Section §2 we define precisely the interaction rules, and we introduce the notation and some previous results which will be needed later.

In Section §3 we present the model and derive the associate Boltzmann equation when all the agents have the same probability distribution and only the means are different, following the ideas in Ref. 9. We will have in mind the case of Normal distributions with a fixed variance, although the results are proved for an arbitrary symmetric random variable $\mathbb{Y}$ with a continuous distribution.

In Section §4 we study the existence of solutions to the Boltzmann equation using A. Bressan's techniques,[2,13] by considering it as an ordinary differential equation in a abstract space. We denote by $\mathcal{M}_b(\mathbb{R})$ the space of bounded Borel measures on $\mathbb{R}$, and by $\mathbb{P}(\mathbb{R})$ the set of probability measures on $\mathbb{R}$ (we refer to Section §2 for the precise definitions and the needed properties of $\mathbb{P}(\mathbb{R})$ and $\mathcal{M}_b(\mathbb{R})$). The main result in Section §3 is the following theorem:

Theorem 1.1. *For any initial condition $f_0 \in \mathbb{P}(\mathbb{R})$ and any $h > 0$ there exists a unique $f^h \in C([0, +\infty), \mathbb{P}(\mathbb{R})) \cap C^1([0, +\infty), \mathcal{M}_b(\mathbb{R}))$, solution to the Boltzmann equation*

$$\frac{d}{dt}\int \phi \, df_t^h = \int \mathbb{E}[\phi(x') - \phi(x)] \, df_t^h(x) df_t^h(x_*) \qquad \phi \in C_b(\mathbb{R}), \tag{1.1}$$

where $\mathcal{M}_b(\mathbb{R}))$ and $\mathbb{P}(\mathbb{R})$ are endowed with the total variation norm.

Also, we derive an ordinary differential equation for the variance of the agents distribution $Var[f_t^h]$, which enable us to describe its clustering, and is the key point to prove the long time asymptotic behavior of the opinions.

In Section §5 we perform the so-called grazing (or quasi-invariant) limit when the parameter h goes to zero, see for references Pareschi and Toscani's book[37] and the references therein. We obtain a first order mean field equation and we show that its unique solution approximates the solutions to the Boltzmann equation for h small, namely:

Theorem 1.2. *Suppose that f_0 has finite first moment, and let $\mathbb{P}(\mathbb{R})$ be endowed with the weak convergence. Denote $g_\tau^h := f_t^h$ with $\tau = ht$. Then, up to a subsequence as $h \to 0$, $\{g_\tau^h\}$ converges in $C([0, T], \mathbb{P}(\mathbb{R}))$ for any $T > 0$ to $g_\tau \in C([0, +\infty), \mathbb{P}(\mathbb{R}))$, a solution to*

$$\frac{d}{d\tau}\int \phi \, dg_\tau = \int (\Psi_\sigma * g_\tau)(x)\phi'(x) \, dg_\tau(x) \tag{1.2}$$

which is the weak formulation of the first order, nonlocal, mean field equation

$$\partial_\tau g_\tau + \partial_x \Big((\Psi_\sigma * g_\tau) g_\tau \Big) = 0 \tag{1.3}$$

with initial condition $g_0 = f_0$.

It is well-known (see e.g. Ref. 23 and references therein) that a first order equation like (1.3) with a solution-dependent vector-field has a unique solution in $C([0, +\infty), P(\mathbb{R}))$ when the vector-field satisfies some mild properties satisfied in our case.

Moreover, we prove that the expected value of the distribution of the means of the agents is constant in time and that the variance of this distribution decreases to 0. So, the agents reach asymptotically a quasi-consensus state whose expected value equals the expected value of the initial distribution f_0 of the opinion means:

Theorem 1.3. *Suppose that g_0 has finite second moment. The solution g_τ to equation* (1.3) *converges as $\tau \to +\infty$ to the Dirac mass located at $\mathbb{E}[f_0]$:*

$$\lim_{\tau \to +\infty} g_\tau = \delta_{\mathbb{E}[f_0]}.$$

The next result gives an explicit estimate on the convergence time to the quasi-consensus. We assume that agent's opinions are of the form $x + \sigma\mathbb{Y}$ where $\sigma > 0$ is given and $\mathbb{Y}$ is a symmetric random variable with mean 0 and variance 1 having a density $f_\mathbb{Y} \in C^1(\mathbb{R})$. We assume that $f'_\mathbb{Y}$ is bounded and $F(x) > F(0)$ for $x > 0$ (this holds if e.g. $f_\mathbb{Y}(0) > 0$), where F is the cumulative distribution function of $\mathbb{Y}$. Under these assumptions we can prove the next theorem:

Theorem 1.4. *We suppose that g_0 has compact support and let us fix $R_0 > 0$ such that $\operatorname{supp} g_0 \subset [-R_0, R_0]$. Then, as $\tau \to +\infty$, the support of g_τ shrinks to the single point $c = \mathbb{E}[f_0]$, and for any $t \geq 0$,*

$$W_2(g_\tau, \delta_c)^2 = Var[g_\tau] \leq (Var[g_0]) \exp\Big\{\frac{4\tau}{\sigma}\Big(- f_\mathbb{Y}(c/\sigma) + \frac{R_0}{\sigma}\|f'_\mathbb{Y}\|_\infty\Big)\Big\}.$$

Moreover, for any $\varepsilon > 0$, there exists $\tau_\varepsilon > 0$ such that

$$W_2(g_\tau, \delta_c)^2 = Var[g_\tau] \leq (Var[g_0])e^{(-\frac{4f_\mathbb{Y}(c/\sigma)}{\sigma}+\varepsilon)\tau}, \qquad \tau \geq \tau_\varepsilon. \tag{1.4}$$

Here, $W_2(.,.)$ is the Wasserstein distance, see Section §2 for the definitions, and $Var[g_\tau]$ is the variance of the probability measure g_τ namely $Var[g_\tau] = \int (x - \mathbb{E}[g_\tau])^2\, dg_\tau(x)$, being $\mathbb{E}[g_\tau] = \int x\, dg_\tau(x)$ the mean value of g_τ.

Remark 1.1. If $f_\mathbb{Y} \in C^1$ is non-increasing in $(0, +\infty)$, we can give a lower bound for $W_2(g_\tau, \delta_c)$:

$$W_2(g_\tau, \delta_c)^2 = Var[g_\tau] \geq (Var[g_0])e^{(-\frac{4f_\mathbb{Y}(c/\sigma)}{\sigma})\tau},$$

so the estimate given by inequality (1.4) is asymptotically optimal in this case.

Let us note that this lower bound hold in the particular case that $\mathbb{Y} \sim N(0, 1)$.

Let us compare briefly our results and proofs with the formalism of gradient flows in Wasserstein spaces introduced by Jordan, Kinderlehrer and Otto,[32] and fully developed in the monograph Ref. 3. Indeed, Eq. (1.3) can be seen as a gradient flow in the space $\mathbb{P}_2(\mathbb{R})$ of probability measures with finite second moment for the energy functional

$$\phi(\mu) = \int_{\mathbb{R}\times\mathbb{R}} W(x - y)\, d\mu(x)\mu(y) \qquad \mu \in P_2(\mathbb{R}),$$

where the interaction potential $W : \mathbb{R} \to \mathbb{R}$ is such that $W' = -\Psi_\sigma$ (see example 11.2.7 in Ref.[3]). For instance we can take $W(x) = \int_0^x (2F_\sigma(y) - 1)\, dy$ so that $W(0) =$

0 and W is even. Moreover $W''(x) = 2F'_\sigma(x) = \frac{2}{\sigma} f_{\mathbb{Y}}(x/\sigma)$ where $f_{\mathbb{Y}}$ is the density of $\mathbb{Y}$. Thus $\inf_{\mathbb{R}} W'' = 0$ so that W is λ-convex with $\lambda = 0$. It follows that ϕ is λ-convex with $\lambda = 0$ along generalized geodesic (see remark 9.2.5 in Ref. 3).

Making the additional assumption that W is doubling in the sense that $W(x + y) \le C(1 + W(x) + W(y))$ for any $x, y \in \mathbb{R}$, the energy functional ϕ satisfies all the assumptions stated in section 10.4.7 and we can then apply Theorem 11.2.8 and Theorem 11.2.1 in Ref. 3. We deduce that Eq. (1.3) has a unique solution in $\mathbb{P}_2(\mathbb{R})$ obtained as the gradient flow of ϕ and which satisfies, among others properties, that for any minimum point $\bar\mu$ of ϕ, $\phi(\mu_t) - \phi(\bar\mu) \le W_2^2(\mu_0, \bar\mu)/(2t)$ and the map $t \to W_2(\mu_t, \bar\mu)$ is not increasing.

Notice that since we only have $\lambda = 0$, i.e. W is convex but not uniformly strictly convex in $\mathbb{R}$, we cannot deduce directly from this theory that ϕ has a unique minimum point, nor the exponential convergence of the solution μ_t to the minimum. Indeed, the value of $\min \phi$ is a priori not clear though we can conjecture that it should be equal to 0, which corresponds to μ being a Dirac's mass.

However, in view of our Theorem 1.3 and its proof, the support of g_τ shrinks to the point $c = \mathbb{E}[f_0]$. Then, after a sufficiently long time, we can approximate the infimum of W'' on the support of g_τ by its value at c, namely $\frac{2}{\sigma} f_{\mathbb{Y}}(c/\sigma)$. We thus approximately obtain that W is λ-convex, and so that ϕ is λ-convex along generalized geodesic, with $\lambda \approx \frac{2}{\sigma} f_{\mathbb{Y}}(c/\sigma)$. Then Theorem 11.2.8 and Theorem 11.2.1 in Ref. 3 gives in particular for times $\tau \gg 1$ that $W_2(g_\tau, \delta_c) \le W_2(g_0, \delta_c) e^{-\frac{2}{\sigma} f_{\mathbb{Y}}(c/\sigma)\tau}$, which is essentially the content of our Theorem 1.4.

Theorem 1.4 states that this asymptotic behaviour is the correct one and can be justified rigorously. Its proof is quite elemental since it is essentially based on scaling arguments and it is also completely self-contained (in particular it does not refer to the theory developed in Ref. 3). Moreover, we also prove (and observe numerically) that this exponential convergence is optimal in some cases (see Remark 1.1).

In Section §6 we present microscopic simulations of the dynamics. The simulations show a good agreement with the theoretical predictions about the expected value of consensus and the time of convergence. We conclude in Section §7, where we compare the results with the corresponding ones of DeGroot's model in some particular cases, and we describe possible extensions.

2. Preliminaries

2.1. *Notations and definitions*

Given $K \subset \mathbb{R}$, we denote by $\mathbb{P}(K)$ the convex set of probability measures on K. For any $f \in \mathbb{P}(K)$, we write the integral of a function $\phi \in C_b(K)$ against f both as $\int_K \phi(x)\, df(x)$ or $\int_K \phi(x) f(x)\, dx$. Let us stress that $f(x)\, dx$ is only a notation convention since we are not assuming that f has a density, indeed, f could be a Dirac Delta measure.

$\mathcal{M}(K)$ stands for the space of Borel signed measures on K, and $\mathcal{M}_b(K) \subset \mathcal{M}(K)$

denotes the measures with finite total mass. These sets are endowed with the total variation norm,

$$\|f\|_{TV} = \sup \left\{ \int_K \phi \, df : \phi \in C(K) \text{ such that } \|\phi\|_\infty \leq 1 \right\}. \tag{2.1}$$

Let us remark that $\mathcal{M}(K)$ becomes a Banach space with this norm.

We will need to endow the set $\mathbb{P}(K)$ with the weak* topology. We will use also the Wasserstein distances W_p, with $p \geq 1$, between two probability measures μ, ν, which are defined as

$$W_p(\mu,\nu) = \left(\inf_{\alpha \in \Gamma(\mu,\nu)} \int_{K\times K} |x-y|^p d\alpha(x,y) \right)^{1/p} \qquad p \geq 1, \tag{2.2}$$

being $\Gamma(\mu,\nu)$ the collection of all probability measures on $K \times K$ with marginal measures ν and μ on the first and second factor, respectively. Any Wasserstein distance W_p, $p \geq 1$, metrizes the weak*-convergence when K is bounded. When $p = 1$ the Kantorovich-Rubinstein Theorem provides a dual representation of W_1, namely

$$W_1(\nu,\mu) = \sup \left\{ \int_K \varphi \, d(\mu-\nu) : \quad \varphi \text{ is 1-Lipschitz} \right\}. \tag{2.3}$$

See Ref. 24, 51 for details.

Let $\mathbb{Y}$ be a random variable. We say that $\mathbb{Y}$ is symmetric if

$$P(\mathbb{Y} \in [-x,0]) = P(\mathbb{Y} \in [0,x]) \qquad \text{for any } x \geq 0.$$

Notice that a symmetric random variable has expected value 0. We say that $\mathbb{Y}$ has finite k-th moment, $k \in \mathbb{N}$, if $\mathbb{E}(|\mathbb{Y}|^k) < \infty$.

3. Mathematical Modelling

The system we study in this paper consists of a large population of interacting agents characterized only by their internal state, here their *opinion*. The mathematical modelling of the system will be done at two scales, the micro- and the macroscopic scale. At the microscopic scale we will be concerned with the precise definition of the rules governing the updating of the opinion of agents involved in an interaction. At the macroscopic level on the other hand we will adopt the paradigm of statistical physics namely describing the whole population through a probability density function modelling the global distribution of opinion in the population. The evolution in time of this probability distribution will be encoded in a Boltzmann-like equation, namely Eq. (1.1), as classically done in statistical mechanics and in the modelling of complex living system, see e.g. Ref. 9.

The microscopic and macroscopic modelling are presented in the next two subsections. See also Section §6 for a Master Equation approach, where we consider only a discrete range of values for $\{x_i\}$, and we study the discrete random process by computing the expected gain and loss terms, and we obtain the corresponding continuous equation. Let us stress that our informal approach can be made rigorous by using the probabilistic theory of interacting particles, and the hydrodynamic limit can be computed following the ideas in Ref. 19, 33.

We will then go on proving the well-posedness of the Boltzmann equation, and the rest of the paper will be devoted to the studying both theoretically and numerically the long-time behaviour of the solution.

3.1. *Microscopic scale modelling*

Let us consider a population with N agents. We model the opinion of an agent, say agent i, by a real-valued random variable $\mathbb{X}_i$ with mean x_i and variance σ^2, the same variance for every agent. We can thus write

$$\mathbb{X}_i = \sigma \mathbb{Y}_i + x_i$$

where $\mathbb{Y}_i$ is a reduced centered random variable. We assume that $\mathbb{Y}_1, \mathbb{Y}_2, ...$ are independent copies of a reduced centered random variable $\mathbb{Y}$ that we additionally suppose symmetric with finite second moment.

As a result of interactions, agents modify their opinion. We assume that during an interaction the random variables $\mathbb{Y}_1, \mathbb{Y}_2, ...$ and the common variance σ^2 remain unchanged and that only the means $x_1, x_2, ...$ change.

To describe the updating rule for the means we first fix some $h > 0$. Consider two interacting agents with opinions $\mathbb{X} = \sigma\mathbb{Y} + x$, $\mathbb{X}_* = \sigma\mathbb{Y}_* + x_*$ before the encounter, and let us denote by $\mathbb{X}' = \sigma\mathbb{Y} + x'$, $\mathbb{X}'_* = \sigma\mathbb{Y}_* + x'_*$ the new opinions after the interaction. Suppose now that the second agent emits an opinion $m_* \in \mathbb{R}$, that is, m_* is a realization of $\mathbb{X}_*$. The first agent will then update his/her mean opinion x in the following way:

$$x' = \begin{cases} x + h & \text{if } x < m_* \\ x - h & \text{if } x > m_*. \end{cases} \tag{3.1}$$

This interaction rule models the tendency to compromise of the agents since the first agent slightly moves his mean opinion x toward the opinion emitted by the second agent. Notice that the means of the agents opinion are random variables due to the stochastic nature of this updating rule.

3.2. *Macroscopic kinetics and properties*

Let us present a generalized Boltzmann model, obtaining kinetic equations for the distribution function $f_t^h(x)$ which gives the distribution of agents whose opinion has mean x at time $t \geq 0$, hence f_t^h is a probability measure on $\mathbb{R}$. Intuitively $f_t^h(x)dx$ is the proportion of agent whose mean opinion belongs to $[x, x + dx]$ at time t.

Let us note that $f_t^h(x) : [0, \infty) \times \mathbb{R} \to \mathbb{R}_+$ depend on a single state variable x, and we assume that the number of agents is conserved,

$$\int_{\mathbb{R}} f_t^h(x)dx = 1, \qquad t \geq 0.$$

The state x will change by pair interactions, according to the previous microscopic rules (3.1). We assume that encounters between agents occur according to a Poisson process with constant rate, that we can assume equal to one up to re-scaling time, and that the encounters times and the $\mathbb{Y}_1, \mathbb{Y}_2, ...$ are independent.

As usual for binary interactions, we assume that they are statistically independent, in order to avoid a BBGKY hierarchy of coupled equations. The time evolution of f_t^h results from the balance of gain and loss terms, and can be described using a Boltzman type integro-differential equation. By taking a test function $\phi \in C_b(\mathbb{R})$ we get the weak formulation

$$\frac{d}{dt}\int \phi(x)\, df_t^h(x) = \int \mathbb{E}[\phi(x)](G[f] - L[f]),$$

where $\mathbb{E}$ is the mathematical expectation, and $G[f]$, $L[f]$ are the gain and loss term, namely

$$G[f_t^h](t,x) = \iint B(x, x', x_*, x'_*)\, df_t^h(x')df_t^h(x'_*),$$

$$L[f_t^h](t,x) = f_t^h(x)\int df_t^h(x'_*).$$

Here B is the kernel which takes into account the interactions, given initial opinions x, x_*, and post-interaction opinions x', x'_* (see below how it can be computed explicitly).

Let us recall that we take the expected value of the changes since x is a random variable, and that all the random variables considered so far, namely $\mathbb{Y}_1, \mathbb{Y}_2, ..., \mathbb{Y}$, and the means $x_1, x_2, ...$, are supposed to be defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The expectation $\mathbb{E}$ is taken with respect to this probability space.

Finally, we can change variables in order to obtain

$$\frac{d}{dt}\int \phi(x)\, df_t^h(x) = \int \mathbb{E}[\phi(x') - \phi(x)]\, df_t^h(x)df_t^h(x_*) =: (Q[f_t^h], \phi) \tag{3.2}$$

for any test function $\phi \in C_b(\mathbb{R})$, see for instance Ref. 14, 37.

In order to write explicitly the gain and loss terms, or the collision operator $Q[f]$ in the right hand side of (3.2), notice first that

$$P(\mathbb{X}_* \leq x) = P(\sigma\mathbb{Y}_* + x_* \leq x) = P(\sigma\mathbb{Y} \leq x - x_*).$$

We denote by F the cumulative distribution function of $\mathbb{Y}$, namely $F(x) = P(\mathbb{Y} \leq x)$ for $x \in \mathbb{R}$, and by $F_\sigma(\cdot) = F(\cdot/\sigma)$ the cumulative distribution of $\sigma\mathbb{Y}$. Thus

$$P(\mathbb{X}_* \leq x) = F_\sigma(x - x^*). \tag{3.3}$$

As a consequence for any $x, x_* \in \mathbb{R}$,

$$\begin{aligned} F[\phi](x, x_*) &:= \mathbb{E}[\phi(x')] \\ &= \phi(x+h)(1 - F_\sigma(x - x_*)) + \phi(x-h)F_\sigma(x - x_*). \end{aligned}$$

It follows that $Q[f]$ is the measure defined, for a given bounded measure f on $\mathbb{R}$, by

$$\begin{aligned} (Q[f], \phi) &= \int \phi(x+h)(1 - F_\sigma(x - x_*)) + \phi(x-h)F_\sigma(x - x_*)\, df(x)df(x_*) - \int \phi\, df \\ &= \int F[\phi](x, x_*)\, df(x)df(x_*) - \int \phi\, df \\ &=: (Q^+[f], \phi) - (Q^-[f], \phi) \end{aligned} \tag{3.4}$$

for any $\phi \in C_b(\mathbb{R})$. Notice that $Q^-[f] = f$ for any f.

We prove now a few results about $Q[f]$ which will be needed later.

Proposition 3.1. *The collision operator Q satisfies the following properties:*

(1) $Q[f]$ has total mass 0.
(2) Positivity: if $f \geq 0$ then $Q^+[f] \geq 0$.
(3) Regularity: for any $f \in C([0,T], \mathcal{M}_b(\mathbb{R}))$ we have that $Q[f] \in C([0,T], \mathcal{M}_b(\mathbb{R}))$.
(4) Lipschitz continuity: Q is locally Lipschitz for the total variation norm: for any $R > 0$ we have

$$\|Q[f] - Q[g]\|_{TV} \leq (2R+1)\|f - g\|_{TV}$$

for any $f, g \in \mathcal{M}_b(\mathbb{R})$ such that $\|f\|_{TV}, \|g\|_{TV} \leq R$.

Proof. Properties (1) and (2) follows by direct computation. Let us prove the regularity. Given $f \in C([0,T], \mathcal{M}_b(\mathbb{R}))$, let us show that $Q[f] \in C([0,T], \mathcal{M}_b(\mathbb{R}))$. Since Q^- is the identity we only have to show that $Q^+[f] \in C([0,T], \mathcal{M}_b(\mathbb{R}))$. For any $\phi \in C(\mathbb{R})$, $\|\phi\|_\infty \leq 1$, and any $s, t \in [0,T]$, we have

$$\begin{aligned} |(Q^+[f_s] - Q^+[f_t], \phi)| &= |\int F[\phi](x, x_*)(df_s(x)df_s(x_*) - df_t(x)df_t(x_*))| \\ &\leq \|F[\phi]\|_\infty \|f_s \otimes f_s - f_t \otimes f_t\|_{TV} \\ &\leq 2\|\phi\|_\infty \|f_s \otimes f_s - f_t \otimes f_t\|_{TV} \\ &\leq 2\|f_s \otimes f_s - f_t \otimes f_t\|_{TV}. \end{aligned}$$

To conclude, notice that the right hand side goes to 0 as $s \to t$ since for any $\phi, \psi \in C(\mathbb{R})$ with $\|\phi\|_\infty \leq 1$ and $\|\psi\|_\infty \leq 1$,

$$\begin{aligned} |(f_s \otimes f_s - f_t \otimes f_t, \phi \otimes \psi)| &= |(f_s, \phi)(f_s, \psi) - (f_t, \phi)(f_t, \psi)| \\ &= |(f_s, \phi)(f_s - f_t, \psi) + (f_s - f_t, \phi)f_t, \psi)| \\ &\leq \max_{0 \leq \tau \leq T} \|f_\tau\|_{TV} \|f_s - f_t\|_{TV}. \end{aligned}$$

Thus

$$\|f_s \otimes f_s - f_t \otimes f_t\|_{TV} \leq \max_{0 \leq \tau \leq T} \|f_\tau\|_{TV} \|f_s - f_t\|_{TV}.$$

Let us check now the Lipschitz continuity. Take $f, g \in C_b(\mathbb{R})$ with $\|f\|_{TV}, \|g\|_{TV} \leq R$. Then for any $\phi \in C(\mathbb{R})$, $\|\phi\|_\infty \leq 1$,

$$|(Q[f] - Q[g], \phi)| \leq |(Q^+[f] - Q^+[g], \phi)| + |(Q^-[f] - Q^-[g], \phi)| =: A + B$$

where

$$B = |(f - g, \phi)| \leq \|f - g\|_\infty$$

and

$$\begin{aligned} A = | \int F[\phi] \, d(f \otimes f - g \otimes g)(x, x_*)| &\leq 2\|\phi\|_\infty \|f \otimes f - g \otimes g\|_{TV} \\ &\leq 2 \max\{\|f\|_{TV}, \|g\|_{TV}\} \|f - f\|_{TV}. \leq 2R\|f - f\|_{TV}. \end{aligned}$$

Thus

$$\|(Q[f] - Q[g]\|_{TV} \leq (2R + 1)\|f - g\|_{TV},$$

and the proof is finished. □

Given $f \in C([0, T], \mathcal{M}_b(\mathbb{R}))$, let us define $\tilde{Q}[f] : [0, T] \to \mathcal{M}_b(\mathbb{R})$ by $\tilde{Q}[f]_t := Q[f_t]$. For ease of notation we still denote $\tilde{Q}$ by Q. According to the previous proposition, we have:

Corollary 3.1. *For any $f, g \in C([0, T], \mathcal{M}_b(\mathbb{R}))$,*

(1) $Q[f] \in C([0, T], \mathcal{M}_b(\mathbb{R}))$.
(2) If $\|f_t\|_{TV} \leq R$ and $\|g_\tau\|_{TV} \leq R$ for any $t \in [0, T]$,

$$\|(Q[f]_t - Q[g]_s\|_{TV} \leq (2R + 1)\|f_t - g_s\|_{TV}.$$

In particular if $f, g \in C([0, T], \mathbb{P}(\mathbb{R}))$ then

$$\|(Q[f]_t - Q[g]_s\|_{TV} \leq 3\|f_t - g_s\|_{TV}.$$

4. The Boltzmann Equation

Let us study now the existence and uniqueness of solutions to the Boltzmann equation Eq. (3.2).

4.1. *Existence for the Boltzmann equation*

We recall the existence and uniqueness result proved in Theorem 6.1 of Ref. 2 which generalizes Bressan's techniques.[13] The authors in Ref. 2 consider the equation

$$\partial_t f = Q[f] \qquad \text{in } [0, T) \times E \tag{4.1}$$

$$f(0) = f_0 \in S \tag{4.2}$$

where E is a Banach space, S is a closed bounded convex subset of E, and

$$Q : C([0,T], S) \to C([0,T], E)$$

is a causal operator in the sense that $Q[f](t) = Q[f1_{[0,t]}](t)$ for any $f \in C([0,T], E)$.

They assume the following conditions on Q:

- Hölder continuity: for any $f, g \in C([0,T], S)$ and any times $0 \le s \le t \le T$, there exists $\beta \in (0,1)$ such that
$$\|Q[f](t) - Q[g](s)\| \le C\Big(\max_{0\le\tau\le s} \|f(\tau) - g(\tau)\|^\beta + \|f(t) - g(s)\|^\beta + |t-s|^\beta\Big)$$
- Sub-tangent condition: for any $f \in C([0,T], S)$,
$$\liminf_{h\to 0^+} \frac{1}{h} \sup_{0\le t\le T} \{dist(f(t) + hQ[f](t), S)\} = 0$$
- One-sided Lipschitz condition: for any $f, g \in C([0,T], S)$ and any $t \in [0,T]$,
$$\int_0^t \Big[f(s) - g(s), Q[f](s) - Q[g](s)\Big]\, ds \le L \int_0^t \|f(s) - g(s)\|\, ds$$
where $[\Phi, \phi] := \lim_{h\to 0^-} \frac{1}{h}[\|\Phi + h\Phi\| - \|\Phi\|]$.

Under these assumptions it is proved in Ref. 2 that Eq. (4.1) has a unique solution in $C([0,T), S) \cap C^1((0,T), E)$.

Using this result we can prove now Theorem 1.1.

Proof. [Proof of Theorem 1.1]

We apply Theorem 6.1 in Ref. 2 with $E = (\mathcal{M}_b(\mathbb{R}), \|.\|_{TV})$, $S = \mathbb{P}(\mathbb{R})$, and Q the collision operator defined by (3.4). Hölder continuity with $\beta = 1$ is given by Corollary 3.1, which also implies the one-sided Lipschitz condition. Indeed, for any $f, g \in \mathcal{M}_b(\mathbb{R})$, the function $h \to \|f + hg\|_{TV}$ is globally Lipschitz and convex, so, as in Ref. 13,

$$\begin{aligned}\lim_{h\to 0^-} \frac{1}{h}(\|f + hg\|_{TV} - \|f\|_{TV}) := [f; g]^- &\le [f; g]^+ := \lim_{h\to 0^+} \frac{1}{h}(\|f + hg\|_{TV} - \|f\|_{TV})\\ &\le \|g\|_{TV}.\end{aligned}$$

Thus for any $f, g \in C([0,T], P(\mathbb{R}))$ and any $s \in [0,T]$,

$$[f_s - g_s; Q[f]_s - Q[g]_s]^- \le \|Q[f]_s - Q[g]_s\|_{TV} \le 3\|f_s - g_s\|_{TV}.$$

It follows that

$$\int_0^t [f_s - g_s; Q[f]_s - Q[g]_s]^-\, ds \le 3 \int_0^t \|f_s - g_s\|_{TV}.$$

Let us observe that the sub-tangent condition is trivial here since for any $f \in \mathbb{P}(\mathbb{R})$ and any $h \in (0,1)$, the measure $f + hQ[f]$ is a probability measure. Indeed it has total mass 1 because

$$(f + hQ[f], 1) = (f, 1) + h(Q[f], 1) = 1 + 0 = 1$$

and is non-negative because since $Q^+[f] \geq 0$ and $Q^-[f] = f$ for any f, we have

$$f + hQ[f] = f + hQ^+[f] - hQ^-[f] \geq (1-h)f \geq 0.$$

We thus obtain the existence and uniqueness of a solution in $C([0,T), P(\mathbb{R})) \cap C^1((0,T), \mathcal{M}_b(\mathbb{R}))$ for some $T > 0$.

Remark 4.1. In order to make the paper self-contained, let us give a short idea of the main steps in the proof in Ref. 2 adapted to our case.

Starting from the initial condition $f_0 \in \mathcal{M}_b(\mathbb{R})$ and given some small $\varepsilon > 0$, we define an approximate solution $f^\varepsilon \in C([0,+\infty), \mathcal{M}_b(\mathbb{R}))$ by the recurrence formula

$$f_s^\varepsilon = \begin{cases} f_0 + sQ[f_0], & s \in [0,\varepsilon], \\ f_{k\varepsilon}^\varepsilon + (s-k\varepsilon)Q[f_{k\varepsilon}^\varepsilon] & s \in [k\varepsilon, (k+1)\varepsilon],\ k = 1,2,... \end{cases}$$

Notice that since $f_0 \in P(\mathbb{R})$, we have $f_s^\varepsilon \in P(\mathbb{R})$ for any $\varepsilon > 0$ and any $s \geq 0$. Then using the various properties of Q we have for any $k = 0,1,...$ and $s \in (k\varepsilon, (k+1)\varepsilon)$ that

$$\|Q[f_{k\varepsilon}^\varepsilon]\| = \|Q[f_{k\varepsilon}^\varepsilon] - Q[0]\| \leq 3\|f_{k\varepsilon}^\varepsilon\| = 3$$

and

$$\begin{aligned} \|\partial_s f_s^\varepsilon - Q[f_s^\varepsilon]\| &= \|Q[f_{k\varepsilon}^\varepsilon] - Q[f_s^\varepsilon]\| \\ &\leq 3\|f_{k\varepsilon}^\varepsilon - f_s^\varepsilon]\| \\ &\leq 3\varepsilon\|Q[f_{k\varepsilon}^\varepsilon]\| \\ &\leq 9\varepsilon\|f_{k\varepsilon}^\varepsilon\| \\ &= 9\varepsilon. \end{aligned} \tag{4.3}$$

Then

$$\begin{aligned} \frac{d}{ds}\|f_s^\varepsilon - f_s^{\varepsilon'}\| &= \lim_{\delta\to 0} \frac{1}{\delta}\Big(\|f_{s+\delta}^\varepsilon - f_{s+\delta}^{\varepsilon'}\| - \|f_s^\varepsilon - f_s^{\varepsilon'}\|\Big) \\ &\leq \lim_{\delta\to 0} \frac{1}{|\delta|}\Big|\|f_s^\varepsilon - f_s^{\varepsilon'} + \delta(\partial_s f_s^\varepsilon - \partial_s f_s^{\varepsilon'}) + o(\delta)\| - \|f_s^\varepsilon - f_s^{\varepsilon'}\|\Big|. \end{aligned}$$

Using $|\|a+b\| - \|a\|| \leq \|b\|$ and the bound obtained in (4.3) we obtain

$$\begin{aligned} \frac{d}{ds}\|f_s^\varepsilon - f_s^{\varepsilon'}\| &\leq \|\partial_s f_s^\varepsilon - \partial_s f_s^{\varepsilon'}\| \\ &\leq \|Q[f_s^\varepsilon] - Q[f_s^{\varepsilon'}]\| + 9(\varepsilon+\varepsilon') \\ &\leq 3\|f_s^\varepsilon] - f_s^{\varepsilon'}\| + 9(\varepsilon+\varepsilon'). \end{aligned}$$

By integrating we obtain

$$\|f_t^\varepsilon - f_t^{\varepsilon'}\| \leq 3\int_0^t \|f_s^\varepsilon - f_s^{\varepsilon'}\|\, ds + 9(\varepsilon+\varepsilon')t$$

and Gronwall's inequality implies that for $t \in [0,T]$ we get

$$\|f_t^\varepsilon - f_t^{\varepsilon'}\| \leq 9(\varepsilon+\varepsilon')Te^{3T}.$$

It follows that for any $T > 0$, we have a Cauchy sequence $\{f^\varepsilon\}_\varepsilon$ in the complete space $C([0,T], \mathcal{M}_b(\mathbb{R}))$ and thus converges to some $f \in C([0,+\infty), \mathcal{M}_b(\mathbb{R}))$. Moreover, using (4.3) we have for any j that

$$\left\| \varepsilon Q[f^\varepsilon_{j\varepsilon}] - \int_{j\varepsilon}^{(j+1)\varepsilon} Q[f^\varepsilon_s]\, ds \right\| \leq \int_{j\varepsilon}^{(j+1)\varepsilon} \|Q[f^\varepsilon_{j\varepsilon}] - Q[f^\varepsilon_s]\|\, ds \leq 9\varepsilon^2.$$

Thus given some $t \geq 0$ with $k\varepsilon \leq t < (k+1)\varepsilon$,

$$\begin{aligned} f^\varepsilon_t &= (t-k\varepsilon)Q[f^\varepsilon_{k\varepsilon}] + f^\varepsilon_{k\varepsilon} \\ &= (t-k\varepsilon)Q[f^\varepsilon_{k\varepsilon}] + \varepsilon Q[f^\varepsilon_{(k-1)\varepsilon}] + f^\varepsilon_{(k-2)\varepsilon} \\ &= \ldots \\ &= (t-k\varepsilon)Q[f^\varepsilon_{k\varepsilon}] + \varepsilon \sum_{j=0}^{k} Q[f^\varepsilon_{j\varepsilon}] + f_0 \\ &= \int_0^t Q[f^\varepsilon_s]\, ds + f_0 + O(\varepsilon^2). \end{aligned}$$

Letting $\varepsilon \to 0$ using the continuity of Q and the uniform convergence of f^ε to f on compacts, we obtain

$$f_t = f_0 + \int_0^t Q[f_s]\, ds$$

so that $f \in C^1((0,+\infty), \mathcal{M}_b(\mathbb{R}))$ with $\partial_t f_t = Q[f_t]$. Eventually, since Q is Lipschitz, denoting by $\tilde{f}_t$ the solution for another initial condition $\tilde{f}_0$, we have

$$\frac{d}{dt}\|f_t - \tilde{f}_t\| \leq \|Q[f_t] - Q[\tilde{f}_t]\| \leq 3\|f_t - \tilde{f}_t\|$$

so that

$$\|f_t - \tilde{f}_t\| \leq \|f_0 - \tilde{f}_0\| e^{3t}$$

from which we deduce the uniqueness and the continuous dependence on initial conditions. The proof is finished. □

4.2. *Properties of f^h_t*

We denote by $\langle x \rangle_t$ the mean-value of x at time t, namely,

$$\langle x \rangle_t = \int x\, df^h_t(x).$$

Proposition 4.1. *The mean value $\langle x \rangle_t$ is constant in time:*

$$\langle x \rangle_t = \langle x \rangle_{|t=0}.$$

Proof. Let us take $\phi(x) = x$ in (1.1). Then

$$\frac{d}{dt}\langle x \rangle_t = \int \mathbb{E}(x' - x)\, df_t(x) df_t(x_*). \tag{4.4}$$

According to the interaction rule (3.1), we have the following expression for the expected value of the interaction:

$$\begin{aligned}\mathbb{E}(x'-x) &= hP(\mathbb{X}_* \geq x) - hP(\mathbb{X}_* \leq x) = h(1-2P(\mathbb{X}_* \leq x)) \\ &= h(1-2F_\sigma(x-x^*))\end{aligned}$$

where we used (3.3) in the last equality. Therefore

$$\begin{aligned}\frac{1}{h}\frac{d}{dt}\langle x\rangle_t &= \int (1-2F_\sigma(x-x^*))\,df_t(x)df_t(x_*) \\ &= 1-2\int F_\sigma(x-x^*)\,df_t(x)df_t(x_*).\end{aligned}$$

By using that $F_\sigma(-x) = 1-F_\sigma(x)$ (becaue of the symmetry of $\mathbb{Y}$), we see that the integral in the r.h.s is equal to $1/2$. The result is proved. □

From now on we will assume without loss of generality that f_0 has zero mean. Let us note that this correspond to relabel all agents opinions with the same translation. This implies that f_t^h has zero mean value for any t,h by Proposition 4.1. The variance of f_t^h is then given by

$$Var[f_t^h] = \langle x^2\rangle = \int x^2\,df_t^h(x).$$

Proposition 4.2. *There holds*

$$\frac{d}{dt}Var[f_t^h] = h^2 - 4h\int x(F_\sigma * f_t^h)(x)\,df_t^h(x).$$

Proof. Notice that

$$\begin{aligned}\mathbb{E}[(x')^2-x^2] &= \mathbb{E}[(x'-x)(x'+x)] \\ &= h(2x+h)P(x\leq \mathbb{X}_*) - h(2x-h)P(x\geq \mathbb{X}_*) \\ &= h(2x+h) - 4xhF_\sigma(x-x^*).\end{aligned}$$

By taking $\phi(x)=x^2$ as test-function, we obtain

$$\begin{aligned}\frac{d}{dt}Var[f_t^h] &= \frac{d}{dt}\langle x^2\rangle \\ &= 2h\langle x\rangle_{|t=0} + h^2 - 4h\int x(F_\sigma * f_t^h)(x)\,df_t^h(x) \\ &= h^2 - 4h\int x(F_\sigma * f_t^h)(x)\,df_t^h(x),\end{aligned}$$

where we have used that $\langle x\rangle_{|t=0}=0$, and the proof is finished. □

Given $f\in\mathbb{P}(\mathbb{R})$ we define

$$I_\sigma(f) := \int x(F_\sigma * f)(x)\,df(x) = \int xF_\sigma(x-y)\,df(x)df(y).$$

Then

$$\frac{d}{dt}Var[f_t^h] = h^2 - 4hI_\sigma(f_t^h).$$

We also introduce

$$J_\sigma(f) := \int x(\Psi_\sigma * f)(x)\, df(x) = \int x\Psi_\sigma(x-y)\, df(x)df(y)$$

where

$$\Psi_\sigma := 1 - 2F_\sigma.$$

If $\mathbb{X}$ is a real random variable with distribution $f \in \mathbb{P}(\mathbb{R})$ we also denote

$$I_\sigma(\mathbb{X}) := I_\sigma(f),$$

$$J_\sigma(\mathbb{X}) := J_\sigma(f).$$

Proposition 4.3. *We have:*

(1) For any $f \in P(\mathbb{R})$ such that $\mathbb{E}[f] = 0$,

$$J_\sigma(f) = -2I_\sigma(f).$$

(2) Moreover,

$$J_\sigma(f) = \int_{\{x,y\in\mathbb{R}:\, x-y>0\}} (x-y)\Psi_\sigma(x-y)\, df(x)df(y). \tag{4.5}$$

(3) If f has positive variance then, as a function of $\sigma > 0$, $J_\sigma(f)$ is strictly increasing and goes to 0 as $\sigma \to +\infty$. Then if $\mathbb{E}[f] = 0$, $I_\sigma(f)$ is strictly decreasing with σ and goes to 0 as $\sigma \to +\infty$.

(4) The following scaling relations hold: for any $t > 0$ and a real random variable $\mathbb{X}$,

$$\begin{aligned} I_\sigma(t\mathbb{X}) &= tI_{\sigma/t}(\mathbb{X}) \\ J_\sigma(t\mathbb{X}) &= tJ_{\sigma/t}(\mathbb{X}). \end{aligned} \tag{4.6}$$

Proof. Let us prove (4.5). Since $\mathbb{Y}$ is symmetric, we have $F_\sigma(0) = 0$ and then $\Psi_\sigma(0) = 0$. It follows that

$$J_\sigma(f) = \int_{\{x-y>0\}} x\Psi_\sigma(x-y)\, df(x)df(y) + \int_{\{x-y<0\}} x\Psi_\sigma(x-y)\, df(x)df(y).$$

We rewrite the second integral in the right hand side by exchanging x and y, and using that Ψ is odd, we get

$$\begin{aligned} \int_{\{x-y<0\}} x\Psi_\sigma(x-y)\, df(x)df(y) &= \int_{\{y-x<0\}} y\Psi_\sigma(y-x)\, df(x)df(y) \\ &= -\int_{\{x-y>0\}} y\Psi_\sigma(x-y)\, df(x)df(y), \end{aligned}$$

and we obtain (4.5).

Thus

$$\begin{aligned}\frac{d}{d\sigma}J_\sigma(f) &= \int_{\{x,y\in\mathbb{R}:\,x-y>0\}}(x-y)\frac{\partial}{\partial\sigma}\Psi_\sigma(x-y)\,df(x)df(y)\\ &= \frac{2}{\sigma^2}\int_{\{x,y\in\mathbb{R}:\,x-y>0\}}(x-y)^2F'(\frac{x-y}{\sigma})\,df(x)df(y)\end{aligned}$$

This integral is positive, since F' is the (positive) density of $\mathbb{Y}$, except when $f\otimes f$ is concentrated on the diagonal $\{(x,y)\in\mathbb{R}^2 : x=y\}$ which happens only if f is a Dirac mass, that is, if f has zero variance (in that case $J_\sigma(f)=0$ for any σ). Thus if $Var[f]>0$ then $\frac{d}{d\sigma}J_\sigma(f)>0$ and otherwise $J_\sigma(f)=0$ for any σ.

Eventually, since $\lim_{\sigma\to+\infty}\Psi_\sigma(x)=0$, we have by Lebesgue's Dominated Convergence Theorem (if f has a finite first moment) that $\lim_{\sigma\to+\infty}J_\sigma(f)=0$.

The rescaling in point (4) follows by using that if $\mathbb{X}$ has a distribution f, then $t\mathbb{X}$ has distribution $\frac{1}{t}f(x/t)$. Then

$$\begin{aligned}I_\sigma(t\mathbb{X}) &= \int xF_\sigma(x-y)f(x/t)f(y/t)\frac{dxdy}{t^2} = t\int xF_\sigma(t(x-y))f(x)f(y)dxdy\\ &= t\int xF_{\sigma/t}(x-y)f(x)f(y)dxdy = tI_{\sigma/t}(\mathbb{X})\end{aligned}$$

and the proof is finished. □

Remark 4.2. Intuitively, this suggests the following dynamic of $Var[f_t^h]$. Initially, if $Var[f_0]\gg h$ then $Varf[f_t^h]$ decreases strictly with time at a decreasing rate until $Var[f_t^h]\simeq h$, and after this point it will oscillate around h.

When $h\to 0$ we thus expect that $Var[f_t]\to 0$ as $t\to+\infty$ and thus $f_t\to\delta_0$, since $E[f_0]=0$.

Both phenomena are observed in the simulations of agents dynamics, see the last section.

To justify the previous remark, we will perform the grazing limit in the next section and we obtain an ordinary differential equation for $Var[f_t]$ which describes its behavior.

5. The Grazing Limit and the Long Time Behavior of f_t

Let us study the asymptotic behaviour of the solutions to the Boltzmann equation (1.1) when $h\to 0$.

By using a Taylor expansion and by denoting ϕ', ϕ'' the derivatives of the test function, we have, formally,

$$\begin{aligned}\frac{d}{dt}\int \phi\, df_t &\simeq \int \mathbb{E}(x'-x)\phi'(x)\, df_t(x)df_t(x_*) + \frac{1}{2}\int \mathbb{E}[(x'-x)^2]\phi''(x)\, df_t(x)df_t(x_*)\\ &= h\int (1-2\Phi_\sigma(x-x^*))\phi'(x)\, df_t(x)df_t(x_*) + \frac{h^2}{2}\int \phi''(x)\, df_t(x)\\ &= h\int (\Psi_\sigma * f_t)(x)\phi'(x)\, df_t(x) + \frac{h^2}{2}\int \phi''(x)\, df_t(x)\end{aligned}$$

where $\Psi = 1 - 2\Phi_\sigma$. We rescale time considering $\tau := ht$ and $g^h_\tau := f^h_t$. Thus, we expect the long-time evolution of f^h_t to be well-approximated by g_τ, the solution to Eq. (1.2), namely

$$\frac{d}{d\tau}\int \phi\, dg_\tau = \int (\Psi_\sigma * g_\tau)(x)\phi'(x)\, dg_\tau(x).$$

Let us recall that this is the weak formulation of the first order mean field equation (1.3)

$$\partial_\tau g_\tau + \partial_x\Big((\Psi_\sigma * g_\tau)g_\tau\Big) = 0.$$

We are ready to prove Theorem 1.2.

5.1. *Proof of Theorem 1.2*

Before the proof of this result we first recall a useful result of Gabetta, Toscani and Wennberg.[24]

We denote by $\rho(\mu,\nu)$ the Prokhorov distance between $\mu,\nu \in \mathbb{P}(\mathbb{R})$. Following Ref. 24[Eq. (5.6)], we also consider

$$\|\mu\|_* := \sup\Big\{\int_{\mathbb{R}} \phi\, d\mu :\ \|\phi\|_2 := \sup_{x\in\mathbb{R}} |\phi'(x)| + |\phi''(x)| \le 1\Big\}$$

and

$$d(\mu,\nu) := \|\mu-\nu\|_*. \tag{5.1}$$

Notice that since μ and ν have same mass, $d(\mu,\nu)$ does not change if we add a constant to the test-function ϕ. In particular we can assume that $\phi(0) = 0$ i.e. we consider

$$\|\mu\|_* := \sup\Big\{\int_{\mathbb{R}} \phi\, d\mu :\ \|\phi\|_2 := \sup_{x\in\mathbb{R}} |\phi'(x)| + |\phi''(x)| \le 1,\ \phi(0)=0\Big\}. \tag{5.2}$$

According to Lemma 5.3 and Corollary 5.5 in Ref. 24, d defines a distance that metrizes the weak-* convergence (that is, the convergence against functions in $C_b(\mathbb{R})$). Moreover, Corollary 5.5 in Ref. 24 reads as

$$\rho(\mu,\nu) \le \max\{C(d(\mu,\nu))^{1/3}; d(\mu,\nu)\} \qquad \text{for any } \mu,\nu \in \mathbb{P}(\mathbb{R}).$$

Since $\mathbb{P}(\mathbb{R})$ is complete for ρ we deduce that it is also complete for d.

Proof. [Proof of Theorem 1.2.]

The rescaled measure $g^h_\tau := f^h_t$, where $\tau = ht$, solves the Boltzmann equation

$$\frac{d}{d\tau}\int \phi\, g^h_\tau = \frac{1}{h}\int \mathbb{E}[\phi(x') - \phi(x)]\, dg^h_\tau(x)dg^h_\tau(x_*) \tag{5.3}$$

for any test function ϕ. By performing a first order Taylor expansion we obtain

$$\begin{aligned}\frac{d}{d\tau}\int \phi\, g^h_\tau &= \frac{1}{h}\int \Big\{\mathbb{E}[x'-x]\phi'(x) + \frac{1}{2}\mathbb{E}[(x'-x)^2]\phi''(\tilde{x})\Big\}\, dg^h_\tau(x)dg^h_\tau(x_*) \\ &= \frac{1}{h}\int \Big\{h\Psi_\sigma(x-x^*)\phi'(x) + \frac{h^2}{2}\phi''(\tilde{x})\Big\}\, dg^h_\tau(x)dg^h_\tau(x_*)\end{aligned}$$

where $\tilde{x} = x + \theta h$ for some $\theta \in (-1,1)$ is an intermediate point between x and x'. Then

$$\frac{d}{d\tau}\int \phi\, g^h_\tau = \int (\Psi_\sigma * g^h_\tau)(x)\phi'(x)\, dg^h_\tau(x) + R^h_\tau$$

where

$$R^h_\tau := \frac{h}{2}\int \phi''(\tilde{x})\, dg^h_\tau(x).$$

Integrating the above expression between τ' and τ, we obtain

$$\int \phi\, d(g^h_\tau - g^h_{\tau'}) = \int_{\tau'}^{\tau}\int (\Psi_\sigma * g^h_s)(x)\phi'(x)\, dg^h_s(x)ds + \int_{\tau'}^{\tau} R^h_s\, ds. \tag{5.4}$$

Since $\|\Psi_\sigma\|_\infty \le 1$, also $\|\Psi_\sigma * g^h_s\|_\infty \le 1$, and $|R^h_\tau| \le \frac{h}{2}\|\phi''\|_\infty$. Therefore, we get

$$\Big|\int \phi\, d(g^h_\tau - g^h_{\tau'})\Big| \le (\|\phi'\|_\infty + \frac{h}{2}\|\phi''\|_\infty)|\tau - \tau'|. \tag{5.5}$$

Taking the supremum over ϕ we obtain

$$d(g^h_\tau, g^h_{\tau'}) \le |\tau - \tau'| \qquad \text{for any } \tau, \tau' \ge 0 \text{ and any } h < 1,$$

where the distance d is defined in (5.1). This means that the sequence of continuous probability measure valued functions $g^h : [0,+\infty) \to \mathbb{P}(K)$ is uniformly equi-continuous. Moreover for any fixed τ, the sequence $(g^h_\tau)_h$ is bounded since for any admissible ϕ in the definition of $\|.\|_*$,

$$\Big|\int \phi\, dg^h_\tau\Big| \le \int |x|\, dg^h_\tau \le \tau + \int |x|\, df_0.$$

To prove the second inequality we use $|x|$ as a test function in (5.3), and we get

$$\begin{aligned}\frac{d}{d\tau}\int |x|\, dg^h_\tau &= \frac{1}{h}\int \mathbb{E}[|x'| - |x|]\, dg^h_\tau(x)dg^h_\tau(x_*) \\ &= \frac{1}{h}\int (|x+h| - |x|)(1 - \Phi_\sigma(x - x_*))\, dg^h_\tau(x)dg^h_\tau(x_*) \\ &\quad + \frac{1}{h}\int (|x-h| - |x|)\Phi_\sigma(x - x_*)\, dg^h_\tau(x)dg^h_\tau(x_*) \\ &\le \frac{1}{h}\int ((|x| + h) - |x|)\, dg^h_\tau(x)dg^h_\tau(x_*) \\ &= 1.\end{aligned}$$

Thus, Arzela-Ascoli theorem, together with a diagonal argument, ensure the existence of $g \in C([0,+\infty);\mathbb{P}(\mathbb{R}))$ and a subsequence $(h_n)_n$ converging to 0 such that $g^{h_n} \to g$ in $C([0,T];\mathbb{P}(\mathbb{R}))$ for any $T > 0$, which implies that

$$\lim_{h\to 0}\left(\max_{0\le s\le T}\max_{Lip(\phi)\le 1}|(g_s^h - g_s,\phi)|\right) = 0. \tag{5.6}$$

It remains to pass to the limit in (5.4) with $\tau' = 0$ and a given ϕ, we get

$$\begin{aligned}\int \phi\, dg_\tau^h - \int \phi df_0 &= \int_0^\tau \int (\Psi_\sigma * g_s^h)(x)\phi'(x)\, dg_s^h(x)ds + \int_0^\tau R_s^h ds \\ &= \int_0^\tau \int (\Psi_\sigma * g_s^h)(x)\phi'(x)\, dg_s^h(x)ds + O(h).\end{aligned} \tag{5.7}$$

Now,

$$\begin{aligned}&\int_0^\tau \int (\Psi_\sigma * g_s^h)(x)\phi'(x)\, dg_s^h(x)ds - \int_0^\tau \int (\Psi_\sigma * g_s)(x)\phi'(x)\, dg_s(x)ds \\ &= \Big(\int_0^\tau \int (\Psi_\sigma * g_s^h)(x)\phi'(x)\, dg_s^h(x)ds - \int_0^\tau \int (\Psi_\sigma * g_s^h)(x)\phi'(x)\, dg_s(x)ds\Big) \\ &\quad + \Big(\int_0^\tau \int (\Psi_\sigma * g_s^h)(x)\phi'(x)\, dg_s(x)ds - \int_0^\tau \int (\Psi_\sigma * g_s)(x)\phi'(x)\, dg_s(x)ds\Big) \\ &=: A_h + B_h.\end{aligned} \tag{5.8}$$

For any $x \in \mathbb{R}$ the function $x_* \to \Psi_\sigma(x - x_*)$ has Lipschitz constant less than $Lip(\Psi_\sigma) < \infty$. It follows from (5.6) that, for any s and x, we have $(\Psi_\sigma * g_s^h)(x) \to (\Psi_\sigma * g_s)(x)$. Moreover, $|(\Psi_\sigma * g_s^h)(x)| \le \|\Psi_\sigma\| \le 1$ for any x. We deduce that $B_h \to 0$ by Lebesgue's Dominated Convergence Theorem.

We also deduce that $(\Psi_\sigma * g_s^h)\phi'$ has Lipchitz constant bounded uniformly in h. Hence, it follows from (5.6) that $\lim_{h\to 0}|(g_s^h - g_s, (\Psi_\sigma * g_s^h)\phi')| = 0$ uniformly in $s \in [0,\tau]$. We deduce that $A_h \to 0$. We can thus pass to the limit $h \to 0$ in (5.7) to obtain for any $\tau > 0$ and any ϕ that

$$\int \phi\, dg_\tau - \int \phi\, df_0 = \int_0^\tau \int (\Psi_\sigma * g_s)(x)\phi'(x)\, dg_s(x)ds,$$

which implies the weak formulation (1.2).

Notice eventually that it is well-known that equations like (1.3) has a unique solution so that the whole sequence $(g^h)_h$ converges to the solution g. The theorem is proved. □

As for f_t^h, it is easily seen that

$$E[g_\tau] = E[f_0] \qquad t \ge 0.$$

We can thus assume without loss of generality that f_0 has zero mean (by shifting all opinions if necessary), and thus g_τ has zero mean for any $\tau \ge 0$. It is then also easily seen from (1.2) that

$$\frac{d}{dt}Var[g_\tau] = 2\int x(\Psi_\sigma * g_\tau)(x)\,dg_\tau(x) = 2J_\sigma(g_\tau).$$

Since by Prop. 4.3, $J_\sigma(g_\tau) \le 0$ with equality if and only if $Var[g_\tau] = 0$, we obtain that $Var[g_\tau]$ decreases to 0 with time, so that

$$g_\tau \to \delta_0.$$

This proves Theorem 1.3.

5.2. *Proof of Theorem 1.4*

We can prove now an estimate on the rate of convergence of g_τ.

We keep on assuming that $\mathbb{Y}$ is a symmetric and we also assume that $\mathbb{Y}$ has a density $f_{\mathbb{Y}} \in C^1(\mathbb{R})$ such that $f'_{\mathbb{Y}}$ is bounded and $F(x) > F(0)$ for $x > 0$.

In order to prove Theorem 1.4 we need some auxiliary results and notations. Let us call

$$\alpha_{R,\sigma} = \sup\Big\{\int_0^{+\infty} x\Psi_\sigma(x)\,dg(x) : g \in \mathbb{P}(\mathbb{R}), \mathrm{supp}(g) \subset [-R,R],$$
$$g \text{ symmetric}, \mathbb{E}[g] = 0, Var[g] = 1\Big\},$$

and

$$\alpha_{\infty,\sigma} = \sup\left\{\int_0^{+\infty} x\Psi_\sigma(x)\,dg(x) : g \in \mathbb{P}(\mathbb{R}),\ g \text{ symmetric},\ \mathbb{E}[g] = 0,\ Var[g] = 1\right\}.$$

Proposition 5.1. *There hold that $\alpha_{R,\sigma} < 0$ increases as $R \uparrow +\infty$ to $\alpha_{\infty,\sigma} = 0$. Moreover*

$$\Big|\alpha_{R,\sigma} + \frac{1}{\sigma}f_{\mathbb{Y}}(0)\Big| \le \frac{R}{2\sigma^2}\|f'_{\mathbb{Y}}\|_\infty \tag{5.9}$$

if $R \ge 1$, and

$$\Big|\alpha_{R,\sigma} + \frac{1}{\sigma}f_{\mathbb{Y}}(0)\Big| \le \frac{R^3}{2\sigma^2}\|f'_{\mathbb{Y}}\|_\infty \tag{5.10}$$

if $R < 1$.

Proof. Since $\Psi_\sigma(x) \le 0$ if $x \ge 0$ we have $\alpha_{R,\sigma}$, $\alpha_{\infty,\sigma} \le 0$.

Consider the probability measures $g_n = \frac{1}{2n^2}(\delta_n + \delta_{-n}) + (1 - \frac{1}{2n^2})\delta_0$, $n \in \mathbb{N}$. Notice that $\mathbb{E}[g_n] = 0$ and $Var[g_n] = \int x^2\,dg_n(x) = \frac{1}{2n^2}(n^2 + (-n)^2) = 1$. Then

$$0 \ge \alpha_{\infty,\sigma} \ge \int_0^{+\infty} x\Psi_\sigma(x)\,dg_n(x) = n\Psi_\sigma(n)\frac{1}{2n^2} = \frac{1}{2n}\Psi_\sigma(n)$$

which goes to 0 as $n \to +\infty$.

On the other hand, given $R > 0$, take a minimizing sequence $(g_n)_n$ for $\alpha_{R,\sigma}$. Since $[-R, R]$ is compact, a subsequence of the g_n converges in $C([-R, R])'$ to some g. Then $g \in \mathbb{P}(\mathbb{R})$ is admissible since

$$\mathbb{E}[g] = \int x\, dg(x) = \int_{-R}^{R} x\, dg(x) = \lim_n \int_{-R}^{R} x\, dg_n(x) = 0$$

and in the same way $Var[g] = \lim Var[g_n] = 1$. Then $\alpha_{R,\sigma} = \int_0^{+\infty} x\Psi_\sigma(x)\, dg(x)$. Since $\Psi_\sigma(x) < 0$ if $x > 0$, we have $\alpha_{R,\sigma} = 0$ iff g is supported in $(-\infty, 0]$ i.e. $g = \delta_0$ since $E[g] = 0$, but this contradicts $Var[g] = 1$. Thus $\alpha_{R,\sigma} < 0$.

To prove (5.9) we just write for an admissible g that

$$\begin{aligned}\int_0^{+\infty} x\Psi_\sigma(x)\, dg(x) &= \int_0^{+\infty} x\Big(\Psi'_\sigma(0)x + \frac{1}{2}\Psi''_\sigma(\theta(x)x^2)\Big)\, dg(x)\\ &= -\frac{2}{\sigma} f_{\mathbb{Y}}(0) \int_0^{+\infty} x^2\, dg(x) + \frac{1}{2}\int_0^{+\infty} \Psi''_\sigma(\theta(x)x)x^3\, dg(x)\end{aligned}$$

where $\theta(x) \in [0, 1]$. Notice that

$$\int_0^{+\infty} x^2\, dg(x) = \frac{1}{2} Var[g] = \frac{1}{2}$$

since g is symmetric and $\mathbb{E}[g] = 0$. Moreover, since $\Psi''_\sigma(x) = -\frac{2}{\sigma^2} f'_{\mathbb{Y}}(x/\sigma)$ with $f'_{\mathbb{Y}}$ bounded, and g is supported in $[-R, R]$,

$$\Big|\int_0^{+\infty} \Psi''_\sigma(\theta(x)x)x^3\, dg(x)\Big| \le \frac{2}{\sigma^2}\|f'_{\mathbb{Y}}\|_\infty R \int_0^{+\infty} x^2\, dg(x) = \frac{R}{\sigma^2}\|f'_{\mathbb{Y}}\|_\infty$$

if $R \ge 1$, which proves (5.9), and

$$\Big|\int_0^{+\infty} \Psi''_\sigma(\theta(x)x)x^3\, dg(x)\Big| \le \frac{2}{\sigma^2}\|f'_{\mathbb{Y}}\|_\infty R^3 \int_0^{+\infty} dg(x) = \frac{R^3}{\sigma^2}\|f'_{\mathbb{Y}}\|_\infty$$

if $R < 1$, which proves (5.10). The proof is now complete. □

Remark 5.1. Let us note that this bound can be improved for particular density functions. In the particular case when $\mathbb{Y} \sim N(0, 1)$, we obtain

$$\Big|\alpha_{R,\sigma} + \frac{1}{\sigma\sqrt{2\pi}}\Big| \le \frac{e^{-1/2}R}{2\sqrt{2\pi}\sigma^2}\mathbf{1}\{R \ge 1\} + \frac{e^{-R^2/2}R^3}{2\sqrt{2\pi}\sigma^2}\mathbf{1}\{R < 1\}. \tag{5.11}$$

since $f_{\mathbb{Y}}(0) = 1/\sqrt{2\pi}$ and $\|f'_{\mathbb{Y}}\|_\infty = |f'_{\mathbb{Y}}(\pm 1)| = \frac{e^{-1/2}}{\sqrt{2\pi}}$, and, for $R < 1$, the maximum of $|f'_{\mathbb{Y}}|$ in $[0, R]$ is reached in $x = R$.

The quantity $\alpha_{R,\sigma}$ is an useful tool to estimate $J_\sigma(f)$:

Lemma 5.1. *For any $R > 0$ and $\sigma > 0$, let us define*

$$A_{R,\sigma} := \sup\{J_\sigma(f) :\ f \in \mathbb{P}(\mathbb{R}),\ supp(f) \subset [-R, R],\ \mathbb{E}[f] = 0,\ Var[f] = 1\}.$$

Then,

$$A_{R,\sigma} \le \sqrt{2}\alpha_{\sqrt{2}R,\sigma/\sqrt{2}} = -\frac{2f_{\mathbb{Y}}(0)}{\sigma} + C(R, \sigma), \qquad |C(R, \sigma)| \le 2\|f'_{\mathbb{Y}}\|_\infty \frac{R}{\sigma^2}.$$

Remark 5.2. When $\mathbb{Y} \sim N(0, \sigma^2)$ we obtain

$$A_{R,\sigma} \leq -\frac{\sqrt{2}}{\sigma\sqrt{\pi}} + C(R, \sigma),$$

where

$$|C(R,\sigma)| \leq \frac{\sqrt{2}}{\sqrt{\pi}\sigma^2}\Big(Re^{-1/2}\mathbf{1}\{R \geq 1\} + e^{-R^2/2}R^3\mathbf{1}\{R < 1\}\Big).$$

Proof. Let $f \in \mathbb{P}(\mathbb{R})$ be admissible in the definition of $A_{R,\sigma}$. We can assume by density that $f \in L^1(\mathbb{R})$. Then, by (4.5),

$$\begin{aligned} J_\sigma(f) &= \int_{\{x,y\in\mathbb{R}:\, x-y>0\}} (x-y)\Psi_\sigma(x-y)f(x)f(y)\,dxdy \\ &= \int_0^{+\infty} u\Psi_\sigma(u)\int f(v)f(v-u)\,dv\,du \end{aligned}$$

where we have changed variables $u = x - y$, $v = x$.

Let $\check{f}(x) := f(-x)$ and $\tilde{g} := f * \check{f}$. Then,

$$J_\sigma(f) = \int_0^{+\infty} u\Psi_\sigma(u)\tilde{g}(u)\,du. \tag{5.12}$$

Notice that $\tilde{g} \in \mathbb{P}(\mathbb{R})$ is symmetric and supported in $[-2R, 2R]$ with $\mathbb{E}[\tilde{g}] = 0$ and $Var[\tilde{g}] = 2Var[f] = 2$. In fact $\tilde{g}$ is the distribution of the random variable $X + Y$ where X and Y are independent and have distribution f and $\check{f}$ respectively. Thus,

$$\begin{aligned} A \leq \sup\Big\{\int_0^{+\infty} x\Psi_\sigma(x)\,d\tilde{g}(x) :\ & \tilde{g} \in \mathbb{P}(\mathbb{R}), \\ & \mathrm{supp}(\tilde{g}) \subset [-2R, 2R],\ \tilde{g} \text{ symmetric},\ \mathbb{E}[\tilde{g}] = 0,\ Var[\tilde{g}] = 2\Big\}. \end{aligned}$$

Next, let us consider $g(t) = \sqrt{2}\tilde{g}(\sqrt{2}t)$ i.e. g is the distribution of the random variable $\frac{1}{\sqrt{2}}X$ if X has distribution $\tilde{g}$. Then $g \in \mathbb{P}(\mathbb{R})$ is symmetric, supported in $[-R\sqrt{2}, R\sqrt{2}]$, $\mathbb{E}[g] = 0$ and $Var[g] = 1$. Since

$$\begin{aligned} \int_0^{+\infty} x\Psi_\sigma(x)\,d\tilde{g}(x) &= \sqrt{2}\int_0^{+\infty} x\Psi_\sigma(\sqrt{2}x)\,dg(x) \\ &= \sqrt{2}\int_0^{+\infty} x\Psi_{\sigma/\sqrt{2}}(x)\,dg(x) \end{aligned}$$

we obtain the result and the Lemma is proved. □

We can now prove Theorem 1.4:

Proof. [Proof of Theorem 1.4.] For ease of notation we write the time variable t instead of τ. Denote by X_t the generalized inverse of the cumulative distribution function of g_t. Then

$$\partial_t X_t(r) = (\Psi * g_t)(X_t(r)) \qquad t > 0,\, r \in [0, 1].$$

Then g_t is supported in $[X_t(0^+), X_t(1^-)]$. Notice that for any t, the set

$$\{y \in [X_t(0^+), X_t(1^-)] : \; y > X_t(0^+)\}$$

has positive measure for g_t unless $X_t(0^+) = X_t(1^-)$, i. e., unless g_t is a Dirac mass.

Thus,

$$X_t(0^+) = X_0(0^+) + \int_0^t \int_{[X_t(0^+), X_t(1^-)]} \Psi_\sigma(X_s(0^+) - y)\, dg_s(y)\, ds$$

is strictly increasing while $Var[g_t] > 0$. In the same way $X_t(1^-)$ strictly decreases. We can thus find $R_t \searrow 0$ such that g_t is supported in $[-R_t, R_t]$.

By using the scaling relation (4.6) and denoting by $\mathbb{Y}_t$ a random variable with distribution g_t, we have

$$\begin{aligned}\frac{d}{dt}Var[g_t] &= 2J_\sigma(\mathbb{Y}_t)\\ &= 2\sqrt{Var[g_t]}\,J_{\sigma/\sqrt{Var[g_t]}}\Big(\frac{\mathbb{Y}_t}{\sqrt{Var[g_t]}}\Big).\end{aligned}$$

Let us note that the variable $\dfrac{\mathbb{Y}_t}{\sqrt{Var[g_t]}}$ has variance equal to 1 and it is supported in the interval $[-R_t/\sqrt{Var[g_t]}, R_t/\sqrt{Var[g_t]}]$. Thus, Lemma 5.1 gives

$$\begin{aligned}\frac{d}{dt}Var[g_t] &\le 2\sqrt{Var[g_t]}\Big(-\frac{2f_{\mathbb{Y}}(0)}{\sigma}\sqrt{Var[g_t]} + 2\|f'_{\mathbb{Y}}\|_\infty\sqrt{Var[g_t]}\frac{R_t}{\sigma^2}\Big)\\ &= \frac{4Var[g_t]}{\sigma}\Big(-f_{\mathbb{Y}}(0) + \frac{R_t}{\sigma}\|f'_{\mathbb{Y}}\|_\infty\Big)\end{aligned}$$

Using that $R_t \le R_0$ we obtain

$$Var[g_t] \le Var[g_0]\exp\Big\{\frac{4t}{\sigma}\Big(-f_{\mathbb{Y}}(0) + \frac{R_0}{\sigma}\|f'_{\mathbb{Y}}\|_\infty\Big)\Big\}$$

We can obtain a better estimate in the long-run. Fix some $\varepsilon > 0$. Since $R_t \searrow 0$ we can find a time $t_\varepsilon > 0$ such that $\frac{4R_t}{\sigma^2}\|f'_{\mathbb{Y}}\|_\infty \le \varepsilon$ for $t \ge t_\varepsilon$. Then for $t \ge t_\varepsilon$,

$$\frac{d}{dt}Var[g_t] \le \Big(-\frac{4f_{\mathbb{Y}}(0)}{\sigma} + \varepsilon\Big)Var[g_t]$$

which gives

$$\frac{d}{dt}Var[g_t] \le C_\varepsilon \exp\Big\{\Big(-\frac{4f_{\mathbb{Y}}(0)}{\sigma} + \varepsilon\Big)t\Big\} \qquad t \ge t_\varepsilon.$$

The proof is finished. □

If $f_{\mathbb{Y}} \in C^1$ is non-increasing in $(0, +\infty)$, we can prove a lower bound for $Var[g_t]$ as mentioned in Remark 1.1. Let us observe that $\Psi'' = -2f'$, and then

$$\int_0^{+\infty} \Psi''_\sigma(\theta(u)u)u^3\, d\tilde{g}_t(u) \ge 0.$$

We call $\check{g}_t(x) := g_t(-x)$ and $\tilde{g}_t := g_t * \check{g}_t$ as before, and recall that $\tilde{g}_t \in \mathbb{P}(\mathbb{R})$ is symmetric with $\mathbb{E}[\tilde{g}_t] = 0$ and $Var[\tilde{g}_t] = 2Var[g_t]$.

Hence, putting together all of this and recalling (5.12) we get

$$\begin{aligned}
\frac{d}{dt}Var[g_t] =& 2J_\sigma(\mathbb{Y}_t) \\
=& 2\int_0^{+\infty} u\Psi_\sigma(u)\, d\tilde{g}_t(u) \\
=& 2\int_0^{+\infty} u\Big(\Psi'_\sigma(0)u + \frac{1}{2}\Psi''_\sigma(\theta(u)u)u^2\Big)\, d\tilde{g}_t(u) \\
=& -\frac{4}{\sigma}f_{\mathbb{Y}}(0)\int_0^{+\infty} u^2\, d\tilde{g}_t(u) + \int_0^{+\infty} \Psi''_\sigma(\theta(u)u)u^3\, d\tilde{g}_t(u) \\
\geq& -\frac{4}{\sigma}f_{\mathbb{Y}}(0)\int_0^{+\infty} u^2\, d\tilde{g}_t(u) \\
=& -\frac{4}{\sigma}f_{\mathbb{Y}}(0)Var[g_t].
\end{aligned}$$

where we used in the last equality that $\int_0^{+\infty} u^2\, d\tilde{g}_t(u) = \frac{1}{2}Var[\tilde{g}_t] = Var[g_t]$.

Thus, the asymptotic upper bound obtained in Theorem 1.4 is optimal if $f_{\mathbb{Y}} \in C^1$ is non-increasing in $(0, +\infty)$. In the examples in the next section we will see that the simulations when $\mathbb{Y} \sim N(0,1)$ are close to this lower bound.

6. Examples and Simulations

In this section we present some numerical agent-based simulations to illustrate the convergence results stated in Theorem 1.3 and in Theorem 1.4. We will consider a finite populations of N agents interacting following the interaction rules presented before. Notice that we have obtained the limit nonlocal equation (1.3) from the Boltzmann equation (1.1) which assumed implicitly an infinite population. Since we will consider a large but finite population of N agents in the simulations, we first need to find the correct time scaling relating N and h. We will do so by obtaining the nonlocal equation (1.3) with an informal argument based on the Master Equation where the time scaling will be apparent (see (6.2) below). We will then proceed with the numerical experiments.

6.1. *The master equation and the time scaling*

We consider a large population of N agents, $N \gg 1$. The opinion of an agent is a random variable of the form $x + \sigma\mathbb{Y}$ where $x \in \mathbb{R}$ and the random variable $\mathbb{Y}$ is symmetric and has variance 1. The mean x of the opinion varies from an agent to another whereas σ is the same for all agent. We denote by x_i the mean opinion of the i-th. agent.

We want to write the Master Equation giving the evolution of the distribution of agents and deduce from it the equation obtained in Theorem 1.2. The derivation we present is informal since we are mainly interested in obtaining the right scaling relation (see (6.2) below) between the number of agents N and the step h to compare the simulations for different values of h. A rigorous study of the system from a probabilistic point of view shoud be possible following Kipnis and Landim,[33] but it is out of the scope of the present paper.

So, we fix some small parameter h, and we assume that the possible mean opinions x are of the form hj with $j \in \mathbb{Z}$. We get a partition of $\mathbb{R}$ into small intervals $I_j = [(j-1/2)h, (j+1/2)h]$ of lengths h, centered at jh. We denote

$$s(j,t) = \frac{\#\{i | x_i \in I_j\}}{N}$$

the proportion of agents with mean opinion jh.

We assume that interactions follow a Poisson process of rate 1, and that only the first agent in the interaction updates his mean x_i. Then, the Master Equation reads

$$s(j, t+dt) = s(j,t) + \frac{1}{N}\Big(G(j,t) - L(j,t)\Big)dt, \tag{6.1}$$

where G and L are the gain and loss function given by

$$\begin{aligned}
G(j,t) &= s(j-1,t)\sum_{k\in\mathbb{Z}} s(k,t)P(hk + \sigma\mathbb{Y} \geq (j-1/2)h) \\
&\quad + s(j+1,t)\sum_{k\in\mathbb{Z}} s(k,t)P(hk + \sigma\mathbb{Y} \leq (j+1/2)h) \\
&= s(j-1,t)\sum_{k\in\mathbb{Z}} s(k,t)(1 - F_\sigma((j-k-1/2)h)) \\
&\quad + s(j+1,t)\sum_{k\in\mathbb{Z}} s(k,t)F_\sigma((j-k+1/2)h)
\end{aligned}$$

and

$$\begin{aligned}
L(j,t) &= s(j,t)\sum_{k\in\mathbb{Z}} s(k,t)\Big(P(kh + \sigma\mathbb{Y} \geq h(j+1/2)) + P(kh + \sigma\mathbb{Y} \leq h(j-1/2))\Big) \\
&= s(j,t)\sum_{k\in\mathbb{Z}} s(k,j)[1 - F_\sigma(h(j-k+1/2)) + F_\sigma(h(j-k-1/2))].
\end{aligned}$$

These expressions follow noticing that $P(hk + \sigma\mathbb{Y} \geq (j-1/2)h)$ is the probability that an agent whose mean opinion belongs to I_k emits an opinion lying on the right of I_j.

If $f_t(x)$ is the distribution of agents means, we can approximate

$$s(j,t) \approx \int_{(j-1/2)h}^{(j+1/2)h} f_t(x)\,dx \approx hf_t(jh),$$

$$s(j\pm 1, t) \approx hf_t(jh \pm h) \approx hf_t(jh) \pm h^2\partial_x f_t(jh),$$

so that

$$G(j,t)-L(j,t)\approx 2h^2 f_t(jh)\sum_{k\in\mathbb{Z}} f_t(kh)[F_\sigma(j-k+1/2)h)-F_\sigma(j-k-1/2)h)]$$
$$+h^3\partial_x f_t(jh)\sum_{k\in\mathbb{Z}} f_t(kh)[F_\sigma((j-k+1/2)h)+F_\sigma((j-k-1/2)h)-1].$$

Thus, approximating the sum by Riemann integrals, using a Taylor expansion of F, and discarding the lower order term, we get

$$\frac{1}{h^2}(G(j,t)-L(j,t))\approx 2f_t(jh)(f*F'_\sigma)(jh)+\partial_x f_t(jh)(f*(2F_\sigma-1))(jh)$$
$$\approx \partial_x\Big(f_t(f_T*(2F_\sigma-1))\Big)\Big|_{x=jh}.$$

Coming back to (6.1) we obtain

$$\frac{N}{h}\frac{f(jh,t+\Delta t)-f(jh,t)}{\Delta t}\approx \partial_x\Big(f(f*(2F_\sigma-1))\Big)\Big|_{x=jh}.$$

It follows that the correct time scale is

$$\tau=\frac{h}{N}t, \tag{6.2}$$

since then

$$\partial_\tau g_\tau+\partial_x\left(\Big(g_\tau*(1-2F_\sigma)\Big)g_\tau\right)=0.$$

6.2. *Simulations*

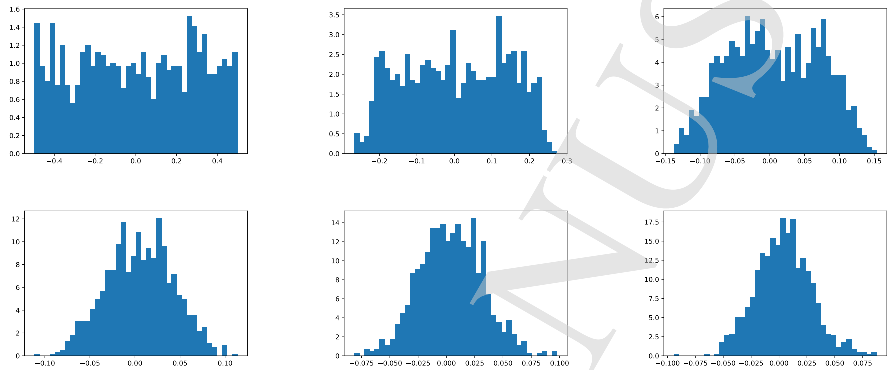
We present here some agent based simulations of the dynamics performed in Python. We consider a list of agents indexed from 1 to N. Each one has a Normal distribution with variance $\sigma^2=1$ and their means x_i are initially uniformly distributed on $[-1/2,1/2]$. In each time step we update all the agents. We take one of them following the index order, and we match this agent i with a different one, j, selected independently using a random uniform distribution among the other agents. Agent j generates an opinion m_j at random following her/his normal $N(x_j,\sigma^2)$, and agent i updates his/her mean following the interaction rule (3.1), namely

$$x'_i=x_i+sign(m_j-x_i)\times h.$$

Different values of N where considered. We present here only the simulations for $N=10^3$, the ones for $N=10^4$ and $N=10^5$ are similar. The time scale considered is $\tau=ht/N$ which is the natural time scale in view of (6.2). This means that time step $\Delta\tau$ corresponds to N/h interactions.

As observed in the figures in Table 1, with $h=10^{-3}$, the distributions of means converges to some quasi-consensus value, and fluctuate around this value due to the step h.

Table 1. Evolution of the distribution of agents means $x_1, .., x_N$ during one simulation with $N = 1000$ and $h = 10^{-3}$. From left to right and top to bottom, $\tau = 0, .., 6$. Axis scales are different in each figure.



Remark 6.1. Let us remark that different simulations for the same parameters show a distribution of quasi-consensus values which resembles a Normal distribution (see Figure 1 which displays the histogram of the final quasi-consensus value corresponding to 100 simulations starting with the same initial condition). We conjecture that some Central Limit Theorem is valid here, since the quasi-consensus value of the means is obtained in the random process defined by the random selection of opinions in each interaction, and, when $h \to 0$, the average in the Boltzmann equation implies the convergence to the expected value of this distribution proved in Theorem 1.3.

In Figures 2, 3, and Figure 4 we analyze the time to quasi-consensus for different values of h. We considered the cases $h = 10^{-2}, 10^{-3}, 10^{-4}$. These curves were obtained averaging over 100 simulations for each value of h.

We first plot in Figure 2 the time-evolution of R_τ, introduced at the beginning of the proof of Theorem 1.4, such that $R_\tau \downarrow 0$ and the distribution of agents means at time τ is supported in $[-R_\tau, R_\tau]$.

We then plot in Figure 3 and Figure 4 the time evolution of the variance (Figure 3) and of the logarithm of the variance (Figure 4) of the distribution of agents means at time τ. The dotted, dashed, and dash-dotted curves (blue, red, and yellow in the online version) correspond to $h = 10^{-2}$, $h = 10^{-3}$, and $h = 10^{-3}$. The solid curves (green and magenta in the online version) in Figures 3 represents the theoretical bounds obtained in Theorem 1.4 and Remark 1.1 for the case of Normal opinions considered here. Recall that in the simulations $\sigma = 1$ and the initial distribution of means $\{x_i\}_{i=1}^N$, is uniform in $[-1/2, 1/2]$ so that $R_0 = 1/2$ and the initial variance is $Var(Unif(-1/2, 1/2)) = 1/12$).

Fig. 1. Histogram of the quasi-consensus value corresponding to 100 simulation with uniform $Unif(-1/2, 12)$ initial distribution of x's ($N = 1000$ agents, $h = 10^{-3}$).

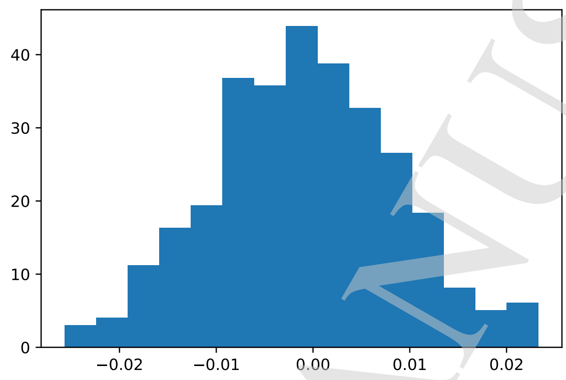
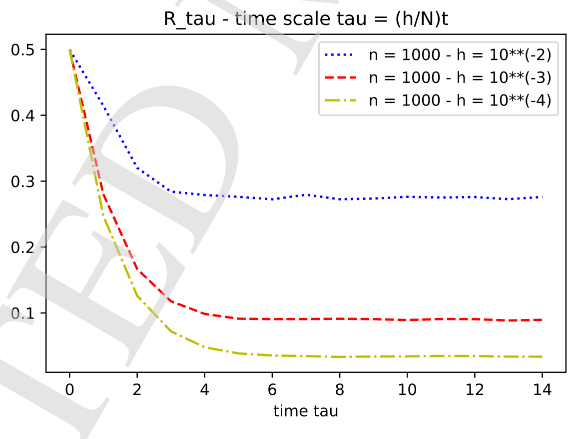


Fig. 2. Time evolution of R_τ defined such that the distribution of agents means at time τ is sopported in $[-R_\tau, R_\tau]$.



We distinguish between the *upper bound* proved in Theorem 1.4 and given by

$$
\begin{aligned}
Var[g_t] &\leq Var[g_0] \exp\Big\{\frac{2\sqrt{2}}{\sigma\sqrt{\pi}}\Big(-1 + e^{-1/2}\frac{R_0}{\sigma}\Big)t\Big\} \\
&= \frac{1}{12} \exp\Big\{\frac{2\sqrt{2}}{\sqrt{\pi}}\Big(-1 + \frac{1}{2}e^{-1/2}\Big)t\Big\} \qquad t \geq 0,
\end{aligned}
$$

Fig. 3. Time evolution of the variance of the distribution of agents means x_i, $i = 1, .., n = 1000$ for different values of h ($h = 10^{-2}$ dotted in blue, $h = 10^{-3}$ dashed in red, $h = 10^{-4}$ dot-dashed in yellow. We also plot the theoretical bounds obtained in Theorem 1.4 and Remark 1.1 in solid lines (see text for more details).

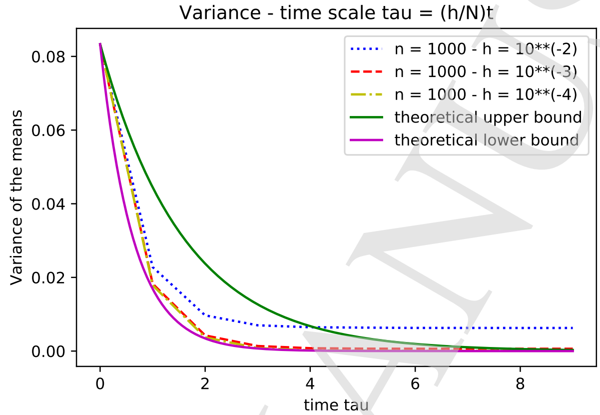
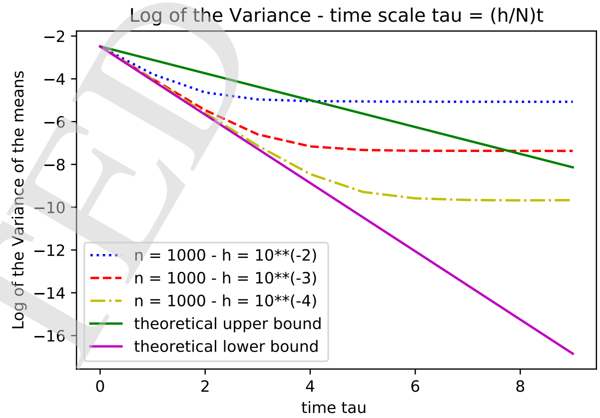


Fig. 4. Time evolution of the logarithm of the variance of the distribution of agents means x_i, $i = 1, .., n = 1000$ for different values of h ($h = 10^{-2}$ dotted in blue, $h = 10^{-3}$ dashed in red, $h = 10^{-4}$ dot-dashed in yellow. We also plot the theoretical bounds obtained in Theorem 1.4 and Remark 1.1 in solid lines (see text for more details).



and the *lower bound* in Remark 1.1 given by

$$\begin{aligned} Var[g_t] &\leq Var[g_0] \exp\Big\{ -\frac{2\sqrt{2}}{\sigma\sqrt{\pi}} t \Big\} \\ &= \frac{1}{12} \exp\Big\{ -\frac{2\sqrt{2}}{\sqrt{\pi}} t \Big\} \end{aligned} \tag{6.3}$$

We can observe a very good agreement between agent based simulations and the theoretical results. All the curves deviate from the theoretical estimates when the support of the means are small enough, of order $O(h)$.

Remark 6.2. Let us observe that in the simulations only one opinion is changed in each interaction, but the results are similar if both agents update their means. The only difference in this case is that the convergence time is reduced by a factor of 2.

7. Final Remarks

7.1. *A comparison with DeGroot model*

As mentioned in the introduction, DeGroot introduced the first behavioral model considered in the aggregation of experts opinions. The procedure was different, since each agent i assigns a weight w_j to agent j, and then updates his probability measure as a mean of the other ones. In the case of Normal distributions with the same variance, agent i obtain a new Normal distribution with different mean and variance, $N(x_i, \sigma^2)$ is changed to

$$\sum_{j=1}^{N} w_j N(x_j, \sigma^2) = N\left(\sum_{j=1}^{N} w_j x_j, \Big(\sum_{j=1}^{N} w_j^2\Big)\sigma^2\right).$$

In particular, if all the agents assign the same weight $w_j = \frac{1}{N}$ to another agent, they reach a consensus in the probability measure $\delta_{\bar{x}}$ where

$$\bar{x} = \sum x_i/N,$$

while in our case the agents goes to the Normal distribution $N(\frac{1}{N}\sum_{i=1}^{N} x_i, \sigma^2)$.

Observe that in this example the variance of the consensus opinion is smaller than the original one. It is possible to show examples where the variance increases in DeGroot's model.

For instance, let us assume that N agents have a Dirac Delta concentrated at x_i, for each $1 \leq i \leq N$, and they assign the same weights $w_j = \frac{1}{N}$ to other experts. Hence, they reach a consensus at the probability measure defined by

$$\frac{1}{N}\sum_{i=1}^{N} w_i \delta_{x_i},$$

which has positive variance except that $x_i = x_j$ for each pair i, j.

However, since agent i can only emit the opinion x_i with probability one, the updating dynamics introduced in this work is the same as the one studied in,[39] and the agents reach a consensus at the probability measure $\delta_{\bar{x}}$, with

$$\bar{x} = \sum x_i/N.$$

7.2. *Possible extensions and future works*

We can extend our model to non-homogeneous population in the following ways:

- each agent has its own value of σ, either positive or 0,
- each agent has individual rates of persuasion and conviction, as in Ref. 38.

We can model the first situation assuming that the population is characterized at time t by the distribution of (x, σ) via a probability measures f_t. In that case f_t is a probability measure over the (x, σ) space $\mathbb{R} \times [0, +\infty)$. Assuming that the parameter σ of an agent is not modified during an interaction, the evolution of f_t is then given by the Boltzmann-like equation

$$\frac{d}{dt} \int \phi(x, \sigma)\, df_t(x, \sigma) = \int \mathbb{E}[\phi(x', \sigma) - \phi(x, \sigma)]\, df_t(x, \sigma) df_t(x_*, \sigma_*).$$

It is then easily seen that as $h \to 0$ this equation is well-approximated, at least formally, by the transport equation

$$\partial_t f_t + \partial_x \left(\Big(\int \Psi_{\sigma_*}(x - x_*)\, df_t(x_*, \sigma_*) \Big) f_t \right) = 0.$$

which reduces to (1.3) when there is only one possible value of σ. For instance if there are only a finite number of positive values for σ, namely $\sigma = 0$ and $\sigma_1, .., \sigma_N > 0$, then we can write f_t as

$$f_t = \alpha_0 f_t^0 + \sum_{i=1}^{N} \alpha_N f_t^i$$

where $\alpha_0, \alpha_1, .., \alpha_N \in [0, 1]$ are the proportion of agents with variance $\sigma = 0$, $\sigma = \sigma_1$, .., $\sigma = \sigma_N$, and $f^0, \ldots, f^N$ are the distribution of x in the subpopulations with $\sigma = 0$, $\sigma = \sigma_1$, .., $\sigma = \sigma_N$ respectively. We then obtain the Boltzmann equation

$$\frac{d}{dt} \int \phi(x)\, df_t^i(x) = \sum_{j=0}^{N} \alpha_i \int \mathbb{E}[\phi(x') - \phi(x)]\, df_t^i(x) df_t^j(x_*)$$

and its approximation

$$\partial_t f_t^i + \partial_x \Big(\Big(\sum_j \alpha_j \int \Psi_j(x - x_*)\, df_t^j(x_*) \Big) f_t^i \Big) = 0$$

The case of heterogeneous agents with different rates of persuasion and conviction will be studied in a future work.

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