



Local existence conditions for an equations involving the $p(x)$ -Laplacian with critical exponent in \mathbb{R}^N

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Abstract. The purpose of this paper is to formulate sufficient existence conditions for a critical equation involving the $p(x)$ -Laplacian of the form (0.1) below posed in \mathbb{R}^N . This equation is critical in the sense that the source term has the form $K(x)|u|^{q(x)-2}u$ with an exponent q that can be equal to the critical exponent p^* at some points of \mathbb{R}^N including at infinity. The sufficient existence condition we find are local in the sense that they depend only on the behaviour of the exponents p and q near these points. We stress that we do not assume any symmetry or periodicity of the coefficients of the equation and that K is not required to vanish in some sense at infinity like in most existing results. The proof of these local existence conditions is based on a notion of localized best Sobolev constant at infinity and a refined concentration-compactness at infinity.

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In this paper we address the existence problem for the $p(x)$ -Laplace operator with a source that has critical growth in the sense of the Sobolev embeddings. To be precise, we consider the equation

$$-\Delta_{p(x)}u + \lambda k(x)|u|^{p(x)-2}u = f(x, u) \quad \text{in } \mathbb{R}^N, \quad (0.1)$$

where $\lambda > 0$, the $p(x)$ -Laplacian operator $-\Delta_{p(x)}$ is defined as usual as $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ and the source term f has the form $f(x, u) = K(x)|u|^{q(x)-2}u$ for some nonnegative continuous function K that has a limit $K(\infty)$ at infinity. The exponents $p, q : \mathbb{R}^N \rightarrow [1, +\infty)$ are Log-Hölder continuous functions having a limit $p(\infty)$ and $q(\infty)$ at infinity and satisfying $1 < \inf_{\mathbb{R}^N} p \leq \sup_{\mathbb{R}^N} p < N$ and $1 \leq q(x) \leq p^*(x) := Np(x)/(N - p(x))$, $x \in \mathbb{R}^N$. We will assume that equation (0.1) is critical at infinity in the sense that $q(\infty) = p^*(\infty)$. Notice that the critical set $\mathcal{A} := \{x \in \mathbb{R}^N : q(x) = p^*(x)\}$

can be non-empty as well. The main purpose of this paper is to find conditions on the coefficients p, q, k, K to obtain the existence of a non-trivial solution to (0.1) without any periodicity or symmetry assumptions of these coefficients and without requiring K to vanish at infinity.

The $p(x)$ -Laplacian, and more generally variable exponent spaces, have been the subject of an intense research activity in the fields of both partial differential equations and harmonic analysis. This interest comes from the possible applications of these spaces as well as the new variety of phenomenon that appear in comparison to the traditional setting of constant exponent spaces.

From the point of view of applications, the $p(x)$ -Laplacian appears in two main applied problems. The first one deals with fluid mechanic where this operator is useful in the modelling of the so-called electrorheological fluids. These fluids have the peculiarity of modifying their mechanical properties in presence of external factors such as electromagnetic field. We refer to M. Ružička's book [34] where this theory is fully developed. The second application comes from the field of image processing where the $p(x)$ -Laplacian is used to design image restauration processes that behave differently according to the smoothness of the image. This way noise can be removed from an image preserving the boundaries. We refer to Chen et al.'s paper [16].

From the points of view of partial differential equations, we mention as an example one striking features of the variable exponent setting and refer to [24] and [32] for general overviews of recent results concerning PDE involving the $p(x)$ -Laplacian. It is well-known that, in the constant exponent setting, the immersion of the Sobolev space $W_0^{1,p}(U)$, with U a smooth bounded domain, into $L^{Np/(N-p)}(U)$ is never compact. However in the variable setting the immersion of $W^{1,p(\cdot)}(U)$ into $L^{q(\cdot)}(U)$ can be compact even if the critical set $\{x \in \bar{U} : q(x) = p^*(x)\}$ is non-empty provided that this set is “small” and we have a control on how the exponent q reaches p^* (see [30]). Moreover it was recently proved in [10] that there exists an extremal for the embedding of $W_0^{1,p(\cdot)}(U)$ into $L^{q(\cdot)}(U)$ with $q(\cdot) \leq p^*(\cdot)$, if q is subcritical in a sufficiently big ball.

In the constant exponent setting, equations like (0.1) in a bounded domain with critical exponent, or in an unbounded domain with subcritical or critical exponent, have been widely studied since the seminal paper by Aubin [7] and Brezis-Nirenberg [14] who dealt with equations involving the Laplacian operator. The initial motivation for the study of this kind of equations comes from their appearance in differential geometry (the so-called Yamabé problem). Aubin's and Brezis-Nirenberg's results and methods have been extended in various directions to deal with critical equations involving the 1-Laplacian, the p -Laplacian, the bilaplacian, ... with various boundary conditions, either in domains of \mathbb{R}^N or on Riemannian manifolds (see e.g. [18, 35–37] and the references therein.)

In contrast there are relatively few results in the variable exponent setting. Concerning the subcritical case, we mention the papers [1, 3, 19–21, 40, 41]. For the critical case, we refer to [2, 4–6, 27, 42, 43]. The authors of these papers

prove existence and multiplicity of solutions for equation similar to (0.1). The major difficulty in proving the existence of solutions is the possibility of a loss of mass at infinity and, in the case of critical problem, the possibility of concentration, either around some points or at infinity. The authors of the aforementioned papers circumvent this problem usually assuming one or various of the following conditions: (i) some symmetry on the coefficients of the equation (in most cases the coefficients are supposed radial) since it is well-known since Strauss' paper [38] (see also [28]) that the presence of symmetry improves Sobolev embeddings, (ii) some periodicity condition since the periodicity virtually reduces the problem to a problem set in an bounded domain, (iii) perturbing f adding some subcritical term, (iv) requiring that K vanishes at infinity in the sense that K belongs to some $L^{r(\cdot)}(\mathbb{R}^N)$ space or that $\lim_{|x| \rightarrow +\infty} K(x) = 0$.

To deal with the concentration phenomenon, the famous concentration-compactness principle (CCP) due to Lions [29], originally formulated for critical problems in bounded domain and later extended to deal with critical problem in unbounded domains by Chabrowski [15], is of prime importance since it describes the concentration by a weighted sum of Dirac masses and the loss of mass by measures “supported at infinity”. The CCP allows to formulate existence conditions for critical equations that are local in the sense that they rely only on the behaviour of the coefficients of the equation near the possible concentration points.

The CCP has recently been extended to the variable exponent setting in [13] (and independently in [39]) for a bounded domain, and in [42] and [43] for an unbounded domain. However in these versions, the Sobolev constant used in the statement to compare the weights of the Dirac masses is a global Sobolev constant which does not reflect the local behaviour of the exponents p and q near the concentrations points. A refined version of the CCP for bounded domain was then later proved in [10] introducing the notion of localized Sobolev constant. This notion was then used in [9] to formulate existence conditions for an equation like (0.1) in a bounded domain relying only on the local behaviour near concentration points. We also mention the papers [11] and [12] where the same program as been carried for equations on a bounded domain with nonlinear boundary conditions. A systematic and self-contained treatment of critical Sobolev immersion and critical equations in bounded domain can be found in the book [8].

Our purpose in this paper is twofold. First we want to prove a refined CCP at infinity in the spirit of [15] together with a notion of localized Sobolev constant at infinity so as to capture the behaviour of p and q at infinity. We then want to apply this CCP to provide sufficient local existence conditions for the equation 0.1 without any of the assumptions usually done. In particular we will not require any periodicity or symmetry of the coefficients nor will we require the function K to vanish in some sense at infinity. Instead we will see that, analogously to what we did in [9–12], if K does not vanish at infinity, then, to prevent the loss of mass or concentration at infinity, it is enough to require that p and q have a local minimum, resp. maximum, at infinity,

and that the hessian matrix of p or q at infinity (see definition 5.5 below) is non-zero.

The paper is organized as follows. In the first section we collect some basic definitions and results concerning the variable exponent Lebesgue and Sobolev spaces that will be needed in this paper. We state the main assumptions on the exponents p and q in the second section. We then introduce the notion of localized best Sobolev constant at infinity in the next section and study its relation with the usual Sobolev constant. As an application we prove in the fourth section a refined version of the concentration-compactness principle at infinity that we apply in the next section to obtain a sufficient condition of existence for equation 0.1. We collect in “Appendix A” a result concerning a compact embedding, which we believe has independent interest, and in “Appendix B” the lengthy test-function computations used to prove the local existence conditions.

1. Preliminaries on variable exponent spaces

In this section we review some preliminary results regarding Lebesgue and Sobolev spaces with variable exponents that will be used throughout this paper. All these results and a comprehensive study of these spaces with an exhaustive bibliography can be found in the books [17] and [33].

1.1. Lebesgue spaces

Consider a measurable function $p : \mathbb{R}^N \rightarrow [1, +\infty]$. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^N)$ is the space of measurable functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ such that the following Luxembourg norm is finite:

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \phi_{p(x)} \left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\} < \infty,$$

where for $t \geq 0$, $\phi_p(t) = t^p$ if $p < \infty$ and $\phi_\infty(t) = 0$ if $t \leq 1$ and $\phi_\infty(t) = \infty$ if $t > 1$ (see Def. 3.2.1 and Remark 3.2.3 in [17]).

The following Hölder inequality holds (see e.g. [17][lemma 3.2.20]):

Proposition 1.1. *Let $p, q, s : \mathbb{R}^N \rightarrow [1, +\infty]$ be measurable functions such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Let $f \in L^{p(\cdot)}(\mathbb{R}^N)$ and $g \in L^{q(\cdot)}(\mathbb{R}^N)$. Then $fg \in L^{s(\cdot)}(\mathbb{R}^N)$ with

$$\|fg\|_{L^{s(\cdot)}(\mathbb{R}^N)} \leq \left(\left(\frac{s}{p} \right)^+ + \left(\frac{s}{q} \right)^+ \right) \|f\|_{L^{p(\cdot)}(\mathbb{R}^N)} \|g\|_{L^{q(\cdot)}(\mathbb{R}^N)}. \quad (1.1)$$

It is usually convenient to study the so-called modular $\rho(u) := \int_{\mathbb{R}^N} |u|^{p(x)} dx$ instead of the norm $\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}$. The following result provide some relations between the two (see [17][Chap 2-1]):

Proposition 1.2. *Assume that p is bounded. For $u \in L^{p(\cdot)}(\mathbb{R}^N)$ and $\{u_k\}_{k \in \mathbb{N}} \subset L^{p(\cdot)}(\mathbb{R}^N)$, we have*

$$\begin{aligned} u \neq 0 &\Rightarrow \left(\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} = \lambda \Leftrightarrow \rho\left(\frac{u}{\lambda}\right) = 1 \right). \\ \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} < 1 (= 1; > 1) &\Leftrightarrow \rho(u) < 1 (= 1; > 1). \\ \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} > 1 &\Rightarrow \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^-} \leq \rho(u) \leq \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^+}. \\ \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} < 1 &\Rightarrow \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^+} \leq \rho(u) \leq \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^-}. \\ \lim_{k \rightarrow \infty} \|u_k\|_{L^{p(\cdot)}(\mathbb{R}^N)} = 0 &\Leftrightarrow \lim_{k \rightarrow \infty} \rho(u_k) = 0. \\ \lim_{k \rightarrow \infty} \|u_k\|_{L^{p(\cdot)}(\mathbb{R}^N)} = \infty &\Leftrightarrow \lim_{k \rightarrow \infty} \rho(u_k) = \infty. \end{aligned}$$

Assume now that $1 < p^- \leq p^+ < \infty$, where $p^- := \text{ess - inf}_{\mathbb{R}^N} p$ and $p^+ := \text{ess - sup}_{\mathbb{R}^N} p$. Then $L^{p(\cdot)}(\mathbb{R}^N)$ is a separable and reflexive Banach space, and the smooth functions with compact support are dense in $L^{p(\cdot)}(\mathbb{R}^N)$.

1.2. Sobolev spaces

Consider a measurable function $p : \mathbb{R}^N \rightarrow [1, +\infty]$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^N)$ is defined by

$$W^{1,p(\cdot)}(\mathbb{R}^N) = \{u \in W^{1,1}_{\text{loc}}(\mathbb{R}^N) : u \in L^{p(\cdot)}(\mathbb{R}^N) \text{ and } |\nabla u| \in L^{p(\cdot)}(\mathbb{R}^N)\}.$$

The corresponding norm for this space is

$$\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} = \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} + \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)}.$$

As with the variable Lebesgue spaces, if $1 < p^- \leq p^+ < \infty$ then $W^{1,p(\cdot)}(\mathbb{R}^N)$ is a separable and reflexive Banach space and the smooth functions with compact support are dense in $W^{1,p(\cdot)}(\mathbb{R}^N)$.

As usual, we denote the conjugate exponent of $p(\cdot)$ by $p'(x) = p(x)/(p(x) - 1)$ and the Sobolev exponent by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

The standard Sobolev embedding theorem and Poincaré inequality still hold provided that p satisfies a regularity condition called Log-Hölder continuity (see [17]) defined by the following condition: there exists $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{\ln(e + 1/|x - y|)}, \quad x, y \in \mathbb{R}^N, x \neq y.$$

Proposition 1.3. *Let $p : \mathbb{R}^N \rightarrow [1, +\infty)$, $p^+ < N$, be a Log-Hölder continuous function and let $q : \mathbb{R}^N \rightarrow [1, +\infty)$ be a measurable function such that $q \leq p^*$. Then, for any bounded smooth domain $U \subset \mathbb{R}^N$, $\|u\| := \|\nabla u\|_{L^{p(\cdot)}(U)}$ is a norm in $W_0^{1,p(\cdot)}(U)$ equivalent to the usual norm. Moreover there is a continuous embedding from $W^{1,p(\cdot)}(U)$ into $L^{q(\cdot)}(U)$, which is also compact if q is strictly subcritical in the sense that $\inf_U p^* - q > 0$.*

We denote by $D^{1,p(\cdot)}(\mathbb{R}^N)$ the closure of $C_c^\infty(\mathbb{R}^N)$ for the norm $\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)}$. Let us assume that $p(\cdot)$ is Log-Hölder continuous function and also that there exists $p(\infty) := \lim_{|x| \rightarrow +\infty} p(x)$ and a constant $C' > 0$ such that

$$|p(x) - p(\infty)| \leq \frac{C'}{\ln(e + |x|)} \quad \text{for all } x \in \mathbb{R}^N.$$

Under these conditions,

Proposition 1.4. *There is a continuous embedding from $D^{1,p(\cdot)}(\mathbb{R}^N)$ into $L^{p^*(\cdot)}(\mathbb{R}^N)$.*

Once again we refer to the book [17] for full proofs of these results.

2. Assumptions on the exponents p and q

In this paper, the exponents p and q will always be measurable functions $p, q : \mathbb{R}^N \rightarrow [1, +\infty)$ satisfying the following assumptions:

(H1) p and q have a modulus of continuity ρ in the sense that for any $x, h \in \mathbb{R}^N$,

$$p(x + h) = p(x) + \rho(h), \quad \text{and} \quad q(x + h) = q(x) + \rho(h)$$

with $\lim_{h \rightarrow 0} \rho(h) \ln |h| = 0$.

(H2) there exist real numbers $p(\infty)$ and $q(\infty)$ such that

$$\lim_{|x| \rightarrow +\infty} |p(x) - p(\infty)| \ln |x| = 0 \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} |q(x) - q(\infty)| \ln |x| = 0 \tag{2.1}$$

(H3) denoting $p^- := \inf_{\mathbb{R}^N} p$, $p^+ := \sup_{\mathbb{R}^N} p$, we assume that

$$1 < p^- \leq p^+ < N \quad \text{and} \quad p \leq q \leq p^* := \frac{Np}{N - p}.$$

We denote $\mathcal{A} := \{x \in \mathbb{R}^N : q(x) = p^*(x)\}$ the critical set which can be empty or not.

We assume that ∞ is critical in the sense that $q(\infty) = p^*(\infty)$.

Assumption (H1) is slightly stronger than the usual Log-Hölder continuity assumption, which is natural to assume to have the Sobolev embeddings theorems. We need this stronger version to be able to perform some test-function computation. Assumption (H2) can be thought in the same way as (H1) as a stronger version of a Log-Hölder continuity at infinity. Log-Hölder continuity at infinity (also called Log-Hölder decay condition) is a classical assumption when dealing for instance with the maximal function in \mathbb{R}^N (see [17][Chap.4]). Assumption (H3) says that we are assuming that the infinity is critical. We stress that the critical set \mathcal{A} can contain other points that ∞ where $q = p^*$. The case where infinity is subcritical, in the sense that $q(\infty) < p^*(\infty)$, will be treated elsewhere.

3. Localized best Sobolev constant at infinity

Given a smooth open subset $U \subset \mathbb{R}^N$ we denote by $S(p(\cdot), q(\cdot), U)$ the best Sobolev constant for the embedding of $W_0^{1,p(\cdot)}(U)$ into $L^{q(\cdot)}(U)$, namely

$$S(p(\cdot), q(\cdot), U) := \inf_{u \in W_0^{1,p(\cdot)}(U), u \neq 0} \frac{\|\nabla u\|_{L^{p(\cdot)}(U)}}{\|u\|_{L^{q(\cdot)}(U)}}.$$

Let $x_0 \in \mathcal{A}$ be a critical point i.e. $q(x_0) = p^*(x_0)$. Taking in the previous definition a ball $U = B_{x_0}(\varepsilon)$ centered at x_0 with small radius $\varepsilon > 0$, and noticing that $S(p(\cdot), q(\cdot), B_{x_0}(\varepsilon))$ is non-decreasing in ε , we can consider the best localized Sobolev constant S_{x_0} at x_0 defined by

$$S_{x_0} = \lim_{\varepsilon \rightarrow 0} S(p(\cdot), q(\cdot), B_{x_0}(\varepsilon)) = \sup_{\varepsilon > 0} S(p(\cdot), q(\cdot), B_{x_0}(\varepsilon)). \tag{3.1}$$

This notion was introduced in [10] to study precisely the concentration phenomenon at x_0 and obtain a refined concentration-compactness principle.

To study the Sobolev embedding at infinity, we consider in an analogous way, the best Sobolev constant $S_R = S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R)$ in $\mathbb{R}^N \setminus B_R$, where we denote $B_R := B_0(R)$, $R > 0$. Noticing that S_R is non-decreasing in R , we define the localized best Sobolev constant S_∞ at infinity taking the limit of S_R as $R \rightarrow +\infty$:

Definition 3.1. The localized best Sobolev constant S_∞ at infinity is defined as

$$S_\infty = \lim_{R \rightarrow +\infty} S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R) = \sup_{R > 0} S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R). \tag{3.2}$$

Since the localized best Sobolev constant S_{x_0} depends only on the behaviour of p and q near a critical point x_0 , it is natural to try to compare it with the usual best Sobolev constant $K(N, r)$, $r \in [1, N)$, corresponding to the embedding of the constant-exponent space $D^{1,r}(\mathbb{R}^N)$, the closure of $C_c^\infty(\mathbb{R}^N)$ for the norm $\|\nabla u\|_{L^r(\mathbb{R}^N)}$, into $L^{Nr/(N-r)}(\mathbb{R}^N)$, namely

$$K(N, r)^{-r} := \inf_{u \in C_c^\infty(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^r dx}{\left(\int_{\mathbb{R}^N} |u|^{\frac{Nr}{N-r}} dx \right)^{\frac{N-r}{N}}}. \tag{3.3}$$

It was proved in [10] that $S_{x_0} \leq K(N, p(x_0))^{-1}$ with equality if p has a local minimum at x_0 and q a local maximum. The next two results show that these two properties still hold for S_∞ .

Proposition 3.2. Assume that ∞ is critical in the sense that $q(\infty) = p(\infty)^*$. Then

$$S_\infty \leq K(N, p(\infty))^{-1}, \tag{3.4}$$

where $K(N, p(\infty))$ is defined in (3.3) (with $r = p(\infty)$) and S_∞ in (3.2).

Proof. Given $\phi \in C_c^\infty(\mathbb{R}^N)$, $\phi \neq 0$, we consider the rescaled function ϕ_λ defined for $\lambda > 0$ small by

$$\phi_\lambda(x) = \lambda^{\frac{-N}{p^*(x\lambda)}} \phi\left(\frac{x - x_\lambda}{\lambda}\right), \tag{3.5}$$

where the points $x_\lambda \in \mathbb{R}^N$ are such that $|x_\lambda| \geq 1/\lambda$. It then follows from assumptions (H1) and (H2) that

$$q(x_\lambda + \lambda y) = q(\infty) + \varepsilon_\lambda^1(y), \quad p(x_\lambda + \lambda y) = p(\infty) + \varepsilon_\lambda^2(y), \tag{3.6}$$

and

$$p(x_\lambda + \lambda y) = p(x_\lambda) + \varepsilon_\lambda^3(y), \tag{3.7}$$

where the functions ε_λ^i , $i = 1, 2, 2$, satisfy

$$|\varepsilon_\lambda^i(y)| \leq \varepsilon_\lambda \quad \text{with} \quad \lim_{\lambda \rightarrow 0} \varepsilon_\lambda \ln \lambda = 0 \tag{3.8}$$

uniformly for y in a compact set.

Fix some $R > 0$. Since $\lim_{\lambda \rightarrow +\infty} |x_\lambda| = +\infty$, we have that $\text{supp } \phi_\lambda \subset \mathbb{R}^N \setminus B_R$ for $\lambda > 0$ small enough. It follows that

$$S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R) \leq \liminf_{\lambda \rightarrow 0} \frac{\|\nabla \phi_\lambda\|_{p(\cdot)}}{\|\phi_\lambda\|_{q(\cdot)}}. \tag{3.9}$$

We now estimate each terms in the right hand side of this inequality. First, in view of (3.6),

$$\int_{\mathbb{R}^N} |\phi_\lambda(x)|^{q(x)} dx = \int_{\mathbb{R}^N} \lambda^N \left(1 - \frac{q(\infty) + \varepsilon_\lambda^1(y)}{(p(\infty) + \varepsilon_\lambda^2(y))^*} \right) |\phi(y)|^{q(\infty) + \varepsilon_\lambda^1(y)} dy.$$

Recalling that ϕ has compact support and that $q(\infty) = p(\infty)^*$, we deduce using (3.8) that

$$\lim_{\lambda \rightarrow +\infty} \int_{\mathbb{R}^N} |\phi_\lambda(x)|^{q(x)} dx = \int_{\mathbb{R}^N} |\phi|^{q(\infty)} dx.$$

According to the definition of the $\|\cdot\|_{q(\cdot)}$ norm we can then write

$$1 = \int_{\mathbb{R}^N} \left(\frac{\phi_\lambda(x)}{\|\phi_\lambda\|_{q(\cdot)}} \right)^{q(x)} dx = (1 + o(1)) \|\phi_\lambda\|_{q(\cdot)}^{-q(\infty) + o(1)} \int_{\mathbb{R}^N} |\phi|^{q(\infty)} dx.$$

We then deduce that

$$\lim_{\lambda \rightarrow 0} \|\phi_\lambda\|_{q(\cdot)} = \|\phi\|_{q(\infty)}. \tag{3.10}$$

We treat the gradient term analogously. First, as $\lambda \rightarrow 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \phi_\lambda(x)|^{p(x)} dx &= \int_{\mathbb{R}^N} \lambda^N \left(1 - \frac{p(x_\lambda + \lambda y)}{p^*(x_\lambda)} \right) -p(x_\lambda + \lambda y) |\nabla \phi|^{p(x_\lambda) + \lambda y} dy \\ &\rightarrow \int_{\mathbb{R}^N} |\nabla \phi|^{p(\infty)} dy. \end{aligned}$$

Then, writing $\int_{\mathbb{R}^N} \left(\frac{|\nabla \phi_\lambda(x)|}{\|\nabla \phi_\lambda\|_{p(\cdot)}} \right)^{p(x)} dx = 1$ as above, we deduce that

$$\lim_{\lambda \rightarrow 0} \|\nabla \phi_\lambda\|_{p(\cdot)} = \|\nabla \phi\|_{p(\infty)}. \tag{3.11}$$

Coming back to (3.9), we thus deduce in view of (3.10) and (3.11) that when $q(\infty) = p(\infty)^*$, there holds that

$$S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R) \leq \frac{\|\nabla \phi\|_{p(\infty), \mathbb{R}^N}}{\|\phi\|_{p(\infty)^*, \mathbb{R}^N}}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$ and any $R > 0$. It follows that for any $R > 0$,

$$S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R) \leq K^{-1}(N, p(\infty)).$$

Taking the limit $R \rightarrow +\infty$ yields (3.4). □

The next result gives a sufficient condition on p and q to have the equality in (3.4). The proof follows the line of [10] with an additional difficulty. Indeed the proof there ultimately relies on Hölder inequality in a ball $B_{x_0}(\varepsilon)$. Here the analogous of this ball is the ball centered at infinity $\mathbb{R}^N \setminus B_R$ which has infinite volume. To overcome this difficulty, we adapt a trick apparently due to Nekvinda [31] which amounts to say that, for a given constant $c \in (0, 1)$ and a variable exponent $\varepsilon(x)$, an integral like $\int_{\mathbb{R}^N \setminus B_R} c^{\varepsilon(x)} dx$ can be made arbitrarily small provided that ε converge fast enough to $+\infty$ at infinity.

Theorem 3.3. *If ∞ is a local maximum of q and a local minimum of p in the sense that*

$$p(x) \geq p(\infty) \quad \text{and} \quad q(x) \leq q(\infty) \tag{3.12}$$

for $|x|$ large, where $p(\infty)$ and $q(\infty)$ are given in assumption (H2), then, if $q(\infty) = p^*(\infty)$,

$$S_\infty = K(N, p(\infty))^{-1}, \tag{3.13}$$

where $K(N, p(\infty))$ is defined in (3.3) and S_∞ in (3.2).

Proof. Notice that the result is trivial if p is constant in $\mathbb{R}^N \setminus B_R$ for some $R > 0$. We thus assume from now on that the set $\{x \in \mathbb{R}^N \setminus B_R \text{ s.t. } p(x) \neq p(\infty)\}$ has positive measure for any $R > 0$ big.

In view of (3.4), we only have to prove that

$$S_\infty \geq K^{-1}(N, p(\infty)). \tag{3.14}$$

As a first step, we claim that for any $R > 0$ and $u \in C_c^\infty(\mathbb{R}^N \setminus B_R)$,

$$\|u\|_{L^{q(\cdot)}(\mathbb{R}^n \setminus B_R)} = R^{\frac{N}{q(\infty)}}(1 + o(1))\|u_R\|_{L^{q_R(\cdot)}(\mathbb{R}^N \setminus B_1)} \tag{3.15}$$

and

$$\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n \setminus B_R)} = R^{\frac{N}{p(\infty)^*}}(1 + o(1))\|\nabla u_R\|_{L^{p_R(\cdot)}(\mathbb{R}^N \setminus B_1)}, \tag{3.16}$$

where $u_R(x) := u(Rx)$, $p_R(x) := p(Rx)$, $q_R(x) := q(Rx)$, and $o(1)$ does not depend on u and goes to 0 as $R \rightarrow \infty$ uniformly in $y \in \mathbb{R}^N \setminus B_R$. We only prove (3.15) since the proof of (3.16) is similar. We have

$$\begin{aligned} \|u\|_{L^{q(\cdot)}(\mathbb{R}^n \setminus B_R)} &= \inf \left\{ \lambda > 0 \text{ s.t. } \int_{\mathbb{R}^n \setminus B_R} \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 \text{ s.t. } \int_{\mathbb{R}^n \setminus B_1} \left| \frac{u_R(x)}{\lambda R^{-N/q_R(y)}} \right|^{q_R(y)} dy \leq 1 \right\}. \end{aligned}$$

Notice that

$$R^{-\frac{N}{q_R(y)} + \frac{N}{q(\infty)}} = \exp \left\{ \frac{N}{q_R(y)q(\infty)} \left(q(Ry) - q(\infty) \right) \ln R \right\}$$

goes to 1 as $R \rightarrow \infty$ uniformly in $y \in \mathbb{R}^N \setminus B_1$ in view of (2.1) and recalling that q is bounded. It follows that

$$\|u\|_{L^{q(\cdot)}(\mathbb{R}^n \setminus B_R)} = \inf \left\{ \lambda > 0 \text{ s.t. } \int_{\mathbb{R}^n \setminus B_1} \left| \frac{u_R(x)}{\lambda R^{-N/q(\infty)}(1+o(1))} \right|^{q_R(y)} dy \leq 1 \right\},$$

from which we deduce (3.15).

It follows from (3.15), (3.16) and the fact that the map $u \in C_c^\infty(\mathbb{R}^N \setminus B_R) \rightarrow u_R \in C_c^\infty(\mathbb{R}^N \setminus B_1)$ is bijective, that when $q(\infty) = p(\infty)^*$,

$$S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R) = (1 + o(1))S(p_R(\cdot), q_R(\cdot), \mathbb{R}^N \setminus B_1)$$

where $o(1) \rightarrow 0$ as $R \rightarrow \infty$. To conclude the proof, it thus suffices to prove that

$$\liminf_{R \rightarrow \infty} S(p_R(\cdot), q_R(\cdot), \mathbb{R}^N \setminus B_1) \geq S(p(\infty), q(\infty), \mathbb{R}^N \setminus B_1).$$

This will hold if we can prove that for any $u \in C_c^\infty(\mathbb{R}^N \setminus B_1)$,

$$\|\nabla u\|_{L^{p(\infty)}(\mathbb{R}^N \setminus B_1)} \leq (1 + o(1))\|\nabla u\|_{L^{p_R(\cdot)}(\mathbb{R}^N \setminus B_1)} \tag{3.17}$$

and

$$\|u\|_{L^{q(\infty)}(\mathbb{R}^N \setminus B_1)} \geq (1 + o(1))\|u\|_{L^{q_R(\cdot)}(\mathbb{R}^N \setminus B_1)}. \tag{3.18}$$

We prove (3.17) (the proof of (3.18) is similar). Fix some $c \in (0, 1)$. Using Hölder inequality (1.1) and the hypothesis that ∞ is a local minimum of p , we have

$$\begin{aligned} \|\nabla u\|_{L^{p(\infty)}(\mathbb{R}^N \setminus B_1)} &= c^{-1} \|c \nabla u\|_{L^{p(\infty)}(\mathbb{R}^N \setminus B_1)} \\ &\leq c^{-1} \left(\left(\frac{p(\infty)}{p_R} \right)^+ + \left(\frac{p(\infty)}{s_R} \right)^+ \right) \\ &\quad \times \|\nabla u\|_{L^{p_R(\cdot)}(\mathbb{R}^N \setminus B_1)} \|c\|_{L^{s_R(\cdot)}(\mathbb{R}^N \setminus B_1)}, \end{aligned}$$

where s_R is defined by $\frac{1}{s_R} = \frac{1}{p(\infty)} - \frac{1}{p_R}$ and $\left(\frac{p(\infty)}{p_R} \right)^+ = \sup_{\mathbb{R}^N \setminus B_1} \frac{p(\infty)}{p_R}$. It follows from (H2) that

$$\|\nabla u\|_{L^{p(\infty)}(\mathbb{R}^N \setminus B_1)} \leq c^{-1}(1 + o(1))\|\nabla u\|_{L^{p_R(\cdot)}(\mathbb{R}^N \setminus B_1)} \|c\|_{L^{s_R(\cdot)}(\mathbb{R}^N \setminus B_1)}.$$

We now estimate $\|c\|_{L^{s_R(\cdot)}(\mathbb{R}^N \setminus B_1)}$. We write c in the form $c = e^{-\gamma}$, $\gamma > 0$. Then

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_1} c^{s_R(x)} dx &= \int_{\mathbb{R}^N \setminus B_1} \exp\left(-\gamma \frac{p(\infty)p_R(x)}{p_R(x) - p(\infty)}\right) dx \\ &\leq \int_{\mathbb{R}^N \setminus B_1} \exp\left(-\frac{\gamma}{p_R(x) - p(\infty)}\right) dx. \end{aligned}$$

The last inequality follows from the fact that $p_R \geq p(\infty) \geq 1$. Fix some $\delta > 0$ small such that $\frac{\gamma}{\delta} > N$. Then for R big (depending on δ) we have in view of assumption (2.1) that

$$0 \leq p_R(x) - p(\infty) \leq \frac{\delta}{\ln R|x|} \quad \text{for } |x| \geq 1.$$

Then

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_1} c^{s_R(x)} dx &\leq \int_{\mathbb{R}^N \setminus B_1} (R|x|)^{-\frac{\gamma}{\delta}} dx = R^{-\frac{\gamma}{\delta}} \omega_{N-1} \int_1^{+\infty} r^{N-1-\frac{\gamma}{\delta}} dr \\ &= \frac{R^{-\frac{\gamma}{\delta}} \omega_{N-1}}{\frac{\gamma}{\delta} - N} \end{aligned}$$

which goes to 0 as $R \rightarrow \infty$. Recall that we assume that the set $\{x \in \mathbb{R}^N \setminus B_1 \text{ s.t.}$

$s_R(x) \neq \infty\}$ has positive measure for any $R > 0$ big. It follows that $\int_{\mathbb{R}^N \setminus B_1} c^{s_R(x)} dx > 0$ so that in view of [17][Lemma 3.2.5],

$$\|c\|_{L^{s_R(\cdot)}(\mathbb{R}^N \setminus B_1)} \leq \max \left\{ \left(\int_{\mathbb{R}^N \setminus B_1} c^{s_R(x)} dx \right)^{\frac{1}{s_R}}; \left(\int_{\mathbb{R}^N \setminus B_1} c^{s_R(x)} dx \right)^{\frac{1}{s_R^+}} \right\}.$$

As $\int_{\mathbb{R}^N \setminus B_1} c^{s_R(x)} dx \rightarrow 0$ as $R \rightarrow \infty$ and $\frac{1}{s_R^+} = \left(\frac{1}{s_R}\right)^- = \inf_{\mathbb{R}^N \setminus B_1} \frac{1}{p(\infty)} - \frac{1}{p_R} = 0$, we obtain $\|c\|_{L^{s_R(\cdot)}(\mathbb{R}^N \setminus B_1)} \leq 1$. In conclusion, for any $c \in (0, 1)$ we have for R big enough depending on c only that

$$\|\nabla u\|_{L^{p(\infty)}(\mathbb{R}^N \setminus B_1)} \leq c^{-1}(1 + o(1))\|\nabla u\|_{L^{p_R(\cdot)}(\mathbb{R}^N \setminus B_1)}.$$

(3.17) follows. □

Observe that if, in the proof of prop 3.2, we take $\lambda \rightarrow +\infty$ with $|x_\lambda| \gg \lambda$, we can easily show that $S_\infty = 0$ if ∞ is subcritical in the sense that $q(\infty) < p(\infty)^*$. The details are given in the proof of the next proposition below. Conversely it seems natural to expect that $S_\infty > 0$ if ∞ is critical. The proof happens to be non-trivial. To see the difficulty, notice that the proof of this fact in the constant exponent case goes as follows: we assume that $K(n, p)^{-1} = 0$ so that there exists a sequence $(u_k)_k$ of compactly supported smooth function such that $\int |\nabla u_k|^p dx \rightarrow 0$ and $\int |u_k|^{p^*} dx = 1$. Using the invariance by translation and dilatation of the norm, we can assume that each u_k is supported in the unit ball B and then use the embedding $W_0^{1,p}(B) \hookrightarrow L^{p^*}(B)$ to obtain a contradiction.

Since the invariance by translation and dilatation is lost in the variable exponent setting, this scheme of proof does not work anymore. Instead we will translate the problem to the sphere using the stereographic projection.

In the course of the proof we will need a Hardy inequality which is essentially a particular case of a general result proved in [22].

Proposition 3.4. *For any $v \in D^{1,p(\cdot)}(\mathbb{R}^N)$, we have that $|x|^{-1}v \in L^{p(\cdot)}(\mathbb{R}^N)$ and there exists a constant $C > 0$ independent of v such that*

$$\||x|^{-1}v\|_{L^{p(\cdot)}(\mathbb{R}^N)} \leq C\|\nabla v\|_{L^{p(\cdot)}(\mathbb{R}^N)}. \tag{3.19}$$

Proof. Let $I_1 u(x) = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-1}} dy$ be the fractional integral of u . As a particular case of [22][Thm 4.1], we have the existence of a constant $C > 0$ such that for any $u \in L^{p(\cdot)}(\mathbb{R}^N)$, $|x|^{-1}I_1 u \in L^{p(\cdot)}(\mathbb{R}^N)$ with

$$\||x|^{-1}I_1 u\|_{L^{p(\cdot)}(\mathbb{R}^N)} \leq C\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)}.$$

Taking $u = |\nabla v|$ with $v \in D^{1,p(\cdot)}(\mathbb{R}^N)$, we obtain

$$\| |x|^{-1} I_1(|\nabla v|) \|_{L^{p(\cdot)}(\mathbb{R}^N)} \leq C \| \nabla v \|_{L^{p(\cdot)}(\mathbb{R}^N)}.$$

Independently, it is well-known (see e.g. [23][Lemma 7.14]) that for any $v \in C_c^\infty(\mathbb{R}^N)$, $|v| \leq C(N)I_1(|\nabla v|)$ with a constant $C(N)$ depending only on N . The result easily follows. \square

Our result is the following:

Proposition 3.5. *There holds that $S_\infty > 0$ if and only if $q(\infty) = p(\infty)^*$.*

Proof. Assume that $q(\infty) < p^*(\infty)$. As in the proof of prop. 3.2, we fix some $\phi \in C_c^\infty(\mathbb{R}^N)$, $\phi \not\equiv 0$, and consider for $\lambda \rightarrow +\infty$ the rescaled functions ϕ_λ defined by (3.5), where the points $x_\lambda \in \mathbb{R}^N$ are such that $\lim_{\lambda \rightarrow +\infty} \frac{|x_\lambda|}{\lambda} = +\infty$. It then follows from assumption (H1)–(H2) that for y in a compact set, we can write (3.6)–(3.8) and, since $\text{supp } \phi_\lambda \subset \mathbb{R}^N - B_R$ for λ big, also (3.9). Then as before, we have

$$\int_{\mathbb{R}^N} |\phi_\lambda(x)|^{q(x)} dx = \lambda^N \left(1 - \frac{q(\infty) + \varepsilon_\lambda}{p(\infty) + \varepsilon_\lambda} \right)^* \int_{\mathbb{R}^N} |\phi(y)|^{q(\infty) + \varepsilon_\lambda} dy.$$

As $\liminf_{\lambda \rightarrow +\infty} 1 - \frac{q(\infty) + \varepsilon_\lambda}{p(\infty) + \varepsilon_\lambda} > 0$, we have $\int_{\mathbb{R}^N} |\phi_\lambda(x)|^{q(x)} dx \rightarrow \infty$. On the other hand, as before $\lim_{\lambda \rightarrow +\infty} \int_{\mathbb{R}^N} |\nabla \phi_\lambda(x)|^{p(x)} dx = \int_{\mathbb{R}^N} |\nabla \phi|^{p(\infty)} dy$. We conclude that $S_\infty = 0$.

Assume now that $q(\infty) = p(\infty)^*$. Let $\Phi : S^N \setminus \{P\} \rightarrow \mathbb{R}^N$ be the stereographic projection centered at the north pole P of the sphere S^N . It is well-known (see e.g. [26][lemma 3.4]) that the pull-back by Φ^{-1} of the standard metric h of S^N is the metric conformal to the Euclidean metric ξ given by

$$(\Phi^{-1})^* h = \phi \xi \quad \text{with} \quad \phi(x) = 4(1 + |x|^2)^{-2}.$$

In particular for any function $u : S^N \setminus \{P\} \rightarrow \mathbb{R}$ there hold

$$\int_{S^N} u dv_h = \int_{\mathbb{R}^N} u \circ \Phi^{-1} dv_{(\Phi^{-1})^* h} = \int_{\mathbb{R}^N} (u \circ \Phi^{-1}) \phi^{N/2} dx$$

and

$$|\nabla u|_h^2 = |\nabla(u \circ \Phi^{-1})|_\xi^2 \circ \Phi.$$

To a function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ with compact support, we associate a function $\hat{u} : S^N \rightarrow \mathbb{R}$ with compact support in $S^N - \{P\}$ defined by

$$\hat{u} = (u \phi^\beta) \circ \Phi \quad \text{where} \quad 2\beta := -\frac{N}{q(\infty)} = 1 - \frac{N}{p(\infty)}. \tag{3.20}$$

We also define the exponent \hat{p} and \hat{q} on S^n by

$$\hat{p} = p \circ \Phi \quad \text{and} \quad \hat{q} = q \circ \Phi. \tag{3.21}$$

We first claim that if $u \in L^{q(\cdot)}(\mathbb{R}^N)$ is supported in $\mathbb{R}^N \setminus B_R$ then

$$\int_{S^N} |\hat{u}|^{\hat{q}(\cdot)} dv_h = (1 + o_R(1)) \int_{\mathbb{R}^N} |u|^{q(\cdot)} dx, \tag{3.22}$$

where $\lim_{R \rightarrow +\infty} o_R(1) = 0$ uniformly in u . Indeed

$$\int_{\mathbb{S}^N} |\hat{u}|^{\hat{q}(\cdot)} dv_h = \int_{\mathbb{R}^N} |u\phi^\beta|^{q(\cdot)} dv_{\phi\xi} = \int_{\mathbb{R}^N \setminus B_R} |u|^{q(\cdot)} \phi^{\beta q(\cdot) + N/2} dx$$

with $\beta q(x) + N/2 = \frac{n}{2q(\infty)}(q(\infty) - q(x))$. Using (H2), it is easy to see that $\lim_{|x| \rightarrow +\infty} \phi(x)^{\beta q(x) + N/2} = 1$ which proves the claim.

We now claim that there exists a constant $C > 0$ such that for any $u \in C^1(\mathbb{R}^N)$,

$$\int_{\mathbb{S}^N} |\nabla \hat{u}|_h^{\hat{p}(\cdot)} dv_h \leq C \int_{\mathbb{R}^N} |\nabla u|^{p(\cdot)} dx + C \int_{\mathbb{R}^N} \frac{|u|^{p(\cdot)}}{|x|^{p(\cdot)}} dx. \tag{3.23}$$

To prove this we begin writing that

$$\begin{aligned} \int_{\mathbb{S}^N} |\nabla \hat{u}|_h^{\hat{p}(\cdot)} dv_h &= \int_{\mathbb{R}^N} |\nabla(u\phi^\beta)|_{\phi\xi}^{p(\cdot)} dv_{\phi\xi} = \int_{\mathbb{R}^N} |\nabla(u\phi^\beta)|_\xi^{p(\cdot)} \phi^{\frac{N-p(\cdot)}{2}} dx \\ &\leq C \int_{\mathbb{R}^N} |\nabla u|^{p(\cdot)} \phi^{\beta p(\cdot) + \frac{N-p(\cdot)}{2}} dx + C \int_{\mathbb{R}^N} |u|^{p(\cdot)} |\nabla \phi|^{p(\cdot)} \phi^{(\beta-1)p(\cdot) + \frac{N-p(\cdot)}{2}} dx \\ &\leq C \int_{\mathbb{R}^N} |\nabla u|^{p(\cdot)} \phi^{\frac{N}{2p(\infty)}(p(\infty) - p(x))} dx \\ &\quad + C \int_{\mathbb{R}^N} |u|^{p(\cdot)} \left(\frac{|\nabla \phi|}{\phi}\right)^{p(\cdot)} \phi^{\frac{N}{2p(\infty)}(p(\infty) - p(x))} dx. \end{aligned}$$

As before, using (H2) we see that $\lim_{|x| \rightarrow +\infty} \phi(x)^{\frac{N}{2p(\infty)}(p(\infty) - p(x))} = 1$. We thus obtain

$$\int_{\mathbb{S}^N} |\nabla \hat{u}|_h^{\hat{p}(\cdot)} dv_h \leq \int_{\mathbb{R}^N} |\nabla u|^{p(\cdot)} dx + \int_{\mathbb{R}^n} |u|^{p(\cdot)} \left(\frac{|\nabla \phi|}{\phi}\right)^{p(\cdot)} dx.$$

Moreover direct computation shows that $\frac{|\nabla \phi|}{\phi} = \frac{4|x|}{1+|x|^2} \leq \frac{C}{|x|}$. The result follows.

We can now prove that $S_\infty > 0$. Assume on the contrary that $S_\infty = 0$. In particular $S_R = 0$ for any $R > 0$. We can thus find smooth functions u_R , $R > 0$, compactly supported in $\mathbb{R}^N \setminus B_R$ such that

$$\lim_{R \rightarrow +\infty} \|\nabla u_R\|_{p(\cdot)} = 0 \quad \text{and} \quad \|u_R\|_{q(\cdot)} = 1. \tag{3.24}$$

Then in view of (3.22) and (3.23), the functions \hat{u}_R defined on \mathbb{S}^N from u_R according to (3.20) satisfies

$$\int_{\mathbb{S}^N} |\hat{u}_R|^{\hat{q}} dv_h = 1 + o_R(1) \tag{3.25}$$

and

$$\int_{\mathbb{S}^N} |\nabla \hat{u}_R|_h^{\hat{p}(\cdot)} dv_h \leq o_R(1) + C \int_{\mathbb{R}^N} \frac{|u_R|^{p(\cdot)}}{|x|^{p(\cdot)}} dx$$

Using Hardy inequality (3.19), we see that the integral in the right hand side is $o_R(1)$ since

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u_R|^{p(\cdot)}}{|x|^{p(\cdot)}} dx &\leq \max \left\{ \| |x|^{-1} u_R \|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^+}, \| |x|^{-1} u_R \|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^-} \right\} \\ &\leq C \max \left\{ \| \nabla u_R \|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^+}, \| \nabla u_R \|_{L^{p(\cdot)}(\mathbb{R}^N)}^{p^-} \right\} \\ &= o_R(1). \end{aligned}$$

We thus deduce that

$$\int_{\mathbb{S}^N} |\nabla \hat{u}_R|^{\hat{p}(\cdot)} dv_h = o_R(1). \tag{3.26}$$

We now obtain a contradiction between (3.25) and (3.26). To see this denote by $\Psi : \mathbb{S}^N \setminus \{S\} \rightarrow \mathbb{R}^N$ the stereographic projection from the south pole S , and consider the functions $\hat{u}_R \circ \Psi^{-1}$ and the exponents $\hat{p} \circ \Psi^{-1}$ and $\hat{q} \circ \Psi^{-1}$. Recall that the \hat{u}_R are supported in ball of \mathbb{S}^N centered at P whose radius becomes smaller and smaller as $R \rightarrow +\infty$. Since the pullback by Ψ^{-1} of h restricted to these balls is a metric in $B_1 \subset \mathbb{R}^N$ bounded above and below by $(1 + \varepsilon)\xi$ and $(1 - \varepsilon)\xi$ for arbitrary small ε , we obtain as before that

$$\int_{B_1} |\nabla(\hat{u}_R \circ \Psi^{-1})|^{\hat{p} \circ \Psi^{-1}} dx = o_R(1),$$

and

$$\int_{B_1} |\hat{u}_R \circ \Psi^{-1}|^{\hat{q} \circ \Psi^{-1}} dx \geq 1 + o_R(1).$$

Recalling that $\hat{u}_R \circ \Psi^{-1} \in C_c^\infty(B_1)$ for R big, this contradicts the Poincare inequality in B_1 . □

As an easy but useful corollary we have that

Corollary 3.6. *If $q(\infty) = p(\infty)^*$, then $D^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$.*

Proof. As $R \rightarrow +\infty$, we have that $S_R := S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R) \rightarrow S_\infty > 0$. We can thus take $R > 0$ such that $S_R > 0$. Let $\eta : \mathbb{R}^N \rightarrow [0, 1]$ be a C^∞ function with compact support in B_{2R} and such that $\eta = 1$ in B_R . We then descompose a function $u \in C_c^\infty(\mathbb{R}^N)$ as $u = \eta u + (1 - \eta)u$. Since $(1 - \eta)u$ is supported in $\mathbb{R}^N \setminus B_R$, we have

$$\begin{aligned} \|u\|_{L^{q(\cdot)}(\mathbb{R}^N)} &\leq \|\eta u\|_{L^{q(\cdot)}(\mathbb{R}^N)} + \|(1 - \eta)u\|_{L^{q(\cdot)}(\mathbb{R}^N)} \leq \|u\|_{L^{q(\cdot)}(B_{2R})} \\ &\quad + S_R^{-1} \|\nabla((1 - \eta)u)\|_{L^{p(\cdot)}(\mathbb{R}^N)} \\ &\leq \|u\|_{L^{q(\cdot)}(B_{2R})} + S_R^{-1} \|u\|_{L^{p(\cdot)}(B_{2R})} + S_R^{-1} \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N \setminus B_R)} \end{aligned}$$

We conclude using the embedding $D^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{p^*(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{q(\cdot)}(B_{2R}) \cap L^{p(\cdot)}(B_{2R})$. □

4. Concentration–compactness principle at infinity

This section is devoted to the description of the default of compactness of a sequence of functions $(u_n)_n \subset D^{1,p(\cdot)}(\mathbb{R}^N)$. This will be done by establishing a version of Lions’concentration-compactness principle (CCP) in that setting.

The CCP originally due to P.L. Lions [29] was established in the framework of constant-exponent spaces over a bounded domain. It explains the possible loss of compactness of the u_n by their weak convergence to Dirac masses. The weights of the Dirac masses are related to the best sobolev constant (3.3). It was later extended by Chabrowski [15] to unbounded domain, still for constant-exponent spaces, by introducing measures supported at infinity related to the best sobolev constant (3.3). An extension to variable exponent spaces over a bounded domain was done independently in [13] and [39], and over unbounded domain by [42]. In those papers the weights of the Dirac masses are related to the Sobolev constant $\inf_{u \in C_c^\infty(\mathbb{R}^N)} \frac{\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)}}{\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)}}$. Notice that this Sobolev constant is global in the sense that it depends on the behaviour of p and q in all \mathbb{R}^N . This is not satisfactory since a Dirac mass is supported at a single point and thus the weights should depend only on the behaviour of p and q near that point. A refined version of the CCP was proven in [10] where the weights of the Dirac masses are related to the localized Sobolev constant (3.1).

The purpose of this section is to state a CCP in \mathbb{R}^N where the weight of the ‘‘Dirac mass at infinity’’ is related to the localized Sobolev constant at infinity (3.2).

Theorem 4.1. *Assume that ∞ is critical in the sense that $q(\infty) = p(\infty)^*$. Let $(u_n)_n \subset D^{1,p(\cdot)}(\mathbb{R}^N)$ be a weakly convergent to $u \in D^{1,p(\cdot)}(\mathbb{R}^N)$. Then there exist two bounded measures μ and ν , an at most enumerable set of indices I , points $x_i \in \mathcal{A}$ (the critical set defined in (H3)), and positive real numbers $\mu_i, \nu_i, i \in I$, such that the following convergence hold weakly in the sense of measures,*

$$|\nabla u_n|^{p(x)} dx \rightharpoonup \mu \geq |\nabla u|^{p(x)} dx + \sum \mu_i \delta_{x_i}, \tag{4.1}$$

$$|u_n|^{q(x)} dx \rightharpoonup \nu := |u|^{q(x)} dx + \sum \nu_i \delta_{x_i}, \tag{4.2}$$

$$S_{x_i} \nu_i^{\frac{1}{p(x_i)^*}} \leq \mu_i^{\frac{1}{p(x_i)}} \quad \text{for all } i \in I, \tag{4.3}$$

where S_{x_i} is the localized Sobolev constant at the point x_i defined in (3.1). Moreover, if we define

$$\nu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{q(x)} dx,$$

$$\mu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^{p(x)} dx,$$

then

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} dx = \mu(\mathbb{R}^N) + \mu_\infty, \tag{4.4}$$

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q(x)} dx = \nu(\mathbb{R}^N) + \nu_\infty, \tag{4.5}$$

$$S_\infty \nu_\infty^{\frac{1}{q(\infty)}} \leq \mu_\infty^{\frac{1}{p(\infty)}}, \tag{4.6}$$

where S_∞ is the localized Sobolev constant at infinity defined in (3.2).

The first part of the theorem, namely (4.1)–(4.3), was proved in [10]. The 2nd part, namely (4.4)–(4.6) was proved in [42] but with a global Sobolev constant in (4.3). The main contribution of the present theorem is inequality (4.6) with the presence of the localized Sobolev constant at infinity. This will allow us to formulate in the next sections a local condition at infinity for the existence of a solution to (0.1).

Proof. Observe first that since ∞ is critical, we have in view of corollary 3.6 that $(u_n)_n$ is bounded in $L^{q(\cdot)}(\mathbb{R}^N)$ which implies the existence of the finite measure ν . We then refer to [10] for the proof of (4.1)–(4.3) and concentrate on (4.4)–(4.6).

A detailed proof of (4.4)–(4.5) can be found in [42][thm 2.5]. We briefly sketch it for the sake of completeness.

Consider a smooth function $\phi : [0, +\infty) \rightarrow [0, 1]$ such that $\phi \equiv 0$ in $[0, 1]$ and $\phi \equiv 1$ in $[2, +\infty)$. Then $\phi_R(x) := \phi(|x|/R)$ is smooth and satisfies $\phi_R(x) = 1$ for $|x| \geq 2R$, $\phi_R(x) = 0$ for $|x| \leq R$ and $0 \leq \phi_R(x) \leq 1$. We then write that

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} dx = \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} \phi_R dx + \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} (1 - \phi_R) dx. \tag{4.7}$$

Observe first that

$$\int_{\{|x|>2R\}} |\nabla u_n|^{p(x)} dx \leq \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} \phi_R^{p(x)} dx \leq \int_{\{|x|>R\}} |\nabla u_n|^{p(x)} dx$$

so that

$$\mu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} \phi_R^{p(x)} dx. \tag{4.8}$$

In the same way

$$\nu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q(x)} \phi_R^{q(x)} dx. \tag{4.9}$$

On the other hand, since $1 - \phi_R$ is smooth with compact support, we have by definition of μ that for R fixed,

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} (1 - \phi_R) |\nabla u_n|^{p(x)} dx = \int_{\mathbb{R}^N} (1 - \phi_R) d\mu.$$

Noticing that $\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^N} \phi_R d\mu = 0$ by dominated convergence, we obtain

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} (1 - \phi_R) |\nabla u_n|^{p(x)} dx = \mu(\mathbb{R}^N). \tag{4.10}$$

Plugging (4.8) and (4.10) into (4.7) yields (4.4). The proof of (4.5) is similar.

We prove (4.6). By definition of $S_R = S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R)$ we can write that

$$\begin{aligned} S_R \|u_n \phi_R\|_{L^{q(\cdot)}(\mathbb{R}^N \setminus B_R)} &\leq \|\nabla(u_n \phi_R)\|_{L^{p(\cdot)}(\mathbb{R}^N \setminus B_R)} \\ &\leq \|(\nabla u_n) \phi_R\|_{L^{p(\cdot)}(\mathbb{R}^N \setminus B_R)} \\ &\quad + \|u_n \nabla \phi_R\|_{L^{p(\cdot)}(B_{2R} \setminus B_R)}. \end{aligned} \tag{4.11}$$

Observe that since $u_n \rightarrow u$ in $L^p_{loc}(\mathbb{R}^N)$ we have

$$\begin{aligned} &\limsup_{n \rightarrow +\infty} \int_{B_{2R} \setminus B_R} |u_n|^{p(x)} |\nabla \phi_R|^{p(x)} dx \\ &= \int_{B_{2R} \setminus B_R} |u|^{p(x)} |\nabla \phi_R|^{p(x)} dx \\ &\leq C \| |u|^{p(\cdot)} \|_{L^{\frac{p^*(\cdot)}{p(\cdot)}}(B_{2R} \setminus B_R)} \| |\nabla \phi_R|^{p(\cdot)} \|_{L^{\frac{N}{p(\cdot)}}(\mathbb{R}^N)} \\ &\leq C \| |u|^{p(\cdot)} \|_{L^{\frac{p^*(\cdot)}{p(\cdot)}}(B_{2R} \setminus B_R)}. \end{aligned}$$

Since $u \in L^{p^*(\cdot)}$ we deduce that

$$\lim_{R \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \|u_n \nabla \phi_R\|_{L^{p(\cdot)}(B_{2R} \setminus B_R)} = 0. \tag{4.12}$$

Independently, letting $p_R^+ = \sup_{|x| \geq R} p(x)$ and $p_R^- = \inf_{|x| \geq R} p(x)$, we have

$$\begin{aligned} \|(\nabla u_n) \phi_R\|_{L^{p(\cdot)}(\mathbb{R}^N \setminus B_R)} &\leq \max \left\{ \left(\int_{\mathbb{R}^N} |(\nabla u_n) \phi_R|^{p(x)} dx \right)^{\frac{1}{p_R^+}}, \right. \\ &\quad \left. \left(\int_{\mathbb{R}^N} |(\nabla u_n) \phi_R|^{p(x)} dx \right)^{\frac{1}{p_R^-}} \right\} \end{aligned}$$

Given $\varepsilon > 0$ we then have for R big that

$$\begin{aligned} \|(\nabla u_n) \phi_R\|_{L^{p(\cdot)}(\mathbb{R}^N \setminus B_R)} &\leq \max \left\{ \left(\int_{\mathbb{R}^N} |(\nabla u_n) \phi_R|^{p(x)} dx \right)^{\frac{1}{p(\infty) - \varepsilon}}, \right. \\ &\quad \left. \left(\int_{\mathbb{R}^N} |(\nabla u_n) \phi_R|^{p(x)} dx \right)^{\frac{1}{p(\infty) + \varepsilon}} \right\}. \end{aligned}$$

Analogously,

$$\begin{aligned} \|u_n \phi_R\|_{L^{q(\cdot)}(\mathbb{R}^N \setminus B_R)} &\geq \min \left\{ \left(\int_{\mathbb{R}^N} |u_n \phi_R|^{q(x)} dx \right)^{\frac{1}{q^+}}, \right. \\ &\quad \left. \left(\int_{\mathbb{R}^N} |u_n \phi_R|^{q(x)} dx \right)^{\frac{1}{q^-}} \right\} \\ &\geq \min \left\{ \left(\int_{\mathbb{R}^N} |u_n \phi_R|^{q(x)} dx \right)^{\frac{1}{q(\infty)-\varepsilon}}, \right. \\ &\quad \left. \left(\int_{\mathbb{R}^N} |u_n \phi_R|^{q(x)} dx \right)^{\frac{1}{q(\infty)+\varepsilon}} \right\}. \end{aligned}$$

Assume for instance that there exists R_0 such that $\int_{\mathbb{R}^N} |(\nabla u_n) \phi_{R_0}|^{p(x)} dx < 1$ and $\int_{\mathbb{R}^N} |u_n \phi_{R_0}|^{q(x)} dx < 1$ so that $\int_{\mathbb{R}^N} |(\nabla u_n) \phi_R|^{p(x)} dx < 1$ and $\int_{\mathbb{R}^N} |u_n \phi_R|^{q(x)} dx < 1$ for all $R > R_0$. The others cases can be handled similarly. Then (4.11) gives for R big,

$$\begin{aligned} S_R \left(\int_{\mathbb{R}^N} |u_n \phi_R|^{q(x)} dx \right)^{\frac{1}{q(\infty)-\varepsilon}} &\leq \left(\int_{\mathbb{R}^N} |(\nabla u_n) \phi_R|^{p(x)} dx \right)^{\frac{1}{p(\infty)+\varepsilon}} \\ &\quad + \|u_n \nabla \phi_R\|_{L^{p(\cdot)}(B_{2R} \setminus B_R)}. \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and then as $R \rightarrow \infty$, we obtain in view of (4.8), (4.9) and (4.12) that

$$S_\infty \nu_\infty^{\frac{1}{q(\infty)-\varepsilon}} \leq \mu_\infty^{\frac{1}{p(\infty)+\varepsilon}}$$

for any $\varepsilon > 0$. We thus deduce (4.6). □

5. Local existence condition for equation (0.1)

In this section we use the CCP at infinity Theorem 4.1 to obtain local existence condition for an equation like (0.1) namely

$$-\Delta_{p(x)} u + \lambda k(x) |u|^{p(x)-2} u = K(x) |u|^{q(x)-2} u \quad \text{in } \mathbb{R}^N.$$

Besides assumptions (H1), (H2), (H3) we will also assume in this section that

- (H4) the nonlinearity is superlinear in the sense that $q^- > p^+$ where $p^+ = \sup_{\mathbb{R}^N} p$ and $q^- = \inf_{\mathbb{R}^N} q$.
- (H5) $k : \mathbb{R}^N \rightarrow [0, +\infty)$ is continuous bounded and such that the embedding from $D^{1,p(\cdot)}(\mathbb{R}^N)$ into $L^{p(\cdot)}(\mathbb{R}^N, k dx)$ is compact.
- (H6) $K : \mathbb{R}^N \rightarrow [0, +\infty)$ is continuous and has a limit at infinity $K(\infty) := \lim_{|x| \rightarrow +\infty} K(x)$.

A sufficient condition on k for the compact embedding in (H5) to hold is given in prop. 5.7 in ‘‘Appendix A’’. It follows in particular that the infimum $\inf_{u \in D^{1,p(\cdot)}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx}{\int_{\mathbb{R}^N} |u|^{p(x)} k dx}$ is positive. Then there exists $C, \lambda_0 > 0$ such that

for any $u \in D^{1,p(\cdot)}(\mathbb{R}^N)$ and any $\lambda > -\lambda_0$,

$$A \int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx + \lambda \int_{\mathbb{R}^N} k|u|^{p(x)} dx \geq C \int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx \tag{5.1}$$

where $A := \frac{1/p^+ - 1/q^-}{1/p^- - 1/q^+} \in (0, 1]$. Notice that $A = 1$ only when p is constant. In that case we can take for λ_0 the first eigenvalue in the problem $-\Delta_p u = \lambda k|u|^{p-2}u$ and the restriction $\lambda > -\lambda_0$ becomes standard. From now on we will assume that $\lambda > -\lambda_0$.

We will look for a solution of (0.1) as a critical point of the associated functional

$$\mathcal{F}(u) := \int_{\mathbb{R}^N} \frac{|\nabla u|^{p(x)}}{p(x)} + \lambda k(x) \frac{|u|^{p(x)}}{p(x)} - K(x) \frac{|u|^{q(x)}}{q(x)} dx, \quad u \in D^{1,p(\cdot)}(\mathbb{R}^N). \tag{5.2}$$

Recall that a sequence $(u_n)_{n \in \mathbb{N}} \subset D^{1,p(\cdot)}(\mathbb{R}^N)$ is a Palais-Smale sequence (P-S for short) for \mathcal{F} if the sequence $(\mathcal{F}(u_n))_n$ is bounded and $\mathcal{F}'(u_n) \rightarrow 0$ in $D^{1,p(\cdot)}(\mathbb{R}^N)'$. Moreover \mathcal{F} is said to satisfy the P-S condition at level c if any P-S sequence $(u_n)_n \subset D^{1,p(\cdot)}(\mathbb{R}^N)$ for \mathcal{F} such that $\mathcal{F}(u_n) \rightarrow c$ has a subsequence strongly convergent in $D^{1,p(\cdot)}(\mathbb{R}^N)$. We will prove that \mathcal{F} satisfies the P-S condition at level small enough depending on the localized Sobolev constant $S_x, x \in \mathcal{A}$, and S_∞ .

We first prove a preliminary lemma which is more or less classical.

Lemma 5.1. *Let $(u_n)_n \subset D^{1,p(\cdot)}(\mathbb{R}^N)$ be a Palais-Smale sequence for \mathcal{F} . Then, up to a subsequence, there exists $u \in D^{1,p(\cdot)}(\mathbb{R}^N)$ such that $u_n \rightarrow u$ weakly in $D^{1,p(\cdot)}(\mathbb{R}^N)$ and u is a weak solution of (0.1) with $\mathcal{F}(u) \geq 0$.*

Moreover letting $\mu, \nu, \mu_i, \nu_i, \mu_\infty, \nu_\infty$ be as in the concentration-compactness principle Theorem 4.1 when applied to $(u_n)_n$ we have the following estimates:

$$\nu_i \geq S_{x_i}^N K(x_i)^{-\frac{N}{p(x_i)}}, \quad \mu_i \geq S_{x_i}^N K(x_i)^{1-\frac{N}{p(x_i)}} \quad \text{if } K(x_i) > 0, \tag{5.3}$$

$$\mu_i = \nu_i = 0 \quad \text{if } K(x_i) = 0, \tag{5.4}$$

and a similar result at infinity:

$$\nu_\infty \geq S_\infty^N K(\infty)^{-\frac{N}{p(\infty)}}, \quad \mu_\infty \geq S_\infty^N K(\infty)^{1-\frac{N}{p(\infty)}} \quad \text{if } K(\infty) > 0, \tag{5.5}$$

$$\mu_\infty = \nu_\infty = 0 \quad \text{if } K(\infty) = 0. \tag{5.6}$$

Proof. The proof is more or less classical so we will be sketchy. We first prove that $(u_n)_n$ is bounded in $D^{1,p(\cdot)}(\mathbb{R}^N)$. Recalling the definition of a P-S sequence, it is easily seen that

$$\begin{aligned} C + o(1)\|u_n\|_{D^{1,p(\cdot)}(\mathbb{R}^N)} &\geq \mathcal{F}(u_n) - \frac{1}{q^-} \langle \mathcal{F}'(u_n), u_n \rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-} \right) \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} dx \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{q^-} \right) \lambda k|u_n|^{p(x)} dx. \end{aligned}$$

The claim will follow if we can prove that the rhs is greater than $C \int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx$. This is trivial if $\lambda \geq 0$. If $\lambda < 0$, writing that $\frac{1}{p(x)} - \frac{1}{q} \leq \frac{1}{p^-} - \frac{1}{q}$ and using (5.1), we see that the rhs is greater than

$$\left(\frac{1}{p^-} - \frac{1}{q}\right) \left\{ A \int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx + \lambda \int_{\mathbb{R}^N} k|u|^{p(x)} dx \right\} \geq C \int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx$$

from which we deduce the claim.

Up to a subsequence we can thus assume that $(u_n)_n$ weakly converges in $D^{1,p(\cdot)}(\mathbb{R}^N)$ to some u , and then also that the convergence holds in $L^{r(\cdot)}_{loc}(\mathbb{R}^N)$ for any strictly subcritical variable exponent r . Moreover since the embedding from $D^{1,p(\cdot)}(\mathbb{R}^N)$ into $L^{p(\cdot)}(k dx, \mathbb{R}^N)$ is compact, we also have that $u \in L^{p(\cdot)}(k dx, \mathbb{R}^N)$ and $u_n \rightarrow u$ in $L^{p(\cdot)}(k dx, \mathbb{R}^N)$.

Estimates (5.3)–(5.6) are a direct consequence of (4.3), (4.6) and the following:

$$\mu_i = \nu_i K(x_i) \quad \text{for any } i \in I, \tag{5.7}$$

and

$$\mu_\infty = K(\infty)\nu_\infty. \tag{5.8}$$

To prove (5.7), we fix a concentration point x_i , a smooth function $\phi : \mathbb{R}^N \rightarrow [0, 1]$ with compact support in B_2 such that $\phi = 1$ in B_1 , and consider $\phi_\delta(x) := \phi(|x - x_i|/\delta)$. Notice that the sequence $(u_n \phi_\delta)_n$ is bounded in $D^{1,p(\cdot)}(\mathbb{R}^N)$ since $(u_n)_n$ is bounded in $D^{1,p(\cdot)}(\mathbb{R}^N)$ and in $L^{p(\cdot)}(B_{x_i}(2\delta))$. We then write that

$$\begin{aligned} o(1) &= \langle \mathcal{F}'(u_n), u_n \phi_\delta \rangle \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} \phi_\delta dx - \int_{\mathbb{R}^N} K|u_n|^{q(x)} \phi_\delta dx \\ &\quad + \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_\delta dx \\ &\quad + \lambda \int_{\mathbb{R}^N} k|u_n|^{p(x)} \phi_\delta dx. \end{aligned}$$

Since $u_n \rightarrow u$ in $L^{p(\cdot)}_{loc}(\mathbb{R}^N)$ we obtain by passing to the limit $n \rightarrow +\infty$ that

$$\begin{aligned} &\int_{\mathbb{R}^N} \phi_\delta d\mu - \int_{\mathbb{R}^N} K \phi_\delta d\nu + \int_{\mathbb{R}^N} k|u|^{p(x)} \phi_\delta dx \\ &= - \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_\delta dx. \end{aligned} \tag{5.9}$$

Observe that

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} (u_n - u) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_\delta dx \right| \\ &\leq \frac{C}{\delta} \|u_n - u\|_{L^{p(\cdot)}(B_{x_i}(2\delta))} \| |\nabla u_n|^{p(x)-1} \|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\mathbb{R}^N)} \\ &\leq \frac{C}{\delta} \|u_n - u\|_{L^{p(\cdot)}(B_{x_i}(2\delta))} \end{aligned}$$

which goes to 0 as $n \rightarrow +\infty$ for δ fixed. Moreover

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} u |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_\delta \, dx \right| \\ & \leq \frac{C}{\delta} \left\| |\nabla u_n|^{p(x)-1} \right\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\mathbb{R}^N)} \|u\|_{L^{p^*(\cdot)}(B_{x_i}(2\delta))} \|1\|_{L^N(B_{x_i}(2\delta))} \\ & \leq C \|u\|_{L^{p^*(\cdot)}(B_{x_i}(2\delta))} \end{aligned}$$

where the constant C does not depend on n . We thus obtain from (5.9) that

$$\int_{\mathbb{R}^N} \phi_\delta \, d\mu - \int_{\mathbb{R}^N} K \phi_\delta \, d\nu + \lambda \int_{B_{x_i}(2\delta)} k |u|^{p(x)} \phi_\delta \, dx = O\left(\|u\|_{L^{p^*(\cdot)}(B_{x_i}(2\delta))}\right).$$

Letting $\delta \rightarrow 0$ then gives (5.7).

To prove (5.8), we fix $\phi : \mathbb{R}^N \rightarrow [0, 1]$ smooth such that $\phi \equiv 0$ in B_1 and $\phi \equiv 1$ in $\mathbb{R}^N \setminus B_2$, and then consider $\phi_R(x) := \phi(x/R)$. Notice that for a given $R > 0$, the sequence $(u_n \phi_R)_n$ is bounded in $D^{1,p(\cdot)}(\mathbb{R}^N)$ since $(u_n)_n$ is bounded in $D^{1,p(\cdot)}(\mathbb{R}^N)$ and also in $L^{p(\cdot)}(B_{2R} \setminus B_R)$. Then

$$\begin{aligned} o(1) &= \langle \mathcal{F}'(u_n), u_n \phi_R \rangle \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} \phi_R^{p(x)} \, dx - \int_{\mathbb{R}^N} K |u_n|^{q(x)} \phi_R^{q(x)} \, dx \\ &\quad + \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_R \, dx \\ &\quad + \lambda \int_{\mathbb{R}^N} k |u_n|^{p(x)} \phi_R^{p(x)} \, dx. \\ &=: A_n - B_n + C_n + \lambda D_n. \end{aligned} \tag{5.10}$$

In view of (4.8) and (4.9),

$$\lim_{R \rightarrow +\infty} \limsup_{n \rightarrow +\infty} A_n = \mu_\infty \quad \text{and} \quad \lim_{R \rightarrow +\infty} \limsup_{n \rightarrow +\infty} B_n = K(\infty) \nu_\infty.$$

Independently, since $u_n \rightarrow u$ in $L^{p(\cdot)}_{loc}(\mathbb{R}^N)$ we have for a fixed R that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} (u_n - u) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_R \, dx \right| \\ & \leq \frac{C}{R} \|u_n - u\|_{L^{p(\cdot)}(B_{2R} \setminus B_R)} \left\| |\nabla u_n|^{p(\cdot)-1} \right\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\mathbb{R}^N)} \\ & \leq \frac{C}{R} \|u_n - u\|_{L^{p(\cdot)}(B_{2R} \setminus B_R)} \end{aligned}$$

which goes to 0 as $n \rightarrow +\infty$. Moreover,

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} u |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_R \, dx \right| \\ & \leq \|u\|_{L^{p^*(\cdot)}(\mathbb{R}^N \setminus B_R)} \left\| |\nabla u_n|^{p(\cdot)-1} \right\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\mathbb{R}^N)} \|\nabla \phi_R\|_N \\ & \leq C \|u\|_{L^{p^*(\cdot)}(\mathbb{R}^N \setminus B_R)} \end{aligned}$$

which goes to 0 as $R \rightarrow +\infty$ uniformly in n . We deduce that

$$\lim_{R \rightarrow +\infty} \limsup_{n \rightarrow +\infty} C_n = 0.$$

Eventually, since $u_n \rightarrow u$ in $L^{p(\cdot)}(k \, dx, \mathbb{R}^N)$ and $u \in L^{p(\cdot)}(k \, dx, \mathbb{R}^N)$,

$$\begin{aligned} \lim_{R \rightarrow +\infty} \lim_{n \rightarrow +\infty} D_n &= \lim_{R \rightarrow +\infty} \int_{\mathbb{R}^N} k|u|^{p(x)} \phi_R^{p(x)} \, dx \\ &\leq \lim_{R \rightarrow +\infty} \int_{\{|x| \geq R\}} k|u|^{p(x)} \, dx = 0. \end{aligned}$$

Thus letting $n \rightarrow +\infty$ and then $R \rightarrow +\infty$ in (5.10) yields (5.8).

It follows from (5.3)–(5.4) that there are at most a finite number of concentration points. Mimicking the proof of [43][Thm 3.1], we can prove that $\nabla u_n \rightarrow \nabla u$ a.e. and then that u is a weak solution of (0.1). Recalling that $p^+ < q^-$ and that K is non-negative it is then easily seen that $\mathcal{F}(u) \geq 0$. Indeed

$$\begin{aligned} \mathcal{F}(u) &\geq \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \lambda k(x)|u|^{p(x)} \right) \, dx - \frac{1}{q^-} \int_{\mathbb{R}^N} K(x)|u|^{q(x)} \, dx \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{q^-} \right) \left(|\nabla u|^{p(x)} + \lambda k(x)|u|^{p(x)} \right) \, dx. \end{aligned}$$

It is easily proved, in the same way we proved that $(u_n)_n$ is bounded in $D^{1,p(\cdot)}(\mathbb{R}^N)$, that the r.h.s is greater than $C \int_{\mathbb{R}^N} |\nabla u|^{p(x)} \, dx$ which is non-negative. \square

The previous lemma gives easily the following

Proposition 5.2. *The functional \mathcal{F} defined in (5.2) verifies the PS condition at level c for all real numbers c satisfying*

$$c < \inf_{x \in \mathcal{A} \cup \{\infty\}, K(x) > 0} \frac{1}{N} S_x^N K(x)^{1 - \frac{N}{p(x)}}.$$

Proof. Let $(u_n)_n$ be a PS sequence for \mathcal{F} of level c . Up to a subsequence, we can assume that $(u_n)_n$ converges to some u weakly in $D^{1,p(\cdot)}(\mathbb{R}^N)$ and strongly in $L^{p(\cdot)}(\mathbb{R}^N)$. Let $\mu, \nu, \mu_i, \nu_i, \mu_\infty, \nu_\infty$ be as in the concentration-compactness principle Theorem. 4.1 when applied to $(u_n)_n$. Then

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \mathcal{F}(u_n) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \, d\mu + \int_{\mathbb{R}^N} \frac{k(x)}{p(x)} |u|^{p(x)} \, dx - \int_{\mathbb{R}^N} \frac{1}{q(x)} \, d\nu \\ &\geq \mathcal{F}(u) + \sum_i \left\{ \frac{\mu_i}{p(x_i)} - \frac{K(x_i)\nu_i}{p(x_i)^*} \right\} + \frac{\mu_\infty}{p(\infty)} - \frac{K(\infty)\nu_\infty}{q(\infty)}. \end{aligned}$$

Since $\mathcal{F}(u) \geq 0$, we deduce, using (5.7), (5.8) and (5.3)–(5.6), that

$$c \geq \sum_{x \in \{x_i, i \in I\} \cup \{\infty\}, K(x) > 0} \frac{1}{N} S_x^N K(x)^{1 - \frac{N}{p(x)}}$$

from which the result follows. \square

A direct application of the Mountain-pass theorem combined with proposition 5.2 then yields the following existence condition:

Theorem 5.3. *Assume (H1)–(H6) and that there exists $v \in D^{1,p(\cdot)}(\mathbb{R}^N)$ such that*

$$\sup_{t>0} \mathcal{F}(tv) < \inf_{x \in \mathcal{A} \cup \{\infty\}, K(x) > 0} \frac{1}{N} S_x^N K(x)^{1 - \frac{N}{p(x)}}. \tag{5.11}$$

Then there exists $\lambda^ > 0$ such that (0.1) has a non-trivial solution for any $\lambda > -\lambda^*$.*

Notice that if ∞ is subcritical, i.e. $q(\infty) < p(\infty)^*$, then $S_\infty = 0$ in view of prop. 3.5. In that case the right hand side in (5.11) becomes non-positive and the Theorem becomes useless. The reason is that (5.5) reduces to $\mu_\infty, \nu_\infty \geq 0$ which gives no information about the loss of mass at infinity. To solve this problem, the definition of S_∞ must be modified. The results we obtained in that case will be presented elsewhere.

Proof. Notice first that \mathcal{F} has the mountain-pass geometry. Indeed $\mathcal{F}(0) = 0$ and, if $\|u\|_{D^{1,p(\cdot)}(\mathbb{R}^N)} = r$ is small enough, then using the coercivity assumption (5.1) and Cor. 3.6, we have

$$\mathcal{F}(u) \geq \frac{1}{p^+} \int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx + \int_{\mathbb{R}^N} \frac{\lambda}{p(x)} k|u|^{p(x)} dx - \frac{1}{q^-} \max \left\{ \|u\|_{q(\cdot)}^{q^+}, \|u\|_{q(\cdot)}^{q^-} \right\}$$

so that

$$p^+ \mathcal{F}(u) \geq \int_{\mathbb{R}^N} |\nabla u|^{p(x)} + \frac{\lambda p^+}{p(x)} k|u|^{p(x)} dx - C \max \left\{ \|u\|_{D^{1,p(\cdot)}(\mathbb{R}^N)}^{q^+}, \|u\|_{D^{1,p(\cdot)}(\mathbb{R}^N)}^{q^-} \right\}.$$

Notice that the integral in the rhs is greater than $C \int_{\mathbb{R}^N} |\nabla u|^{p(x)} dx$ if $\lambda > -\frac{\lambda_0 p^-}{A p^+} =: -\lambda^*$. Thus $\mathcal{F}(u) \geq Cr^{p^+} - Cr^{q^-}$ which is positive by (H4).

Let $v \in D^{1,p(\cdot)}(\mathbb{R}^N)$ satisfying (5.11). It easily follows from (H4) that $\mathcal{F}(tv) < 0$ for $t > 0$ big. In view of proposition 5.2, we can then apply the Mountain-pass theorem to obtain a critical point of \mathcal{F} . \square

Proposition 5.4. *The infimum in the r.h.s. of (5.11) is attained at some point of $\mathcal{A} \cup \{\infty\}$.*

Proof. We first prove that the function $x \in \mathcal{A} \rightarrow S_x$ is lower semi-continuous. Consider $x_0 \in \mathcal{A}$, $(x_n)_n \subset \mathcal{A}$ such that $x_n \rightarrow x_0$ and fix some $\varepsilon > 0$. There exists $n(\varepsilon) \in \mathbb{N}$ such that $B_{x_n}(\varepsilon/3) \subset B_{x_0}(\varepsilon)$ for $n \geq n(\varepsilon)$. It follows that

$$S(p(\cdot), q(\cdot), B_{x_0}(\varepsilon)) \leq S(p(\cdot), q(\cdot), B_{x_n}(\varepsilon/3)) \leq S_{x_n}$$

for $n \geq n(\varepsilon)$. Then $\liminf_{n \rightarrow +\infty} S_{x_n} \geq S(p(\cdot), q(\cdot), B_{x_0}(\varepsilon))$ for any ε . Letting $\varepsilon \rightarrow 0$ gives $\liminf_{n \rightarrow +\infty} S_{x_n} \geq S_{x_0}$.

To prove the proposition, it thus suffices to prove that this function is also lower semi-continuous at infinity in the sense that for any sequence $x_n \in \mathcal{A}$ such that $|x_n| \rightarrow +\infty$, there holds $\liminf_{n \rightarrow +\infty} S_{x_n} \geq S_\infty$. Fix some $R > 0$ and then $n_0 \in \mathbb{N}$ such that $|x_n| \geq R + 1$. Then for $n \geq n_0$, $B_{x_n}(\varepsilon) \subset \mathbb{R}^N \setminus B_R$ for any $\varepsilon < 1$. It follows that for such n and ε , $S(p(\cdot), q(\cdot), B_{x_n}(\varepsilon)) \geq S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R)$. Taking the limit in ε and then in n gives $\liminf_{n \rightarrow +\infty} S_{x_n} \geq S(p(\cdot), q(\cdot), \mathbb{R}^N \setminus B_R)$. We obtain the result taking the limit $R \rightarrow +\infty$. \square

Assume that the infimum in the right hand side of (5.11) is attained at a point $x_0 \in \mathcal{A}$. According to the result in [9], the condition (5.11) will hold for some v if p and q are C^2 in a neighborhood of x_0 , p (resp. q) has a local minimum (resp. maximum) at x_0 and at least one of the hessian matrix $D^2p(x_0)$ or $D^2q(x_0)$ is non-zero. We will now see that the same kind of result holds if the infimum in the right hand side of (5.11) is attained at infinity. To do this we need the following definition:

Definition 5.5. We say that a function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ has a Taylor expansion of order 2 at ∞ , or equivalently that f is C^2 at infinity, if there exists $f(\infty), a_1, \dots, a_N \in \mathbb{R}$ and $a_{i,j} \in \mathbb{R}, i, j = 1, \dots, N$, such that

$$f(x) = f(\infty) + \sum_{i=1}^N a_i \frac{x_i}{|x|^2} + \frac{1}{2} \sum_{i,j=1}^N a_{i,j} \frac{x_i x_j}{|x|^4} + o\left(\frac{1}{|x|^2}\right), \quad |x| \gg 1. \quad (5.12)$$

We say that a_1, \dots, a_N are the first order partial derivatives of f at ∞ and that the $a_{i,j}$ are the second-order partial derivatives of f at ∞ .

We denote $\partial_i f(\infty) := a_i, \nabla f(\infty) := (\partial_1 f(\infty), \dots, \partial_N f(\infty))$ the gradient of f at $\infty, \partial_{ij} f(\infty) := a_{i,j}$ and $D^2 f(\infty) := (\partial_{ij} f(\infty))_{i,j=1,\dots,N} \in \mathbb{R}^{N \times N}$ the Hessian matrix of f at ∞ .

Notice that the usual algebraic rules of computations of partial derivatives apply for the partial derivatives at ∞ .

This definition is justified by the following argument. Denoting by $\Phi_N : \mathbb{S}^N - \{P\} \rightarrow \mathbb{R}$ the stereographic projection from the north pole P , we can define $\tilde{f} : \mathbb{S}^N \rightarrow \mathbb{R}$ by $\tilde{f} = f \circ \Phi_N$. Then, if there exists $f(\infty) := \lim_{|x| \rightarrow \infty} f(x)$, \tilde{f} is continuous on \mathbb{S}^N , and studying f near ∞ is equivalent to study \tilde{f} near P . To do so we consider the chart $(\mathbb{S}^N - \{S\}, \Phi_S)$ around P , where Φ_S is the stereographic projection from the south pole S . Then, in this chart, assuming \tilde{f} of class C^2 , we can write a Taylor expansion of \tilde{f} at $0 = \Phi_S(N)$:

$$\tilde{f}(x) = a_0 + \sum_{i=1}^N a_i x_i + \frac{1}{2} \sum_{i,j=1}^N a_{ij} x_i x_j + o(|x|^2), \quad x \in \mathbb{S}^N \text{ close to } P, \quad (5.13)$$

where x_i is the i -th component of $\Phi_S(x)$. Notice that $a_i = \partial_i \tilde{f}(0)$ and $a_{ij} = \partial_{ij} \tilde{f}(0)$. Then (5.13) can be rewritten as

$$f(x) = a_0 + \sum_{i=1}^N a_i (\Phi_S(\Phi_N^{-1}(x)))_i + \frac{1}{2} \sum_{i,j=1}^N a_{ij} (\Phi_S(\Phi_N^{-1}(x)))_i (\Phi_S(\Phi_N^{-1}(x)))_j + o(|\Phi_S(\Phi_N^{-1}(x))|^2)$$

for $x \in \mathbb{R}^N, |x| \gg 1$. Since $\Phi_S(\Phi_N^{-1}(x)) = \frac{x}{|x|^2}$, we see that (5.13) is equivalent to (5.12).

Assume that f has a limit at ∞ in the sense that there exists $f(\infty) := \lim_{|x| \rightarrow \infty} f(x)$. We say that f has a local minimum (resp. maximum) at ∞

if $f(x) \geq f(\infty)$ (resp. $f(x) \leq f(\infty)$) for $|x| \gg 1$. If this is the case and if moreover f has a Taylor expansion of order 2, then $\nabla f(\infty) = 0$ and the hessian matrix $D^2 f(\infty)$ is nonnegative (resp. non-positive).

We can now state our existence result for equation (0.1).

Theorem 5.6. *Assume that assumptions (H1)–(H6) hold, that $N \geq 4$, and that among the points of $\mathcal{A} \cup \infty$ at which the infimum in (5.11) is attained, there is one, say x_0 , such that $p(x_0) < \sqrt{N}$ and p and q are C^2 at x_0 and have respectively a local minimum and a local maximum) at x_0 . Assume moreover that*

- if $x_0 \neq \infty$, then $\lambda k(x_0) < 0$ if $p(x_0) \in (1, 2)$ or, if $p(x_0) \in [2, \sqrt{N}]$ with $N \geq 5$, then at least one of the hessian matrices $D^2 p(x_0)$ or $D^2 q(x_0)$ is non-zero,
- if $x_0 = \infty$ then for $|x| \gg 1$, $k(x) = O(|x|^{-\beta})$ for some $\beta > 2$, and at least one of the hessian matrices $D^2 p(\infty)$ or $D^2 q(\infty)$ is non-zero.

Then there exists $\lambda^* > 0$ such that equation (0.1) has a non-trivial solution for $\lambda > -\lambda^*$.

Proof. When $x_0 \neq \infty$, the statement is a consequence of [9]. We thus assume from now on that $x_0 = \infty$. For ease of notation we let $p := p(\infty)$ and $q := q(\infty)$. In view of our assumption we can apply Theorem. 3.3 to obtain

$$S_\infty = K(N, p)^{-1}.$$

It is well-known that the extremals for $K(N, p)^{-1}$ are of the form

$$U_{\lambda, x_0}(x) := c\varepsilon^{\frac{N-p}{p}} U\left(\varepsilon(x - x_0)\right), \quad x_0 \in \mathbb{R}^N, \varepsilon > 0, c \in \mathbb{R}^*,$$

where $U(x) = (1 + |x|^{\frac{p}{p-1}})^{\frac{p-N}{p}}$. Given some direction $\nu \in \mathbb{R}^N$, $|\nu| = 1$, and positive real numbers r_ε as in Remark 5.14 below, consider the function u_ε defined by (5.19) in ‘‘Appendix B’’. Letting $C := K(N, p)^{-\frac{N}{q}} \|U\|_q^{-1}$, it is easily seen that $W := CU$ satisfies $-\Delta_p W = W^{q-1}$. We then consider $w_\varepsilon(x) := Cu_\varepsilon(x)$ as test-function in place of v in (5.11). In view of the definition of \mathcal{F} in (5.2) we then have

$$\begin{aligned} J_\varepsilon(t) := \mathcal{F}(tw_\varepsilon) &= \int_{\mathbb{R}^N} g(x, t) |\nabla u_\varepsilon|^{p(x)} dx + \int_{\mathbb{R}^N} h(x, t) |u_\varepsilon|^{p(x)} dx \\ &\quad - \int_{\mathbb{R}^N} f(x, t) |u_\varepsilon|^{q(x)} dx \end{aligned}$$

with

$$g(x, t) = \frac{(tC)^{p(x)}}{p(x)}, \quad f(x, t) = \frac{(tC)^{q(x)} K(x)}{q(x)}, \quad h(x, t) = \frac{(tC)^{p(x)} k(x)}{p(x)}.$$

Consider

$$J_0(t) := \frac{(tC)^p}{p} \|\nabla U\|_p^p - \frac{(tC)^q K(\infty)}{q} \|U\|_q^q = K(N, p)^{-N} \left(\frac{t^p}{p} - \frac{t^q}{q} K(\infty) \right)$$

and

$$\begin{aligned} J_1(t) &:= \frac{N(tC)^q K(\infty)}{2q^2} A(q, \nu) \|U\|_q^q - \frac{N(tC)^p}{2p^2} A(p, \nu) \|\nabla U\|_p^p \\ &= \frac{n}{2} K(N, p)^{-N} \left(\frac{t^q}{q^2} K(\infty) A(q, \nu) - \frac{t^p}{p^2} A(p, \nu) \right). \end{aligned}$$

where

$$A(p, \nu) := \sum_{i,j=1}^N \partial_{i,j} p(\infty) \nu_i \nu_j = (D^2 p(\infty) \nu, \nu)$$

and

$$A(q, \nu) := \sum_{i,j=1}^N \partial_{i,j} q(\infty) \nu_i \nu_j = (D^2 q(\infty) \nu, \nu).$$

Using prop. 5.8, 5.10 and 5.12 and Remark 5.14 in ‘‘Appendix B’’, we then obtain that

$$J_\varepsilon(t) = J_0(t) + J_1(t) \frac{|\ln \varepsilon|}{|x_\varepsilon|^2} + o\left(\frac{|\ln \varepsilon|}{|x_\varepsilon|^2}\right) \tag{5.14}$$

C^1 uniformly in t for t in a compact subset of $[0, +\infty)$. Observe that J_0 attains its maximum at $t_0 := K(\infty)^{\frac{1}{p-q}}$ with $J_0(t_0) = \frac{1}{N} K(N, p)^{-N} K(\infty)^{-\frac{N-p}{p}}$. Moreover this is a non-degenerate maximum since $J_0''(t_0) = (p-q)K(N, p)^{-N} K(\infty)^{\frac{2-p}{q-p}} \neq 0$. It follows that J_ε attains its maximum at $t_\varepsilon = t_0 + a \frac{|\ln \varepsilon|}{|x_\varepsilon|^2} + o\left(\frac{|\ln \varepsilon|}{|x_\varepsilon|^2}\right)$ with $a = -\frac{J_1'(t_0)}{J_0''(t_0)}$. We thus obtain

$$\sup_{t \geq 0} \mathcal{F}(tw_\varepsilon) = \sup_{t \geq 0} J_\varepsilon(t) = J_\varepsilon(t_\varepsilon) = J_0(t_0) + J_1(t_0) \frac{|\ln \varepsilon|}{|x_\varepsilon|^2} + o\left(\frac{|\ln \varepsilon|}{|x_\varepsilon|^2}\right)$$

where

$$J_1(t_0) = \frac{N}{2} K(N, p)^{-N} K(\infty)^{\frac{p}{p-q}} \left(\frac{1}{q^2} A(q, \nu) - \frac{1}{p^2} A(p, \nu) \right).$$

Recall that for any direction ν , $A(p, \nu) \geq 0 \geq A(q, \nu)$ since $p(\cdot)$ and $q(\cdot)$ have a local minimum and maximum respectively at infinity. It follows in particular that $J_1(t_0) \leq 0$ for any direction ν . For the strict inequality to hold, it thus suffices to assume that there exists a direction ν such that $A(p, \nu) > 0$ or $A(q, \nu) < 0$ i.e. to assume that one of the hessian matrices $D^2 p(\infty)$ or $D^2 q(\infty)$ is non-zero. If this holds we then have that $J_1(t_0) < 0$ and thus (5.11) holds. \square

Appendix A: A compact embedding from $D^{1,p(\cdot)}(\mathbb{R}^N)$ into $L^{p(\cdot)}(\mathbb{R}^N, k \, dx)$

In this section, we give a sufficient condition for assumption (H6) to hold i.e. to have a compact embedding from $D^{1,p(\cdot)}(\mathbb{R}^N)$, the closure of $C_c^\infty(\mathbb{R}^N)$ for the norm $\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)}$, into the weighted Lebesgue space $L^{p(\cdot)}(\mathbb{R}^N, k \, dx)$. Such a result was proved in the constant-exponent framework in [15][lemma 1]. We adapt his proof to the variable exponent setting.

Proposition 5.7. *Consider a measurable function $s : \mathbb{R}^N \rightarrow [1, +\infty]$ such that $s \geq p$ and $\inf_{\mathbb{R}^N} p^* - s > 0$. Assume that $k \in L^1(\mathbb{R}^N) \cap L^{\frac{s(\cdot)}{s(\cdot)-p(\cdot)}}_{loc}(\mathbb{R}^N)$ is non-negative and satisfies*

$$\lim_{|x| \rightarrow \infty} \|k\|_{L^{\frac{s(\cdot)}{s(\cdot)-p(\cdot)}}(Q(x,l))} = 0$$

for some $l > 0$. Here $Q(x,l)$ denotes the cube centered at x with sides of length l parallel to the axis. Then $D^{1,p(\cdot)}(\mathbb{R}^N)$ is compactly embedded into $L^{p(\cdot)}(\mathbb{R}^N, k dx)$.

The proof follows closely the proof [15][lemma 1] with the following useful trick due to P. Hasto [25][Thm 2.4] in the final step. Given a sequence $(q_j)_j$ in $(1, +\infty)$, consider the space $\ell^{(q_j)}$ of sequences $x = (x_j)_j$ such that $\rho(x) := \sum_j |x_j|^{q_j} < \infty$. This space is endowed with the norm $\|x\|_{\ell^{(q_j)}} := \inf \{\lambda > 0 \text{ s.t. } \rho(x/\lambda) \leq 1\}$. A. Nekvinda [31][Thm 4.3] proved that if there exists $q(\infty)$ such that $|q_j - q(\infty)| \leq C/\ln(e+j)$ then $\ell^{(q_j)} \approx \ell^{q(\infty)}$. Using this result Hasto [25][Thm 2.4] proved that given a partition of \mathbb{R}^N by cubes $Q_l, l \geq 1$, of same length satisfying

$$dist(Q_k, 0) < dist(Q_l, 0) \quad \text{if } k < l, \tag{5.15}$$

and if p is log-Holder continuous exponent such that

$$|p(x) - p(\infty)| \leq C/\ln(e + |x|) \quad \text{for all } x \in \mathbb{R}^N, \tag{5.16}$$

then

$$\|g\|_{L^{p(\cdot)}(\mathbb{R}^N)} \approx \|(\|g\|_{L^{p(\cdot)}(Q_j)})_j\|_{\ell^{(q_j)}} \approx \|(\|g\|_{L^{p(\cdot)}(Q_j)})_j\|_{\ell^{p(\infty)}}, \tag{5.17}$$

where $q_j = p^+_{Q_j}$. Actually the first \approx follows from the first part of the proof of [25][Thm 2.4]. It is easily seen that we can take q_j of the form $q_j = p(x_j)$ where $x_j \in Q_j$. In particular we will use this result with $q_j = p^-_{Q_j}$.

Proof. Without loss of generality we may assume that $l = 1$. We first verify that it is sufficient to show that for every $\delta > 0$ there exists $j > 0$ such that

$$\|f - f\chi_{Q(0,j)}\|_{L^{p(\cdot)}(\mathbb{R}^N, k dx)} < \delta \tag{5.18}$$

for all $f \in D^{1,p(\cdot)}(\mathbb{R}^N)$ such that $\|\nabla f\|_{L^{p(\cdot)}(\mathbb{R}^N)} \leq 1$. Here $\chi_{Q(0,j)}$ denotes the characteristic function of the cube $Q(0,j)$. In fact, assume that (5.18) holds and take a bounded sequence $(f_n)_n$ in $D^{1,p(\cdot)}(\mathbb{R}^N)$. Since s is assumed to be strictly subcritical, we can assume that there exists $f \in D^{1,p(\cdot)}(\mathbb{R}^N)$ such that $f_m \rightarrow f$ in $L^{s(\cdot)}(Q(0,R))$ for any $R > 0$ and $\nabla f_m \rightharpoonup \nabla f$ in $L^{p(\cdot)}(\mathbb{R}^N)$. It follows from (5.18) that

$$\|f_m - f\|_{L^{p(\cdot)}(\mathbb{R}^N \setminus Q(0,j), k dx)} < 2\delta.$$

On the other hand we have using Hölder inequality that

$$\begin{aligned} \int_{Q(0,j)} |f_m - f|^{p(\cdot)} k dx &\leq \| |f_m - f|^{p(\cdot)} \|_{L^{\frac{s(\cdot)}{p(\cdot)}}(Q(0,j))} \|k\|_{L^{\frac{s(\cdot)}{s(\cdot)-p(\cdot)}}(Q(0,j))} \\ &\leq C \max \left\{ \|f_m - f\|_{L^{s(\cdot)}(Q(0,j))}^+; \|f_m - f\|_{L^{s(\cdot)}(Q(0,j))}^- \right\} \end{aligned}$$

which goes to 0 as $m \rightarrow +\infty$ since $f_m \rightarrow f$ in $L^{s(\cdot)}(Q(0, j))$, We can thus conclude that $f_m \rightarrow f$ in $L^{s(\cdot)}(\mathbb{R}^N)$.

We now prove (5.18). We fix a function $f \in D^{1,p(\cdot)}(\mathbb{R}^N)$ such that $\|\nabla f\|_{L^{p(\cdot)}(\mathbb{R}^N)} \leq 1$, and consider a partition of \mathbb{R}^N by cubes Q_l , $l \geq 1$, of unit side length satisfying (5.15). Independently, given $\eta > 0$ we fix $j \in \mathbb{N}$ such that

$$\|k\|_{L^1(\mathbb{R}^N \setminus Q(0, j))} \leq \eta$$

and

$$\|k\|_{L^{\frac{s(\cdot)}{s(\cdot)-p(\cdot)}}(Q_l)} < \eta \quad \text{for every } Q_l \text{ outside } Q(0, j).$$

Denote $f_{Q_l} := \int_{Q_l} f(x) dx$ the mean-value of f on Q_l . Then by Sobolev and Poincaré inequality there exists a constant C independent of f and Q_l such that

$$\|f - f_{Q_l}\|_{L^{s(\cdot)}(Q)} \leq C\|\nabla f\|_{L^{p(\cdot)}(Q_l)}$$

and

$$|f_{Q_l}| \leq C\|f\|_{L^{p^*(\cdot)}(Q_l)} \leq C\|f\|_{L^{p^*(\cdot)}(\mathbb{R}^N)} \leq C\|\nabla f\|_{L^{p(\cdot)}(\mathbb{R}^N)} = C.$$

Let $p_l^+ := \max_{Q_l} p$ and $p_l^- := \min_{Q_l} p$. It follows that for any cube Q_l we have

$$\begin{aligned} \int_{Q_l} |f|^{p(\cdot)} k dx &\leq 2^{p^+-1} \int_{Q_l} |f - f_{Q_l}|^{p(\cdot)} k dx + 2^{p^+-1} \int_{Q_l} |f_{Q_l}|^{p(\cdot)} k dx \\ &= C\| |f - f_{Q_l}|^{p(\cdot)} \|_{L^{\frac{s(\cdot)}{p(\cdot)}}(Q_l)} \|k\|_{L^{\frac{s(\cdot)}{s(\cdot)-p(\cdot)}}(Q_l)} + C \int_{Q_l} k dx \\ &\leq C \max \left\{ \|\nabla f\|_{L^{p(\cdot)}(Q_l)}^{p_j^+}; \|\nabla f\|_{L^{p(\cdot)}(Q_l)}^{p_j^-} \right\} \eta + C \int_{Q_l} k dx. \end{aligned}$$

Since $\|\nabla f\|_{L^{p(\cdot)}(\mathbb{R}^N)} \leq 1$, we eventually obtain for any cube Q_l that

$$\int_{Q_l} |f|^{p(\cdot)} r dx \leq C\|\nabla f\|_{L^{p(\cdot)}(Q_l)}^{p_j^-} \eta + C \int_{Q_l} k dx.$$

Summing these inequalities over all the cubes Q_l outside $Q(0, j)$ using (5.17) to handle the first term, we obtain

$$\int_{\mathbb{R}^N \setminus Q(0, j)} |f|^{p(\cdot)} k dx \leq C\eta + C \int_{\mathbb{R}^N \setminus Q(0, j)} k dx \leq C\eta$$

which is (5.18). □

Appendix B: Test-functions computations at infinity

Let $U(x) = (1 + |x|^{\frac{p(\infty)}{p(\infty)-1}})^{-\frac{N-p(\infty)}{p(\infty)}}$. It is well-known that U is an extremal for $K(N, p(\infty))^{-1}$. Given a cut-off function $\eta \in C_c^\infty([0, +\infty), [0, 1])$ supported in $[0, 2]$ such that $\eta \equiv 1$ in $[0, 1]$, points $x_\varepsilon \in \mathbb{R}^N$ of the form $x_\varepsilon := r_\varepsilon \nu$ where

$\nu \in \mathbb{R}^N$, $|\nu| = 1$, and $r_\varepsilon = |x_\varepsilon| \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, and a real number $\alpha > 1$ to be specified later, we consider the test function

$$u_\varepsilon(x) = \varepsilon^{\frac{N-p(\infty)}{p(\infty)}} U(\varepsilon(x - x_\varepsilon))\eta(\varepsilon^\alpha|x - x_\varepsilon|), \quad \varepsilon \ll 1. \tag{5.19}$$

We first compute the expansion of a term like $\int f|u_\varepsilon|^{q(x)} dx$.

Proposition 5.8. *Assume that q and $f \in C(\mathbb{R}^N)$ have a Taylor expansion at ∞ of order 2 with $\nabla f(\infty) = \nabla q(\infty) = 0$ and $p(\infty) \leq \frac{N+2}{2}$. Assume also that r_ε and $\alpha > 1$ are such that*

$$\varepsilon^{-2\alpha} \ll |x_\varepsilon|^2 \ll \frac{|\ln \varepsilon|}{\varepsilon^{\frac{N(\alpha-1)}{p(\infty)-1} - \delta}}. \tag{5.20}$$

for some small $\delta > 0$. Then

$$\int_{\mathbb{R}^N} f|u_\varepsilon|^{q(x)} dx = A_0 + A_1 \frac{|\ln \varepsilon|}{|x_\varepsilon|^2} + o\left(\frac{|\ln \varepsilon|}{|x_\varepsilon|^2}\right), \tag{5.21}$$

where, denoting $A(q, \nu) := \sum_{i,j=1}^n \partial_{i,j} q(\infty) \nu_i \nu_j = (D^2 q(\infty) \nu, \nu)$,

$$A_0 = f(\infty) \int_{\mathbb{R}^N} U^{q(\infty)}(x) dx \quad \text{and}$$

$$A_1 = -\frac{Nf(\infty)}{2q(\infty)} A(q, \nu) \int_{\mathbb{R}^N} U^{q(\infty)}(x) dx.$$

Remark 5.9. Notice first that $|x_\varepsilon| \gg \varepsilon^{-\alpha} \gg |\ln \varepsilon|$ so that $\frac{|\ln \varepsilon|}{|x_\varepsilon|^2} \rightarrow 0$. Independently it is possible to choose r_ε and $\alpha > 1$ satisfying (5.20). Indeed it suffices to choose $\alpha > 1$ so that $\varepsilon^{-2\alpha} \ll \varepsilon^{\delta - \frac{N(\alpha-1)}{p(\infty)-1}}$. Notice now that since $p(\infty) < \frac{N+2}{2}$, choosing $\alpha > \frac{N}{N-2p(\infty)+2}$ implies that $\frac{N(\alpha-1)}{p(\infty)-1} - 2\alpha > 0$ which gives the result. If we make the more restrictive assumption that $p(\infty) \leq \sqrt{N}$, then $\frac{N}{N-2p(\infty)+2} \leq 1$. In that case it thus suffices to take $\alpha > 1$.

Proof. First

$$\int_{\mathbb{R}^N} f|u_\varepsilon|^{q(x)} dx = \int_{B(\varepsilon^{1-\alpha})} f(x_\varepsilon + x/\varepsilon) \varepsilon^{N\left(\frac{q(x_\varepsilon+x/\varepsilon)}{q(\infty)} - 1\right)} U(x)^{q(x_\varepsilon+x/\varepsilon)} dx + R_\varepsilon \tag{5.22}$$

with

$$R_\varepsilon = \int_{B(2\varepsilon^{1-\alpha}) \setminus B(\varepsilon^{1-\alpha})} f(x_\varepsilon + x/\varepsilon) \varepsilon^{N\left(\frac{q(x_\varepsilon+x/\varepsilon)}{q(\infty)} - 1\right)} U(x)^{q(x_\varepsilon+x/\varepsilon)} \eta(\varepsilon^{\alpha-1}|x|) dx.$$

We estimate the different terms appearing in these integrals. All the expansion will be uniform in $|x| \leq 2\varepsilon^{1-\alpha}$ i.e. $\frac{|x|}{\varepsilon} \leq 2\varepsilon^{-\alpha}$. Recall that $|x_\varepsilon| \gg \frac{|x|}{\varepsilon}$ uniformly for $\frac{|x|}{\varepsilon} \leq 2\varepsilon^{-\alpha}$ by (5.20).

First, for $|x| \leq 2\varepsilon^{1-\alpha}$, $|x_\varepsilon + x/\varepsilon|^2 = |x_\varepsilon|^2 \left(1 + O\left(\frac{1}{\varepsilon^\alpha|x_\varepsilon|}\right)\right)$ with $\varepsilon^\alpha|x_\varepsilon| \gg 1$. So,

$$|x_\varepsilon + x/\varepsilon|^{-4} = |x_\varepsilon|^{-4} \left(1 + O\left(\frac{1}{\varepsilon^\alpha|x_\varepsilon|}\right)\right). \tag{5.23}$$

independently, since q has a Taylor expansion at ∞ of order 2 with $\nabla q(\infty) = 0$, we have

$$q(x) = q(\infty) + \frac{1}{2} \sum_{i,j=1}^n \partial_{i,j} q(\infty) \frac{x_i x_j}{|x|^4} + o\left(\frac{1}{|x|^2}\right) \quad \text{for } |x| \gg 1.$$

We thus have uniformly in $|x| \leq 2\varepsilon^{1-\alpha}$ that

$$\begin{aligned} & q(x_\varepsilon + x/\varepsilon) - q(\infty) \\ &= \frac{1}{2} \sum_{i,j=1}^N \partial_{i,j} q(\infty) (x_\varepsilon^i + x^i/\varepsilon)(x_\varepsilon^j + x^j/\varepsilon) |x_\varepsilon|^{-4} \left(1 + O\left(\frac{1}{\varepsilon^\alpha |x_\varepsilon|}\right)\right) + o\left(\frac{1}{|x_\varepsilon|^2}\right) \\ &= \frac{1}{2} A(q, \nu) \frac{1}{|x_\varepsilon|^2} + o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right). \end{aligned} \tag{5.24}$$

It follows in particular that uniformly for $|x| \leq 2\varepsilon^{1-\alpha}$,

$$\begin{aligned} \varepsilon^N \left(\frac{q(x_\varepsilon + x/\varepsilon)}{q(\infty)} - 1\right) &= \exp \left\{ \frac{N}{q(\infty)} \left(q(x_\varepsilon + x/\varepsilon) - q(\infty)\right) \ln \varepsilon \right\} \\ &= 1 - \frac{NA(q, \nu)}{2q(\infty)} \frac{|\ln \varepsilon|}{|x_\varepsilon|^2} + o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right), \end{aligned} \tag{5.25}$$

and also that

$$\begin{aligned} U(x)^{q(x_\varepsilon + \varepsilon x)} &= U(x)^{q(\infty)} \exp \left\{ (q(x_\varepsilon + \varepsilon x) - q(\infty)) \ln U(x) \right\} \\ &= U(x)^{q(\infty)} \left\{ 1 + o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right) \ln U(x) \right\}. \end{aligned}$$

Eventually,

$$f(x_\varepsilon + \varepsilon x) = f(\infty) + \frac{1}{2} \partial_{i,j} f(\infty) \frac{x_\varepsilon^i x_\varepsilon^j}{|x_\varepsilon|^4} + o\left(\frac{1}{|x_\varepsilon|^2}\right) = f(\infty) + o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right).$$

Using the previous expansion, we can rewrite (5.22) as

$$\begin{aligned} \int_{\mathbb{R}^N} f|u_\varepsilon|^{q(x)} dx &= f(\infty) \left(1 - \frac{NA(q, \nu)}{2q(\infty)} \frac{|\ln \varepsilon|}{|x_\varepsilon|^2}\right) \int_{B(\varepsilon^{1-\alpha})} U(x)^{q(\infty)} dx \\ &\quad + o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right) \int_{B(\varepsilon^{1-\alpha})} U(x)^{q(\infty)} \left(1 + \ln U(x)\right) dx + R_\varepsilon. \end{aligned}$$

with

$$|R_\varepsilon| \leq C \int_{B(2\varepsilon^{1-\alpha}) \setminus B(\varepsilon^{1-\alpha})} U(x)^{q(\infty)(1-\gamma)} dx.$$

Here γ is a given small positive number and the above estimate of R_ε holds for $\varepsilon \ll 1$ since $|x_\varepsilon| \gg \frac{|x|}{\varepsilon}$ uniformly for $\frac{|x|}{\varepsilon} \leq 2\varepsilon^{-\alpha}$ by (5.20).

To conclude, notice that $U^{q(\infty)} \ln U$ is integrable in \mathbb{R}^N and that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B(\varepsilon^{1-\alpha})} U(x)^{q(\infty)(1-\gamma)} dx &\leq C \int_{\varepsilon^{1-\alpha}} r^{-\frac{p(\infty)}{p(\infty)-1} N(1-\gamma)} r^{N-1} dr \\ &\leq C \varepsilon^{\frac{N(\alpha-1)}{p(\infty)-1} (1-\gamma p(\infty))} \end{aligned}$$

which is $o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right)$ by (5.20) for a small enough γ . □

We now compute the asymptotic expansion of a term like $\int_{\mathbb{R}^N} g(x) |\nabla u_\varepsilon|^{p(x)} dx$.

Proposition 5.10. *Assume that p and $g \in C(\mathbb{R}^N)$ have a Taylor expansion of order 2 at ∞ with $\nabla p(\infty) = \nabla g(\infty) = 0$ and that $p(\infty) < \sqrt{N}$. Assume also that r_ε and $\alpha > 1$ are such that*

$$\varepsilon^{-2\alpha} \ll |x_\varepsilon|^2 \ll \frac{|\ln \varepsilon|}{\varepsilon^{\frac{(\alpha-1)(N-p(\infty))}{p(\infty)-1}}}. \tag{5.26}$$

Then

$$\int_{\mathbb{R}^N} g|\nabla u_\varepsilon|^{p(x)} dx = B_0 + \frac{|\ln \varepsilon|}{|x_\varepsilon|^2} B_1 + o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right),$$

where, denoting $A(p, \nu) := \sum_{i,j=1}^n \partial_{i,j} p(\infty) \nu_i \nu_j$,

$$B_0 = g(\infty) \int_{\mathbb{R}^N} |\nabla U|^{p(\infty)} dx, \quad \text{and}$$

$$B_1 = -\frac{Ng(\infty)}{2p(\infty)} A(p, \nu) \int_{\mathbb{R}^N} |\nabla U|^{p(\infty)} dx.$$

Remark 5.11. We can find r_ε and $\alpha > 1$ such that (5.26). Indeed it suffices to have $\varepsilon^{-2\alpha} \ll \varepsilon^{-\frac{(\alpha-1)(N-p(\infty))}{p(\infty)-1}}$ i.e. $\frac{(\alpha-1)(N-p(\infty))}{p(\infty)-1} - 2\alpha > 0$. Since $N \geq 4$ and $p(\infty) < \sqrt{N} \leq \frac{N+2}{3}$, we thus need $\alpha > \frac{N-p(\infty)}{N-3p(\infty)+2}$. Notice that $\frac{N-p(\infty)}{N-3p(\infty)+2} > 1$.

Proof. Since the proof is very similar to that of the previous proposition, we will be sketchy. Denote $u_\varepsilon = U_\varepsilon \eta_\varepsilon$ with $U_\varepsilon(x) = \varepsilon^{\frac{N-p(\infty)}{p(\infty)}} U(\varepsilon(x - x_\varepsilon))$ and $\eta_\varepsilon(x) := \eta(\varepsilon^\alpha |x - x_\varepsilon|)$. We first write that

$$\int_{\mathbb{R}^N} g(x) |\nabla u_\varepsilon|^{p(x)} dx = \int_{\{\varepsilon^\alpha |x - x_\varepsilon| \leq 1\}} g(x) |\nabla U_\varepsilon|^{p(x)} dx + R_\varepsilon \tag{5.27}$$

where

$$R_\varepsilon = \int_{\{1 \leq \varepsilon^\alpha |x - x_\varepsilon| \leq 2\}} g(x) \left| \eta_\varepsilon \nabla U_\varepsilon + U_\varepsilon \nabla \eta_\varepsilon \right|^{p(x)} dx.$$

Notice that

$$\begin{aligned} |R_\varepsilon| &\leq C \int_{\{1 \leq \varepsilon^\alpha |x - x_\varepsilon| \leq 2\}} |\nabla U_\varepsilon|^{p(x)} + |U_\varepsilon|^{p(x)} \varepsilon^{\alpha p(x)} dx \\ &\leq C \int_{B(2\varepsilon^{1-\alpha}) \setminus B(\varepsilon^{1-\alpha})} \varepsilon^{N \left(\frac{p(x_\varepsilon + x/\varepsilon)}{p(\infty)} - 1 \right)} \\ &\quad \times \left\{ |\nabla U|^{p(x_\varepsilon + x/\varepsilon)} + \varepsilon^{(\alpha-1)p(x_\varepsilon + x/\varepsilon)} U^{p(x_\varepsilon + x/\varepsilon)} \right\} dx. \end{aligned}$$

It follows from (5.25) with p in place of q and from the fact that p has a local minimum at ∞ with $|x_\varepsilon| \gg \varepsilon^{-\alpha}$ that

$$|R_\varepsilon| \leq C \int_{\mathbb{R}^N \setminus B(\varepsilon^{1-\alpha})} |\nabla U|^{p(\infty)} + \varepsilon^{(\alpha-1)p(\infty)} U^{p(\infty)} dx.$$

Using that $|U'(r)| \leq Cr^{-\frac{N-1}{p(\infty)-1}}$ with $p(\infty) < \sqrt{N} \leq \frac{N+1}{2}$, it is then easily seen that

$$|R_\varepsilon| \leq C\varepsilon^{(\alpha-1)\frac{N-p(\infty)}{p(\infty)-1}} + C\varepsilon^{(\alpha-1)p(\infty)} \varepsilon^{(\alpha-1)(\frac{p(\infty)(N-p(\infty))}{p(\infty)-1} - N)} \leq C\varepsilon^{\frac{(\alpha-1)(N-p(\infty))}{p(\infty)-1}}$$

which is $o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right)$ by (5.26).

We can thus rewrite (5.27) as

$$\begin{aligned} \int_{\mathbb{R}^N} g(x) |\nabla u_\varepsilon|^{p(x)} dx &= \int_{B(\varepsilon^{1-\alpha})} g(x_\varepsilon + x/\varepsilon) \varepsilon^{N\left(\frac{p(x_\varepsilon+x/\varepsilon)}{p(\infty)} - 1\right)} |\nabla U|^{p(x_\varepsilon+x/\varepsilon)} dx \\ &\quad + o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right). \end{aligned}$$

As in the previous proposition we then obtain

$$\begin{aligned} \int_{\mathbb{R}^N} g(x) |\nabla u_\varepsilon|^{p(x)} dx &= g(\infty) \left(1 - \frac{NA(p, \nu)}{2p(\infty)} \frac{|\ln \varepsilon|}{|x_\varepsilon|^2}\right) \int_{B(\varepsilon^{1-\alpha})} |\nabla U|^{p(\infty)} dx \\ &\quad + o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right) \int_{B(\varepsilon^{1-\alpha})} |\nabla U|^{p(\infty)} (1 + \ln |\nabla U|) dx \\ &\quad + o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right). \end{aligned}$$

We conclude recalling that $\int_{B(\mathbb{R}^N \setminus \varepsilon^{1-\alpha})} |\nabla U|^{p(\infty)} dx = o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right)$, which we already noticed when estimating R_ε , and noticing that $|\nabla U|^{p(\infty)}(1 + \ln |\nabla U|)$ is integrable. \square

We eventually compute the asymptotic expansion of an expression like $\int_{\mathbb{R}^N} h(x) |u_\varepsilon(x)|^{p(x)} dx$:

Proposition 5.12. *Assume that p has a Taylor expansion of order 2 at ∞ with $\nabla p(\infty) = 0$ and $p(\infty) < \sqrt{N}$, that $h \in L^\infty(\mathbb{R}^N)$ satisfies $h(x) = O(|x|^{-\beta})$ for $|x| \gg 1$, and also that $r_\varepsilon, \alpha > 1$ and $\beta > 2$ are such that*

$$\varepsilon^{-\alpha} \ll |x_\varepsilon| \quad \text{and} \quad |x_\varepsilon|^{2-\beta} \ll \varepsilon^{p(\infty)(1+\gamma)} \tag{5.28}$$

for some small $\gamma > 0$. Then

$$\int_{\mathbb{R}^N} h(x) |u_\varepsilon(x)|^{p(x)} dx = o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right).$$

Remark 5.13. If $\varepsilon^{-\alpha} \ll |x_\varepsilon|$ then to obtain $|x_\varepsilon|^{2-\beta} \ll \varepsilon^{p(\infty)(1+\gamma)}$, it suffices to choose $\beta > 2$ so that $\varepsilon^{\alpha(\beta-2)} \ll \varepsilon^{p(\infty)(1+\gamma)}$. This holds for a small $\gamma > 0$ if $\beta > 2 + p(\infty)/\alpha$.

Proof. First

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} h(x)u_\varepsilon^{p(x)} dx \right| \\ & \leq \int_{B(2\varepsilon^{1-\alpha})} |h(x_\varepsilon + x/\varepsilon)|\varepsilon^N \left(\frac{p(x_\varepsilon + x/\varepsilon) - 1}{p(\infty)} \right) \varepsilon^{-p(x_\varepsilon + x/\varepsilon)} U(x)^{p(x_\varepsilon + x/\varepsilon)} dx \\ & \leq C \int_{B(2\varepsilon^{1-\alpha})} |h(x_\varepsilon + x/\varepsilon)|\varepsilon^{-p(x_\varepsilon + x/\varepsilon)} U(x)^{p(x_\varepsilon + x/\varepsilon)} dx \end{aligned}$$

where the 2nd inequality is a consequence of (5.25). Since $|x_\varepsilon| \gg \varepsilon^{-\alpha}$, (5.23) holds. We thus have uniformly for $x \in B(2\varepsilon^{1-\alpha})$ that $|h(x_\varepsilon + x/\varepsilon)| = O(|x_\varepsilon|^{-\beta})$, and also, for a given $\gamma > 0$ small, that $\varepsilon^{-p(x_\varepsilon + x/\varepsilon)} \leq \varepsilon^{-p(\infty)(1+\gamma)}$. Thus

$$\begin{aligned} \left| \int_{\mathbb{R}^N} h(x)u_\varepsilon^{p(x)} dx \right| & \leq O(|x_\varepsilon|^{-\beta})\varepsilon^{-p(\infty)(1+\gamma)} \int_{B(2\varepsilon^{1-\alpha})} U(x)^{p(x_\varepsilon + x/\varepsilon)} dx \\ & \leq o\left(\frac{\ln \varepsilon}{|x_\varepsilon|^2}\right) \int_{B(2\varepsilon^{1-\alpha})} U(x)^{p(x_\varepsilon + x/\varepsilon)} dx \end{aligned}$$

where we used (5.28) in the second inequality. To conclude, notice that the integral in the r.h.s is finite. Indeed, uniformly for $x \in B(2\varepsilon^{1-\alpha}) \cap \{U \leq 1\}$, we have $p(x_\varepsilon + x/\varepsilon) \geq p(\infty)$ (because p has a local minimum at infinity) and $U \in L^{p(\infty)}(\mathbb{R}^N)$ when $p(\infty) < \sqrt{N}$. \square

Remark 5.14. Summing-up the different remarks following the three previous propositions, we see that their conclusions hold if $p(\infty) < \sqrt{N}$ with $N \geq 4$ and $\beta > 2$. Indeed it suffices to choose $\alpha > \frac{N-p(\infty)}{N-3p(\infty)+2} > 1$ big enough so that $\beta > 2 + p(\infty)/\alpha$, and then to take $r_\varepsilon = |x_\varepsilon|$ so that $r_\varepsilon \gg \varepsilon^{-\alpha}$.

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