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The limit as $p \to +\infty$ of the first eigenvalue for the p-Laplacian with mixed Dirichlet and Robin boundary conditions

Julio D. Rossi [a,*], Nicolas Saintier [a,b]

[a] Departamento de Matemática, FCEyN, Univ. de Buenos Aires, C.P. 1428, Buenos Aires, Argentina

[b] Instituto de Ciencias, Univ. Gral Sarmiento, Juan María Gutierrez 1150, Los Polvorines, Pcia de Buenos Aires, Argentina

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ABSTRACT

We analyze the behavior as $p \to \infty$ of the first eigenvalue of the p-Laplacian with mixed boundary conditions of Dirichlet–Robin type. We find a nontrivial limit that we associate to a variational principle involving L^∞-norms. Moreover, we provide a geometrical characterization of the limit value as well as a description of it using optimal mass transportation techniques. Our results interpolate between the pure Dirichlet case and the mixed Dirichlet–Neumann case.

1. Introduction and description of the main results

Let $U \subset \mathbb{R}^n$ be a smooth, bounded, open and connected set. In order to consider mixed boundary conditions, we split the boundary of U as $\partial U = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $|\Gamma_1| > 0$. In this paper we deal with the first eigenvalue, that we will call λ_p, of the p-Laplacian with Dirichlet condition on Γ_1 and Robin condition on Γ_2 namely the smallest λ such that there is a nontrivial solution to the following problem,

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } U, \\ u = 0 & \text{on } \Gamma_1, \\ |\nabla u|^{p-2}\partial_\nu u + \alpha^p |u|^{p-2}u = 0 & \text{on } \Gamma_2. \end{cases} \tag{1}$$

Here α is a non-negative parameter. Notice that when $\alpha = +\infty$, the boundary condition become $u = 0$ in all ∂U (a pure Dirichlet condition) and when $\alpha = 0$ we have a mixed Dirichlet–Neumann boundary condition.

Our main goal is to compute the limit as $p \to \infty$ of this problem and look at its dependence on the parameter α.

To start our analysis we remark that λ_p has the following variational formulation:

$$\lambda_p = \inf_{u \in X_p} \left\{ \int_U |\nabla u|^p + \alpha^p \int_{\Gamma_2} |u|^p \,:\, \|u\|_{L^p(U)} = 1 \right\} \tag{2}$$

where

$$X_p = \left\{ u \in W^{1,p}_{\Gamma_1}(U),\ u \geq 0 \right\},$$

* Corresponding author. Tel.: +54 34652577160.
E-mail addresses: jrossi@dm.uba.ar (J.D. Rossi), nsaintie@dm.uba.ar (N. Saintier).

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and

$$W^{1,p}_{\Gamma_1}(U) = \left\{ u \in W^{1,p}(U),\ u = 0 \text{ on } \Gamma_1 \right\}.$$

Note that the infimum is attained since we assumed that $|\Gamma_1| > 0$. Also notice that if we regard λ_p as a function of α, $\alpha \in [0, +\infty) \to \lambda_p(\alpha)$, then $\lambda_p(\alpha)$ is non-decreasing with $\lim_{\alpha\to+\infty} \lambda_p = \lambda_{p,D}$ the first Dirichlet eigenvalue for the p-Laplacian in U.

We expect the limit problem of (2) as $p \to \infty$ to be

$$\lambda_\infty = \inf_{u\in X,\ \|u\|_{L^\infty(U)}=1} \max\left\{ \|\nabla u\|_{L^\infty(U)},\ \alpha\|u\|_{L^\infty(\Gamma_2)} \right\} \tag{3}$$

where $X := \left\{ u \in W^{1,\infty}(U),\ u = 0 \text{ on } \Gamma_1,\ u \geq 0 \text{ in } U \right\}$. Notice that when we let $\alpha \to +\infty$ in (3) with $\Gamma_1 = \partial U$ we obtain

$$\lim_{\alpha\to+\infty} \lambda_\infty(\alpha) = \lambda_{\infty,D} = \inf_{u\in W_0^{1,\infty}(U),\ \|u\|_{L^\infty(U)}=1} \|\nabla u\|_{L^\infty(U)}$$

that is the first eigenvalue of the infinity Laplacian, $\Delta_\infty u = DuD^2uDu$ with Dirichlet boundary conditions. This value, $\lambda_{\infty,D}$, turns out to be the limit of $(\lambda_{p,D})^{1/p}$ as $p \to \infty$, see [1]. Our first result says that this kind of limit can be also computed for any nonnegative α.

Theorem 1. *There holds that*

$$\lim_{p\to+\infty} (\lambda_p)^{1/p} = \lambda_\infty.$$

Moreover the positive, normalized extremals for λ_p, u_p converge uniformly in $\bar{U}$ along subsequences $p_j \to \infty$ to $u \in X$ which is a minimizer for (3) and a viscosity solution to

$$\begin{cases} \min\{|Du| - \lambda_\infty u, -\Delta_\infty u\} = 0 & \text{in } U, \\ u = 0 & \text{on } \Gamma_1, \\ \min\{|Du| - \alpha u, -\partial_\nu u\} = 0 & \text{on } \Gamma_2. \end{cases}$$

Our next goal is to characterize this limit value λ_∞. The value of λ_∞ results in the interplay between α, the geometry of U and the sets Γ_1, Γ_2. We consider the (possibly empty) set

$$\mathcal{A} := \left\{ x \in \bar{U},\ d(x, \Gamma_1) \geq \frac{1}{\alpha} + d(x, \Gamma_2) \right\}.$$

Notice that if $\mathcal{A} \neq \emptyset$ then the set

$$\mathcal{A}' := \left\{ x \in \bar{U},\ d(x, \Gamma_1) = \frac{1}{\alpha} + d(x, \Gamma_2) \right\}$$

is also not empty. Indeed the function $f(x) = \frac{1}{\alpha} + d(x, \Gamma_2) - d(x, \Gamma_1)$ is continuous, less than or equal to 0 on $\mathcal{A}$, and greater than or equal to 0 if $d(x, \Gamma_1) \ll 1$ (we are using here the fact that U is connected to apply the mean value theorem). Our next result gives a geometrical characterization of λ_∞.

Theorem 2. *It holds that*

$$\lambda_\infty = \begin{cases} \min_{x\in\bar{U}} \dfrac{1}{d(x, \Gamma_1)}, & \text{if } \mathcal{A} = \emptyset, \\ \min_{x\in\mathcal{A}} \dfrac{1}{\frac{1}{\alpha} + d(x, \Gamma_2)} = \min_{x\in\mathcal{A}'} \dfrac{1}{\frac{1}{\alpha} + d(x, \Gamma_2)}, & \text{if } \mathcal{A} \neq \emptyset. \end{cases} \tag{4}$$

Notice that when $\alpha = +\infty$, which corresponds to pure Dirichlet boundary conditions on the whole ∂U, then $\mathcal{A} = \mathcal{A}' = \emptyset$ and we recover the result of [1], $\lambda_\infty^{-1} = \lambda_{\infty,D}^{-1} = \max_{x\in\bar{U}} d(x, \partial U)$. In the case of Neumann boundary conditions i.e. $\Gamma_1 = \emptyset$ and $\alpha = 0$ then $\mathcal{A} = \emptyset$ and $d(x, \Gamma_1) = d(x, \emptyset) = +\infty$ for any $x \in \bar{U}$ so that $\lambda_\infty = 0$ which is consistent with the fact the 1st eigenvalue of Δ_p with Neumann boundary conditions is 0.

We will first give a simple proof in the case where U is convex by using a test-function argument based proof which we were not able to extend to the general case. In fact the result for a arbitrary connected domain will be a consequence of an optimal mass transport formulation of λ_∞ that we now introduce.

To continue our analysis we have to recall some notions and notations from optimal mass transport theory. Recall that the Monge–Kantorovich distance $W_1(\mu, \nu)$ between two probability measures μ and ν over $\bar{U}$ is defined by

$$W_1(\mu, \nu) = \max_{v\in W^{1,\infty}(U),\ \|\nabla v\|_\infty\leq 1} \int_U v\,(d\mu - d\nu). \tag{5}$$

Recently the authors in [2] relate $\lambda_{\infty,D}$ with the Monge–Kantorovich distance W_1. They proved that

$$\lambda_{\infty,D}^{-1} = \max_{\mu\in P(U)} W_1(\mu, P(\partial U)), \tag{6}$$

where $P(U)$ and $P(\partial U)$ denote the set of probability measures over $\bar{U}$ and ∂U. Notice that the maximum is easily seen to be reached at δ_x where $x \in U$ is a most inner point.

In our case we are also able to give a characterization for λ_∞ in terms of a maximization problem involving W_1 but this time we get an extra term involving the total variation of a measure on Γ_2.

Theorem 3. *It holds that*

$$\frac{1}{\lambda_\infty} = \max_{\sigma\in P(\bar{U})} \inf_{\nu\in P(\partial U)} \left\{ W_1(\sigma, \nu) + \frac{1}{\alpha}\nu(\Gamma_2) \right\}. \tag{7}$$

Moreover, the measures $u_p^{p-1}\,dx$ weakly converge (up to a subsequence) as $p \to +\infty$ to a probability measure f_∞ which attains the maximum in (7).

Notice that when $\alpha = +\infty$, which corresponds to Dirichlet boundary conditions, then we recover the result of [2], who showed that (6) holds.

As a corollary of this characterization in terms of optimal transportation, we can extend the result stated in Theorem 2 for the value of λ_∞ to the case where U is not convex. We prefer to present our results in this order (even if Theorem 2 is not initially proved in its full generality) for readability of the whole paper (the proof of Theorem 2 in the convex case is much simpler).

Let us end the introduction with a brief description of the previous bibliography and the main ideas and techniques used to prove our results. First, as by now classical results, we mention that the limit as $p \to \infty$ of the first eigenvalue $\lambda_{p,D}$ of the p-Laplacian with Dirichlet boundary condition was studied in [3,1] (see also [4] for an anisotropic version). For its dependence with respect to the domain we refer to [5]. The *limit operator* that appears here, the infinity-Laplacian is given by the limit as $p \to \infty$ of the p-Laplacian, in the sense that solutions to $\Delta_p v_p = 0$ with a Dirichlet data $v_p = f$ on $\partial\Omega$ converge as $p \to \infty$ to the solution to $\Delta_\infty v = 0$ with $v = f$ on $\partial\Omega$ in the viscosity sense (see [6–8]). This operator appears naturally when one considers absolutely minimizing Lipschitz extensions in Ω of a boundary data f (see [9,6,10]).

The case of a Steklov boundary condition (here the eigenvalue appears in the boundary condition) has also been investigated recently. Indeed in [11] (see also [12] for a slightly different problem) it is studied the behavior as $p \to +\infty$ of the so-called variational eigenvalues $\lambda_{k,p,S},\ k \geq 1$, of the p-Laplacian with a Steklov boundary condition. In particular it is proved that

$$\lim_{p\to+\infty} \lambda_{1,p,S}^{1/p} = 1 \quad \text{and} \quad \lambda_{2,\infty,S} := \lim_{p\to+\infty} \lambda_{2,p,S}^{1/p} = \frac{2}{\operatorname{diam}(U, \mathbb{R}^n)},$$

where here $\operatorname{diam}(U, \mathbb{R}^n)$ denotes the diameter of U for the usual Euclidean distance in $\mathbb{R}^n$.

For pure Neumann eigenvalues, we quote [13,14]. In those references it is considered the limit for the second eigenvalue (the first one is zero). It is proved that in this case $\lambda_\infty := \lim_{p\to+\infty} \lambda_p^{1/p} = 2/\operatorname{diam}(U)$, where $\operatorname{diam}(U)$ denotes the diameter of U with respect to the geodesic distance in U. In addition, the regularity of λ_∞ as a function of the domain U is studied in [14] and in [13] it is proved that there are no nonzero eigenvalues below λ_∞, so that λ_∞ is indeed the first nontrivial eigenvalue for the infinity-Laplacian with Neumann boundary conditions.

Concerning ideas and methods used in the proofs we use classical variational ideas to obtain the limit of $(\lambda_p)^{1/p}$ and viscosity techniques and to find the limit PDE problem we use viscosity techniques as in [1] (we refer to [8] for the definition of a viscosity solution). The characterization of λ_∞ given in Theorem 2 follows using cones as test functions in the variational formulation. Finally, mass transport techniques (we refer to [15]) and gamma-convergence of functionals are used to show the more general characterization of λ_∞ given in Theorem 3, see [2,14] for similar arguments in different contexts.

The paper is organized as follows. In Section 2 we deal with the limit as $p \to \infty$ and prove Theorem 1. In Section 3 we prove Theorem 2 that characterizes λ_∞ in geometrical terms in the cases of a convex domain U. In Section 4 we use optimal transport ideas to obtain Theorem 3. As a corollary, we eventually prove Theorem 2 for a general connected domain in the last section.

2. Proof of Theorem 1

For the proof of Theorem 1 we will use the following lemma.

Lemma 1. *For any $f, g \in L^\infty(U)$ there holds*

$$\lim_{p\to+\infty} \left(\|f\|_{L^p(U)} + \|g\|_{L^p(U)} \right)^{\frac{1}{p}} = \max\{\|f\|_{L^\infty(U)}, \|g\|_{L^\infty(U)}\}.$$

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Proof. The result is a direct consequence of the inequalities

$$\begin{aligned}\max\{\|f\|_{L^p(U)}^p, \|g\|_{L^p(U)}^p\} &\leq \|f\|_{L^p(U)}^p + \|g\|_{L^p(U)}^p \\ &\leq 2\max\{\|f\|_{L^p(U)}^p, \|g\|_{L^p(U)}^p\}.\end{aligned}$$

In fact, from the previous inequalities, we get

$$\begin{aligned}\lim_{p\to+\infty} \max\{\|f\|_{L^p(U)}, \|g\|_{L^p(U)}\} &\leq \lim_{p\to+\infty} \left(\|f\|_{L^p(U)} + \|g\|_{L^p(U)}\right)^{\frac{1}{p}} \\ &\leq \lim_{p\to+\infty} 2^{\frac{1}{p}} \max\{\|f\|_{L^p(U)}, \|g\|_{L^p(U)}\}.\end{aligned}$$

We conclude using that

$$\lim_{p\to+\infty} \|f\|_{L^p(U)} = \|f\|_{L^\infty(U)}$$

and

$$\lim_{p\to+\infty} \|g\|_{L^p(U)} = \|g\|_{L^\infty(U)}. \qquad \square$$

Now let us proceed with the proof of Theorem 1.

Proof of Theorem 1. Let $u \in X$ then $u \in \cap_p X_p$. From the variational characterization of λ_p we have

$$(\lambda_p)^{1/p} \leq \frac{1}{\|u\|_{L^p(U)}}\left(\int_U |\nabla u|^p + \alpha^p \int_{\Gamma_2} |u|^p\right)^{1/p}.$$

Hence, using the previous lemma we get

$$\limsup_{p\to\infty} (\lambda_p)^{1/p} \leq \max\left\{\|\nabla u\|_{L^\infty(U)},\ \alpha\|u\|_{L^\infty(\Gamma_2)}\right\}$$

for any $u \in X$. Therefore, we conclude that

$$\limsup_{p\to\infty} (\lambda_p)^{1/p} \leq \lambda_\infty.$$

In addition, we get that, for u_p an eigenfunction associated to λ_p in X_p it holds that

$$\limsup_{p\to\infty} \|\nabla u_p\|_{L^p(U)} \leq \lambda_\infty.$$

Therefore, we have that $\{u_p\}$ is uniformly bounded (independently of p) in $W^{1,p}(U)$. Then, for any fixed q we obtain

$$\|\nabla u_p\|_{L^q(U)} \leq \|\nabla u_p\|_{L^p(U)} |U|^{\frac{p-q}{qp}} \leq C$$

with C independent of p. Hence, by a diagonal procedure, we can extract a subsequence $p_j \to \infty$ such that

$$u_{p_j} \to u$$

uniformly in $\overline{U}$ and weakly in every $W^{1,q}(U)$, $q \in \mathbb{N}$. This limit u verifies that

$$\|\nabla u\|_{L^q(U)} \leq \limsup_{p\to\infty} \|\nabla u_p\|_{L^q(U)} \leq \limsup_{p\to\infty} \|\nabla u_p\|_{L^p(U)} |U|^{\frac{p-q}{qp}} \leq \lambda_\infty |U|^{\frac{1}{q}}$$

and then we get

$$\|\nabla u\|_{L^\infty(U)} \leq \lambda_\infty.$$

Moreover, we have

$$\alpha\|u_p\|_{L^q(\Gamma_2)} \leq \left(\alpha^p \|u_p\|_{L^p(\Gamma_2)}^p |\Gamma_2|^{\frac{p-q}{q}}\right)^{1/p} \leq \left(\lambda_p |\Gamma_2|^{\frac{p-q}{q}}\right)^{1/p},$$

then

$$\alpha\|u\|_{L^q(\Gamma_2)} \leq \limsup_{p\to\infty} \alpha\|u_p\|_{L^q(\Gamma_2)} \leq \limsup_{p\to\infty} \alpha\|u_p\|_{L^q(\Gamma_2)} \leq \lambda_\infty |\Gamma_2|^{\frac{1}{q}}$$

and we conclude that

$$\alpha\|u\|_{L^\infty(\Gamma_2)} \leq \lambda_\infty.$$

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Hence

$$\max\left\{\|\nabla u\|_{L^\infty(U)},\ \alpha\|u\|_{L^\infty(\Gamma_2)}\right\}\le\lambda_\infty.$$

Now, we only have to observe that from the uniform convergence we get that $u\in X$, and then we conclude that u is a minimizer of (3). In addition, our previous calculations show that

$$\lambda_\infty\le\liminf_{p\to\infty}(\lambda_p)^{\frac{1}{p}}.$$

Now, concerning the equation verified by the limit of u_p, u, we have that, from the fact that u_p are viscosity solutions to $\Delta_p u=\lambda_p|u|^{p-2}u$ and that $(\lambda_p)^{1/p}$ converges to λ_∞ we conclude as in [1] that the limit u is a viscosity solution to

$$\min\{|Du|-\lambda_\infty u,-\Delta_\infty u\}=0.$$

That $u=0$ on Γ_1 is immediate from uniform convergence in $\overline{U}$ and the fact that u_p verify the same condition.

On Γ_2 we have

$$|\nabla u|^{p-2}\partial_\nu u+\alpha^p|u|^{p-2}u=0,$$

therefore, passing to the limit in the viscosity sense as done in [16] we obtain

$$\min\{|Du|-\alpha u,-\partial_\nu u\}=0.$$

This ends the proof. □

3. Proof of Theorem 2 for convex domains

Along this section we assume that U is convex.

Proof of Theorem 2. Using the variational characterization (3) proved in the previous section, we estimate λ_∞ from above by using as test-function a truncated cone of the form

$$u(x)=\Big(1-a|x-x_0|\Big)_+$$

where $a>0$ and $x_0\in\bar{U}$. Then

$$\begin{aligned}&u\equiv 0\quad\text{on }\Gamma_1\quad\text{iff}\quad a\ge\frac{1}{d(x_0,\Gamma_1)}\\&\|\nabla u\|_{L^\infty(U)}=a,\\&\text{and}\quad\|u\|_{L^\infty(\Gamma_2)}=\Big(1-ad(x_0,\Gamma_2)\Big)_+.\end{aligned}$$

It follows that

$$\lambda_\infty\le\inf\max\{a,\ \alpha[1-ad(x_0,\Gamma_2)]_+\}$$

where the infimum is taken over all the $x_0\in\bar{U}$ and $a>0$ such that $a\ge 1/d(x_0,\Gamma_1)$. Examining the two possibilities for the max, we obtain easily the upper bound for λ_∞.

To prove the lower bound we argue as follows: for any $x_0\in\bar{U}$, and any Lipschitz function $u\in X$ with $u(x_0)=1$, we have

$$1\ge\|u\|_{L^\infty(\Gamma_2)}\ge\Big(1-\|\nabla u\|_\infty d(x_0,\Gamma_2)\Big)_+.$$

Thus

$$\lambda_\infty\ge\inf\max\left\{\|\nabla u\|_{L^\infty(U)},\alpha\Big(1-\|\nabla u\|_{L^\infty(U)}d(x_0,\Gamma_2)\Big)_+\right\}$$

where the infimum is taken over all $u\in W^{1,\infty}(\bar{U})$ such that $u=0$ in Γ_1, $\|u\|_{L^\infty(U)}=1$ and $\|\nabla u\|_{L^\infty(U)}\ge\frac{1}{d(x_0,\Gamma_1)}$ for any $x_0\in\{u=1\}$. From this point the argument concludes as for the previous case just analyzing the possibilities for the max. □

4. Proof of Theorem 3

The proof follows the lines of [2] (see also [14] for the pure Neumann boundary case).

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We begin rewriting the variational formulation (2) of λ_p as

$$1=\sup\left\{\int_U |u|^p : u\in W^{1,p}_{\Gamma_1}(U) \text{ s.t. } \int_U |\nabla u|^p+\alpha^p\int_{\Gamma_2}|u|^p=\lambda_p\right\}.$$

We are thus lead to consider the functions $G_p : C(\bar{U})\times M(\bar{U})\to\mathbb{R},\ p\geq 1$, defined by

$$G_p(v,\sigma)=\begin{cases}-\int v\,d\sigma & \text{if } v\in W^{1,p}_{\Gamma_1}(U),\ \int_U|\nabla v|^p+\alpha^p\int_{\Gamma_2}|v|^p\leq\lambda_p^p, \text{ and } \sigma\in L^{p'}(U) \text{ with } \int_U|\sigma|^{p'}\leq 1,\\ +\infty & \text{otherwise.}\end{cases}$$

Notice that the pair $(u_p, u_p^{p-1}\,dx)$ is an extremal for G_p so that $\min G_p=-1$. Indeed for any admissible pair $(v,\sigma)\in W^{1,p}_{\Gamma_1}(U)\times L^{p'}(U)$, we have

$$\begin{aligned}-G_p(v,\sigma)&=\int_U v\sigma\leq\|v\|_p\|\sigma\|_{p'}\leq\lambda_p^{-1/p}\Big(\int_U|\nabla v|^p+\alpha^p\int_{\Gamma_2}|v|^p\Big)^{1/p}\\ &\leq 1=\int_U u_p^p\end{aligned}$$

(we used successively Hölder's inequality, the definition of λ_p and the fact that v is admissible). In view of Lemma 1, we introduce the formal limit functional $G_\infty : C(\bar{U})\times M(\bar{U})\to\mathbb{R}$ of the G_p by

$$G_\infty(v,\sigma)=\begin{cases}-\int v\,d\sigma & \text{if } v\in W^{1,\infty}_{\Gamma_1}(U), \max\{\|\nabla u\|_\infty,\alpha\|v\|_{L^\infty(\Gamma_2)}\}\leq\lambda_\infty, \text{ and } |\sigma|(\bar{U})\leq 1,\\ +\infty & \text{otherwise.}\end{cases}$$

The convergence of the functionals G_p to G_∞ can be justified using the notion of Γ-convergence. Recall that a sequence of functionals $F_n : X\to[0,+\infty]$ defined over a metric space X is said to Γ-converge to a functional $F_\infty : X\to[0,+\infty]$ if the following two conditions hold:

- for every $x\in X$ and every sequence $(x_n)_n\subset X$ converging to x, $F(x)\leq\liminf F(x_n)$, and
- for any $x\in X$, there exists a sequence $(x_n)_n\subset X$ converging to x such that $F(x)\geq\limsup F(x_n)$.

An easy but important consequence of the definition, that we will use later, is the fact that if x_n is a minimizer of F_n then every cluster point of the sequence (x_n) is a minimizer of F_∞. We refer e.g. to [17,18] for a detailed account on Γ-convergence.

Proposition 4.1. *The functionals G_p Γ-converge as $p\to+\infty$ to G_∞.*

Proof. The proof is very similar to [2] (see also [14] for the pure Neumann boundary case). We briefly sketch it for the reader's convenience.

Assume that $(v_p,\sigma_p)\in C(\bar{U})\times M(\bar{U})$ converges to (v,σ). We have to prove that

$$\liminf_{p\to+\infty} G_p(v_p,\sigma_p)\geq G(v,\sigma). \tag{8}$$

We can assume that $G_p(v_p,\sigma_p)<\infty$. Then we have

$$\int_U v_p\sigma_p\,dx-\int_U v\,d\sigma=\int_U(v_p-v)\sigma_p\,dx+\int_U v\,(\sigma_p\,dx-d\sigma)\to 0$$

as $p\to+\infty$. Indeed the first integral on the right hand side can be bounded by $\|v_p-v\|_\infty\|\sigma_p\|_{p'}|U|^{\frac{1}{p}}=o(1)$. Independently

$$\int_U|\sigma|=\int_U|\sigma_p|\,dx+o(1)\leq\|\sigma_p\|_{p'}|U|^{\frac{1}{p}}+o(1)\leq 1+o(1)$$

so that $\int_U|\sigma|\leq 1$. Moreover taking limit in $\alpha\|v_p\|_{L^p(\Gamma_2)}\leq\lambda_p$ yields $\alpha\|v\|_{L^\infty(\Gamma_2)}\leq\lambda_\infty$. Eventually, for any $\phi\in L^{p'}(U,\mathbb{R}^n)$ such that $\|\phi\|_{p'}\leq 1$ we have

$$\begin{aligned}\int_U\phi\nabla v\,dx&=-\int_U v\operatorname{div}\phi\,dx=-\int_U v_p\operatorname{div}\phi\,dx+o(1)=\int_U\phi\nabla v_p\,dx+o(1)\\ &\leq\|\nabla v_p\|_p+o(1)\leq\lambda_p^{\frac{1}{p}}+o(1)=\lambda_\infty+o(1),\end{aligned}$$

where the $o(1)$ does not depend on ϕ. Taking the supremum over all such ϕ we obtain $\|\nabla v\|_p\leq\lambda_\infty+o(1)$, so that $\|\nabla v\|_\infty\leq\lambda_\infty$. It follows that (v,σ) is admissible for G_∞.

We now fix a pair (v,σ) admissible for G_∞. We have to find some pair (v_p,σ_p) admissible for G_p which converges to (v,σ) and such that

$$\limsup_{p\to+\infty} G_p(v_p,\sigma_p) \le G_\infty(v,\sigma).$$

We define

$$v_p = \frac{\lambda_p^{\frac{1}{p}}}{\lambda_\infty(|U|+|\Gamma_2|)^{\frac{1}{p}}}\, v.$$

Then $v_p \in W^{1,p}(U)$, $v_p \to v$ uniformly, and $\int_U |\nabla v_p|^p + \alpha^p \int_{\Gamma_2} |v_p|^p \le \lambda_p^p$.

In order to define σ_p by regularizing σ by convolution, we first need to adjust a little. Let $\vec{n}$ be the unit inner normal vector to U that we extend in a smooth way to $\mathbb{R}^n$ with compact support in a neighborhood of ∂U. We consider $T_\varepsilon : \bar{U} \to \bar{U}_{2\varepsilon} := \{x\in\bar{U},\ \mathrm{dist}(x,\partial U)\ge 2\varepsilon\}$ defined by $T_\varepsilon(x) = x+2\varepsilon\vec{n}$. Let $\sigma_\varepsilon = T_\varepsilon\sharp\sigma$ be the push-forward of σ by T_ε i.e. $\int f\,d\sigma_\varepsilon = \int f\circ T_\varepsilon\,d\sigma$ for any $f\in C(\bar{U}_{2\varepsilon})$. Observe that supp $\sigma_\varepsilon \subset \bar{U}_{2\varepsilon}$ and also that $\int|\sigma_\varepsilon|\le 1$ since

$$\begin{aligned}\int|\sigma_\varepsilon| &= \sup_{\|\phi\|_{L^\infty(U_{2\varepsilon})}\le 1}\int \phi\,d\sigma_\varepsilon = \sup_{\|\phi\|_{L^\infty(U_{2\varepsilon})}\le 1}\int \phi\circ T_\varepsilon\,d\sigma \\ &\le \int d|\sigma| \le 1.\end{aligned}$$

Moreover

$$\sigma_\varepsilon \to \sigma \quad \text{weakly in the sense of measure.}$$

Indeed for any $\phi\in C(\bar{U})$,

$$\left|\int \phi\,d\sigma_\varepsilon - \int\phi\,d\sigma\right| \le \int\left|\phi(x+2\varepsilon\vec{n})-\phi(x)\right|d\sigma(x) = o(1)$$

since the integrand goes to 0 uniformly in $x\in\bar{U}$. Denote by ρ_ε the usual mollifying functions (i.e. $\rho_\varepsilon(x)=\varepsilon^{-n}\rho(x/\varepsilon)$ where ρ is a smooth function compactly supported in the unit ball of $\mathbb{R}^n$ with $\int\rho=1$). Then

$$\rho_\varepsilon*\sigma_\varepsilon-\sigma_\varepsilon\to 0 \quad \text{weakly in the sense of measure.}$$

This follows from the fact that $\|\phi*\rho_\varepsilon-\phi\|_{L^\infty(U_{2\varepsilon})}\to 0$ for any $\phi\in C(\bar{U})$. Hence

$$\rho_\varepsilon*\sigma_\varepsilon \rightharpoonup \sigma \quad \text{weakly in the sense of measure.} \tag{9}$$

We now regularize σ_ε considering

$$\tilde{\sigma}_\varepsilon := \sigma_\varepsilon*\tilde{\rho}_\varepsilon \in C^\infty(U)$$

with

$$\tilde{\rho}_\varepsilon := \frac{\rho_\varepsilon}{\|\rho_\varepsilon\|_{p'}}, \quad \varepsilon = 1/p.$$

Then $\|\rho_\varepsilon\|_{p'}\to 1$ since $\|\rho_\varepsilon\|_{p'} = \varepsilon^{-n/p}\|\rho\|_{p'}\to\|\rho_\varepsilon\|_1 = 1$. It then follows that $\tilde{\sigma}_\varepsilon\rightharpoonup\sigma$. Moreover $\tilde{\sigma}_\varepsilon$ is admissible for G_p since, by the Holder inequality and recalling (9),

$$\|\tilde{\sigma}_\varepsilon\|_{p'}^{p'} \le \left(\int|\sigma_\varepsilon|\right)^{\frac{1}{p-1}}\int\tilde{\rho}_\varepsilon(x-y)^{p'}\,dx\,d|\sigma_\varepsilon|(y) = \|\tilde{\rho}_\varepsilon\|_{p'}^{p'}\left(\int|\sigma_\varepsilon|\right)^{\frac{p}{p-1}} \le 1.$$

It follows that $(\sigma_\varepsilon, v_p)$ is admissible for G_p and converges to (v,σ). As before we have $G_p(v_p,\sigma_\varepsilon)\to G_\infty(v,\sigma)$. □

Recall that from Theorem 1, u_p converge in $C(\bar{U})$ up to a subsequence to some $u_\infty\in C(\bar{U})$, $\|u\|_\infty = 1$. Moreover, up to a subsequence, the measures $u_p^{p-1}\,dx$ converge weakly to some probability measure σ_∞. Indeed since $\bar{U}$ is compact, it suffices, according to the Prokhorov theorem, to show that

$$\lim_{p\to+\infty}\int_{\bar{U}} u_p^{p-1}\,dx = 1.$$

This follows from

$$\int_{\bar{U}} u_p^{p-1}\,dx \le \|u_p\|_p|U|^{1/p}\to 1$$

and, for $p > n$,

$$1 = \int_{\bar U} u_p^{p-1} u_p \, dx \le \|u_p\|_\infty \int_{\bar U} u_p^{p-1}\, dx = (1+o(1)) \int_{\bar U} u_p^{p-1}\, dx.$$

As a consequence of the Γ-convergence of G_p to G_∞ and the fact that $(u_p, u_p^{p-1}\,dx)$ is a minimizer of G_p, we obtain that $(u_\infty, \sigma_\infty)$ is a minimizer of G_∞ with $G_\infty(u_\infty, \sigma_\infty) = \lim_{p\to+\infty} G_p(u_p, u_p^{p-1}\,dx) = -1$. Since $\sigma_\infty \in P(\bar U)$ and u_∞ is an extremal for λ_∞, we can thus write

$$1 = \max\left\{ \int v\, d\sigma;\ v \in W^{1,\infty}_{\Gamma_1}(U), \max\{\|\nabla v\|_\infty, \alpha\|v\|_{L^\infty(\Gamma_2)}\} = \lambda_\infty, \sigma \in P(\bar U)\right\}$$

i.e.

$$\lambda_\infty^{-1} = \max\left\{ \int v\, d\sigma;\ v \in W^{1,\infty}_{\Gamma_1}(U), \max\{\|\nabla v\|_\infty, \alpha\|v\|_{L^\infty(\Gamma_2)}\} = 1, \sigma \in P(\bar U)\right\}. \tag{10}$$

An approximation argument shows that we can replace $W^{1,\infty}_{\Gamma_1}(U)$ by $C^1(U) \cap C_{\Gamma_1}(\bar U)$ where $C_{\Gamma_1}(\bar U) = \{u \in C(\bar U) : u = 0 \text{ on } \Gamma_1\}$.

Proposition 4.2. *Given $v \in W^{1,\infty}_{\Gamma_1}(U)$, $\max\{\|\nabla v\|_\infty, \alpha\|v\|_{L^\infty(\Gamma_2)}\} \le 1$, there exist $v_k \in C^1(U) \cap C_{\Gamma_1}(\bar U)$, $\max\{\|\nabla v_k\|_\infty, \alpha\|v_k\|_{L^\infty(\Gamma_2)}\} \le 1$, such that $v_k \to v$ uniformly in $\bar U$.*

Proof. The proof uses ideas from [2]. We first extend v in a neighborhood of ∂U by antisymmetric reflection across ∂U so that the extended function $\bar v$ is Lipschitz with $\|\nabla \bar v\|_\infty = \|\nabla v\|_\infty \le 1$. We then apply the same method as in [2] consisting in introducing the function $\theta_\varepsilon(t) = (t - \mathrm{sgn}(t)\varepsilon)1_{|t|\ge\varepsilon}$ and then regularizing $\theta_\varepsilon \circ \bar v$ by convolution with the usual mollifying functions. Observe that $\|\nabla(\theta_\varepsilon \circ \bar v)\|_\infty \le \|\nabla \bar v\|_\infty \le 1$ and that $\theta_\varepsilon \circ \bar v = 0$ in the ε-neighborhood $\{x \in \mathbb{R}^n,\ \mathrm{dist}(x, \Gamma_1) < \varepsilon\}$ of Γ_1 since $\bar v$ is 1-Lipschitz. Note also that $|\theta(t)| = (|t| - \varepsilon)1_{|t|\ge\varepsilon}$ so that on Γ_2, $|\theta_\varepsilon \circ \bar v| \le (\alpha^{-1} - \varepsilon)_+$. Hence $|\theta_\varepsilon \circ \bar v| \le \alpha^{-1}$ in the ε-neighborhood of Γ_2. It follows from these three comments that the regularizing of $\theta_\varepsilon \circ \bar v$ is adequate. □

Denoting by $\mathrm{Res} : C(\bar U) \to C(\Gamma_2)$ the restriction operator, $Au = \nabla u$ the derivation operator with domain $C^1(U)$, and $B(R)$ the closed ball of radius R centered at 0 in $C(\bar U)$, $B = B(1)$, we can rewrite (10) as

$$\frac{1}{\lambda_\infty} = \max_{\sigma\in P(\bar U)} \max_{u \in C(\bar U)} \left\{ (\sigma, u) - (\chi_{B(1/\alpha)} \circ \mathrm{Res})(u) - (\chi_B \circ A)(u) - \chi_{C_{\Gamma_1}(U)}(u)\right\}.$$

Recalling the definition of the Legendre transform, we eventually obtain

$$\frac{1}{\lambda_\infty} = \max_{\sigma\in P(\bar U)} \Big((\chi_{B(1/\alpha)} \circ \mathrm{Res}) + (\chi_B \circ A) + \chi_{C_{\Gamma_1}(U)}\Big)^*(\sigma). \tag{11}$$

The inf-convolution $f \Box g$ of two proper lower semi-continuous (lsc) convex functions $f, g : E \to \mathbb{R}$ (E denotes a normed space—we will take $E = C(\bar U)$ here) is defined by $(f\Box g)(x) = \inf_{y\in E} f(y) + g(x-y)$. This operation is commutative and associative. Moreover $f\Box g$ is a proper lsc convex function with domain $\mathrm{Dom}(f) + \mathrm{Dom}(g)$, and its Legendre transform is $(f\Box g)^* = f^* + g^*$. Eventually, if 0 belongs to the interior of $\mathrm{Dom}(f) - \mathrm{Dom}(g)$ then $(f+g)^* = f^* \Box g^*$ (see [19, Section 3.9 p. 42]). This last assumption is trivially satisfied here since any neighborhood of 0 in $C(\bar U)$ is contained in $C^1(\bar U) + C(\bar U)$.

We can thus rewrite (11) as

$$\begin{aligned}\frac{1}{\lambda_\infty} &= \max_{\sigma\in P(\bar U)} \Big((\chi_{B(1/\alpha)} \circ \mathrm{Res})^* \Box (\chi_B \circ A)^* \Box \chi^*_{C_{\Gamma_1}(U)}\Big)(\sigma)\\ &= \max_{\sigma\in P(\bar U)} \inf\Big((\chi_{B(1/\alpha)} \circ \mathrm{Res})^*(\mu_1) + (\chi_B\circ A)^*(\mu_2) + \chi^*_{C_{\Gamma_1}(U)}(\mu_3)\Big),\end{aligned} \tag{12}$$

where the inf is taken over all triple of measures $\mu_1, \mu_2, \mu_3 \in M(\bar U)$ such that $\sigma = \mu_1 + \mu_2 + \mu_3$. To pursue further we need to compute the various Legendre transforms involved in this expression. This is the content of the next proposition.

Proposition 4.3. *There holds for $\mu \in M(\bar U)$,*

$$\chi^*_{C_{\Gamma_1}(U)}(\mu) = \begin{cases} 0 & \text{if } \operatorname{supp}\mu \subset \Gamma_1 \\ +\infty & \text{otherwise}\end{cases} \tag{13}$$

and

$$\begin{aligned}(\chi_B \circ A)^*(\mu) &= \inf\left\{ \int_{\bar U} |\sigma| : \sigma \in M(\bar U, \mathbb{R}^n) \text{ s.t. } -\operatorname{div}\sigma = \mu \text{ in } \mathcal{D}'(\mathbb{R}^n)\right\}\\ &= \begin{cases} W_1(\mu^+, \mu^-) & \text{if } \mu(\bar U) = 0,\\ +\infty & \text{otherwise}.\end{cases}\end{aligned} \tag{14}$$

Moreover,

$$(\chi_{B(1/\alpha)}\circ \mathrm{Res})^*(\mu)=\begin{cases}\dfrac{1}{\alpha}|\mu|(\Gamma_2) & \textit{if } \operatorname{supp}\mu\subset\Gamma_2\\ +\infty & \textit{otherwise}.\end{cases} \tag{15}$$

Proof. These computations are more or less classical. We sketch them here for the reader's convenience.

First, the definition of the Legendre transform gives

$$\begin{aligned}\chi^*_{C_{\Gamma_1}(U)}(\mu) &= \sup_{u\in C(\bar U)} (\mu,u)-\chi_{C_{\Gamma_1}(U)}(u)\\ &= \sup_{u\in C(\bar U),\,u=0 \text{ on } \Gamma_1}\int_{\bar U} u\,d\mu\end{aligned} \tag{16}$$

from which we deduce (13).

We now prove (14). The second equality in (14) is well-known. It remains to prove the first one. We recall the following result concerning the Legendre transform: if E and F are two normed space, $L:E\to F$ linear with domain $\mathrm{Dom}(L)$ and $f:E\to\mathbb{R}$ is convex, consider the function $(LF)(y)=\inf\{f(x): x\in \mathrm{Dom}(L) \text{ s.t. } Lx=y\}$, $y\in F$. Then Lf is convex with $(Lf)^*=f^*\circ L^*$ in the domain $\mathrm{Dom}(L^*)$ of the adjoint $L^*:F^*\to E^*$ of L.

Notice that the adjoint $A^*: M(\bar U)\to M(\bar U)$ of A is defined by $A^*\mu=-\mathrm{div}\,\mu$ in the weak sense (i.e. $(A^*\mu,u)=(\mu,\nabla u)=\int\nabla u\,d\mu$ for any $u\in \mathrm{Dom}(A)=C^1(\bar U)$) with domain $\mathrm{Dom}(A^*)=\{\mu\in M(\bar U),\ -\mathrm{div}\,\mu\in M_b(\mathbb{R}^n)\}$.

In a similar way as in (16), it can be seen that $\chi^*_B(\sigma)=\int|\sigma|$, so that the inf in (14) can be written as $(A^*\chi^*_B)(\mu)$. Then taking $f=\chi^*_B$, $L=A^*$ and noticing that χ_B is convex lsc (because B is convex and closed), so that $\chi^{**}_B=\chi_B$, we obtain $\chi_B\circ A^{**}=(A^*\chi^*_B)^*$. Observe that $A^{**}=A$ on $\mathrm{Dom}(A)$ so that $\chi_B\circ A=(A^*\chi^*_B)^*$ on $\mathrm{Dom}(A)$.

Observe that $A^*\chi^*_B$, which is the r.h.s. of (14), is lsc for the weak convergence (and thus also for the strong i.e. total variation convergence) in the sense that if $\mu_n,\mu\in M(\bar U)$ verify $\mu_n\to\mu$ weakly then

$$\liminf_{n\to+\infty}(A^*\chi^*_B)(\mu_n)\ge (A^*\chi^*_B)(\mu).$$

Indeed we can assume that $(A^*\chi^*_B)(\mu_n)\le Cste$. Then taking $\sigma_n\in M(\bar U,\mathbb{R}^n)$ s.t. $-\mathrm{div}\,\sigma_n=\mu_n$ and $A^*\chi^*_B(\mu_n)=\int|\sigma_n|+o(1)$, we have $\int|\sigma_n|\le C$. Then applying the Prokhorov theorem to σ_n^+ and σ^-, we have, up to a subsequence, that $\sigma_n\to\sigma$ weakly. In particular $-\mathrm{div}\,\sigma=\mu$ and $\liminf_{n\to+\infty}\int|\sigma_n|\ge\int|\sigma|\ge(A^*\chi^*_B)(\sigma)$ from which we deduce the result.

We thus have that $A^*\chi^*_B$ is convex lsc so that $A^*\chi^*_B=(A^*\chi^*_B)^{**}$. Hence $(\chi_B\circ A)^*=A^*\chi^*_B$ which is exactly (14).

The proof of (15) is similar. We have as before that for any $\mu\in M(\bar U)$,

$$(\chi_{B(1/\alpha)}\circ\mathrm{Res})^*(\mu)=(\mathrm{Res}^*\chi^*_{B(1/\alpha)})(\mu)=\inf\{(\chi^*_{B(1/\alpha)})(\sigma):\mathrm{Res}^*(\sigma)=\mu\}$$

with $\mathrm{Res}: C(\bar U)\to C(\Gamma_2)$ and $\mathrm{Res}^*: C(\Gamma_2)^*=M(\Gamma_2)\to C(\bar U)^*=M(\bar U)$ is given by

$$(\mathrm{Res}^*(\sigma),v)=(\sigma,\mathrm{Res}(v))=(\sigma,v_{|\Gamma_2})=\int_{\Gamma_2}v\,d\sigma$$

for any $\sigma\in C(\Gamma_2)^*$, $v\in C(\bar U)$. Moreover $\chi_{B(1/\alpha)}: C(\Gamma_2)\to\mathbb{R}$ and for any $\sigma\in C(\Gamma_2)^*$,

$$\begin{aligned}\chi^*_{B(1/\alpha)}(\sigma) &= \sup_{v\in C(\Gamma_2)}(\sigma,v)-\chi_{B(1/\alpha)}(v)=\sup_{v\in C(\Gamma_2),\|v\|_{L^\infty(\Gamma_2)}\le 1/\alpha}\int_{\Gamma_2}v\,d\sigma\\ &=\frac{1}{\alpha}\sup_{v\in C(\Gamma_2),\|v\|_{L^\infty(\Gamma_2)}\le 1}\int_{\Gamma_2}v\,d\sigma\\ &=\frac{1}{\alpha}\int_{\Gamma_2}|\sigma|.\end{aligned}$$

Thus

$$(\chi_{B(1/\alpha)}\circ\mathrm{Res})^*(\mu)=\inf\left\{\frac{1}{\alpha}\int_{\Gamma_2}|\sigma|:\sigma\in C(\Gamma_2)^* \text{ s.t. } \int_{\Gamma_2}u\,d\sigma=\int_{\bar U}u\,d\mu \text{ for all } u\in C(\bar U)\right\}.$$

Consider an admissible measure σ. Then for any $A\subset\bar U$,

$$\sigma(A\cap\Gamma_2)=\int_{\Gamma 2}1_A\,d\sigma=\int_{\bar U}1_A\,d\mu=\mu(A).$$

It follows that there cannot exist $A\subset\bar U\setminus\Gamma_2$ s.t. $\mu(A)\neq 0$ i.e. $\operatorname{supp}\mu\subset\Gamma_2$, and then $\sigma=\mu$. Hence $(\chi_{B(1/\alpha)}\circ\mathrm{Res})^*(\mu)=\frac{1}{\alpha}|\mu|(\Gamma_2)$ if $\operatorname{supp}\mu\subset\Gamma_2$. Otherwise there does not exist any admissible σ and the inf is $+\infty$. □

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Using the previous proposition, we can rewrite (16) as

$$\frac{1}{\lambda_\infty} = \max_{\sigma\in P(\bar U)} \inf (\chi_B \circ A)^*(\mu_2) + \frac{1}{\alpha}|\mu_1|(\Gamma_2),$$

where the inf is taken over all triple of measures $\mu_1, \mu_2, \mu_3 \in M(\bar U)$ such that $\sigma = \mu_1 + \mu_2 + \mu_3$, supp $\mu_3 \subset \Gamma_1$, supp $\mu_1 \subset \Gamma_2$, $\mu_2(\bar U) = 0$. Letting $\nu = \mu_1 + \mu_3 = \sigma - \mu_2$, we have $|\mu_1|(\Gamma_2) = |\nu|(\Gamma_2) = \nu^+(\Gamma_2) + \nu^-(\Gamma_2)$ since μ_1 and μ_3 have disjoint support. Moreover, since $\mu_2(\bar U) = 0$ i.e. $(\sigma + \nu^-)(\bar U) = \nu^+(\bar U)$, we have

$$\begin{aligned}(\chi_B \circ A)^*(\mu_2) &= (\chi_B \circ A)^*(\sigma - \nu)\\ &= \inf\Big\{\int_{\bar U} |\tilde\sigma| : \tilde\sigma \in M(\bar U, \mathbb{R}^n) \text{ s.t.} - \operatorname{div}\tilde\sigma = (\sigma + \nu^-) - \nu^+ \text{ in } \mathcal{D}'(\mathbb{R}^n)\Big\}\\ &= W_1(\sigma + \nu^-, \nu^+).\end{aligned}$$

We thus obtain

$$\frac{1}{\lambda_\infty} = \max_{\sigma\in P(\bar U)} \inf_{\nu\in M(\partial U),\,\nu(\partial U)=1} W_1(\sigma + \nu^-, \nu^+) + \frac{1}{\alpha}\nu^+(\Gamma_2) + \frac{1}{\alpha}\nu^-(\Gamma_2).$$

To conclude the proof of (7), it suffices to verify that the inf can be taken over non-negative ν. This is a consequence of the following proposition:

Proposition 4.4. *For any $\sigma \in P(\bar U)$,*

$$\inf_{\nu_1,\nu_2\in M_+(\partial U),\,\nu_2(\partial U)=\nu_1(\partial U)+1} W_1(\sigma + \nu_1, \nu_2) = \inf_{\nu\in P(\partial U)} W_1(\sigma, \nu).$$

The proof of this lemma is based on the following lemma:

Lemma 2. *Consider probability measures $\mu_\varepsilon, \mu \in P(\mathbb{R}^n)$ such that*

$$\lim_{\varepsilon\to 0} W_1(\mu_\varepsilon, \mu) = 0,$$

and a subset $A \subset P(\mathbb{R}^n)$ compact w.r.t. the convergence in distance W_1. Then $\lim_{\varepsilon\to 0} W_1(\mu_\varepsilon, A) = W_1(\mu, A)$ where $W_1(\mu, A) = \inf_{\nu\in A} W_1(\mu, \nu)$.

Observe that the compactness assumption is satisfied for $A = P(K)$ where $K \subset \mathbb{R}^n$ is compact in view of Prokhorov theorem and the fact that W_1 matrices the weak convergence in $P(K)$ (because K is bounded).

Proof of Lemma 2. Consider $\nu_\delta \in A$ s.t. $\lim_{\delta\to 0} W_1(\nu_\delta, \mu) = W_1(\mu, A)$. Then passing to the limit in $W_1(\mu_\varepsilon, A) \le W_1(\mu_\varepsilon, \nu_\delta)$ yields $\limsup_{\varepsilon\to 0} W_1(\mu_\varepsilon, A) \le W_1(\mu, \nu_\delta)$ for any δ, so that $\limsup_{\varepsilon\to 0} W_1(\mu_\varepsilon, A) \le W_1(\mu, A)$.

To prove the opposite inequality we consider $\nu_\varepsilon \in A$ such that $W_1(\mu_\varepsilon, \nu_\varepsilon) = W_1(\mu_\varepsilon, A) + o(1)$. Since A is compact, we can assume up to a subsequence that there exists $\nu \in A$ s.t. $W_1(\nu_\varepsilon, \nu) \to 0$. Since $W_1(\mu_\varepsilon, \mu) \to 0$, we obtain

$$\lim_{\varepsilon\to 0} W_1(\mu_\varepsilon, A) = \lim_{\varepsilon\to 0} W_1(\mu_\varepsilon, \nu_\varepsilon) = W_1(\mu, \nu) \ge W_1(\mu, A)$$

which ends the proof of the lemma. □

We now prove Proposition 4.4.

Proof of Proposition 4.4. The $\le$ inequality is clear (take $\nu_1 = 0$). To prove the opposite inequality, we first assume that supp $\sigma \subset U$. Given any ν_1, ν_2, any transfer plan $\pi \in \Pi(\sigma + \nu_1, \nu_2)$ (i.e. $\pi \in P(\bar U)$ has marginals $\sigma + \nu_1$ and ν_2) can be written as

$$\pi = \tilde\pi + \bar\pi, \quad \tilde\pi \in \Pi(\sigma, \tilde\nu_2),\ \bar\pi \in \Pi(\nu_1, \bar\nu_2)$$

for some decomposition $\nu_2 = \tilde\nu_2 + \bar\nu_2$ with $\tilde\nu_2, \bar\nu_2 \in M_+(\partial U)$, $\tilde\nu_2(\partial U) = 1$, $\bar\nu_2(\partial U) = \nu_1(\partial U)$. It follows that

$$\begin{aligned}W_1(\sigma + \nu_1, \nu_2) &= \inf_{\pi\in\Pi(\sigma+\nu_1,\nu_2)} \int_{\bar U\times\bar U} d(x, y)\, d\pi(x, y)\\ &= \inf_{\nu_2=\tilde\nu_2+\bar\nu_2,\ \tilde\pi\in\Pi(\sigma,\tilde\nu_2),\ \bar\pi\in\Pi(\nu_1,\bar\nu_2)} \int_{\bar U\times\bar U} d(x, y)\, d\tilde\pi(x, y) + \int_{\bar U\times\bar U} d(x, y)\, d\bar\pi(x, y)\\ &\ge \inf_{\nu_2=\tilde\nu_2+\bar\nu_2} W_1(\sigma, \tilde\nu_2) + W_1(\nu_1, \bar\nu_2).\end{aligned}$$

Then

$$\inf_{\nu_1,\nu_2\in M_+(\partial U),\,\nu_2(\partial U)=\nu_1(\partial U)+1} W_1(\sigma + \nu_1, \nu_2) \ge \inf_{\nu_1,\nu_2\in M_+(\partial U),\,\nu_2(\partial U)=\nu_1(\partial U)+1}\ \inf_{\nu_2=\tilde\nu_2+\bar\nu_2} W_1(\sigma, \tilde\nu_2) + W_1(\nu_1, \bar\nu_2)$$

which is clearly greater than or equal to $\inf_{\tilde\nu_2\in P(\partial U)} W_1(\sigma, \tilde\nu_2)$. This proves the $\ge$ inequality when supp $\sigma \subset U$.

In the general case we have supp $\sigma \subset \bar{U}$. We consider $\sigma_\varepsilon = T_\varepsilon \sharp \sigma$ the push-forward of σ under $T_\varepsilon(x) = x - \varepsilon \vec{n}$ where $\vec{n}$ denotes some smooth extension of the unit exterior normal to a neighborhood of ∂U. Then supp $\sigma_\varepsilon \subset U$ so that

$$\inf_{\nu_1,\nu_2 \in M_+(\partial U),\, \nu_2(\partial U)=\nu_1(\partial U)+1} W_1(\sigma_\varepsilon + \nu_1, \nu_2) = W_1(\sigma_\varepsilon, P(\partial U)).$$

To pass to the limit as $\varepsilon \to 0$, we use Lemma 2. Just notice that $\sigma_\varepsilon \to \sigma$ weakly as measure i.e. $W_1(\sigma_\varepsilon, \sigma) \to 0$ since U is bounded, and $A = P(\partial U)$ is compact for the weak convergence. We then have $W_1(\sigma_\varepsilon, P(\partial U)) \to W_1(\sigma, P(\partial U))$. Observe also that the first part of the proof of Proposition 4.4, which does not use the compactness assumption, yields

$$\limsup_{\varepsilon \to 0} \inf_{\nu_1,\nu_2 \in M_+(\partial U),\, \nu_2(\partial U)=\nu_1(\partial U)+1} W_1(\sigma_\varepsilon + \nu_1, \nu_2) \le \inf_{\nu_1,\nu_2 \in M_+(\partial U),\, \nu_2(\partial U)=\nu_1(\partial U)+1} W_1(\sigma + \nu_1, \nu_2).$$

The result follows. □

To end the proof of Theorem 3, we verify that the max in (7) is attained by f_∞, the weak limit as $p \to +\infty$ of the measures $f_p = u_p^{p-1}\,dx$ (which exists up to a subsequence). Notice that u_p is the unique minimizer of the functional $F_p : W^{1,p}_{\Gamma_1}(U) \to \mathbb{R}$ defined by

$$F_p(u) = \frac{1}{p\lambda_p} \int_U |\nabla u|^p + \frac{\alpha^p}{p\lambda_p} \int_{\Gamma_2} |u|^p - (f_p, u).$$

Indeed the associated Euler–Lagrange equation, which has a unique solution since F_p is strictly convex, is the equation $\Delta_p u = \lambda_p f_p$ with the boundary conditions of (1), which admits u_p as a solution.

Writing F_p as

$$F_p(u) = \int_U \left| \frac{\nabla u}{p^{1/p} \lambda_p^{1/p}} \right|^p + \int_{\Gamma_2} \left| \frac{\alpha u}{p^{1/p} \lambda_p^{1/p}} \right|^p - (f_p, u),$$

we can prove, as in Proposition 4.1, that F_p Γ-converge as $p \to +\infty$ to the functional $F_\infty : C(\bar{U}) \to \mathbb{R}$ defined by

$$F_\infty(u) = \begin{cases} -(f_\infty, u), & \text{if } u \in W^{1,\infty}_{\Gamma_1}(U),\ \|\nabla u\|_\infty \le \lambda_\infty, \text{ and } \alpha \|u\|_{L^\infty(\Gamma_2)} \le \lambda_\infty \\ +\infty & \text{otherwise.} \end{cases}$$

Since

$$\inf F_p = F_p(u_p) = \frac{1}{p} - 1,$$

we obtain that

$$F_\infty(u_\infty) = \inf F_\infty = \lim_{p \to +\infty} \inf F_p = -1.$$

Hence

$$-1 = \min \left\{ -(f_\infty, u) + \chi_{B(1/\alpha)}(u_{|\Gamma_1}/\lambda_\infty) + \chi_B(\nabla u/\lambda_\infty) + \chi_{C_{\Gamma_1}(\bar{U})}(u) \right\},$$

i.e.

$$-\frac{1}{\lambda_\infty} = \min \left\{ -(f_\infty, u) + \chi_{B(1/\alpha)}(u_{|\Gamma_1}) + \chi_B(\nabla u) + \chi_{C_{\Gamma_1}(\bar{U})}(u) \right\}.$$

Then

$$\begin{aligned} \frac{1}{\lambda_\infty} &= \max_{u \in C(\bar{U})} \left\{ (f_\infty, u) - \chi_{B(1/\alpha)}(u_{|\Gamma_1}) - \chi_B(\nabla u) - \chi_C(u) \right\} \\ &= \left((\chi_{B(1/\alpha)} \circ \mathrm{Res}) + (\chi_B \circ A) + \chi_{C_{\Gamma_1}(\bar{U})} \right)^* (f_\infty). \end{aligned}$$

Since $f_\infty \in P(\partial U)$, we obtain in view of (11) that f_∞ is extremal in (7).

5. Proof of Theorem 2 for connected domains

Let $\phi(\sigma, \nu) = W_1(\sigma, \nu) + \frac{1}{\alpha}\nu(\Gamma_2)$, $\sigma, \nu \in P(\partial U)$. Since W_1 is convex in (σ, ν) (see e.g. [15, Theorem 4.8]), we see that ϕ is convex. It easily follows that the function $\Phi(\sigma) = \inf_{\nu \in P(\partial U)} \phi(\sigma, \nu)$, $\sigma \in P(\bar{U})$ is also convex. Indeed given $\sigma_1, \sigma_2 \in P(\bar{U})$ and any $\nu_1, \nu_2 \in P(\bar{U})$, we have

$$\begin{aligned} \Phi(t\sigma_1 + (1-t)\sigma_2) &\le \phi(t\sigma_1 + (1-t)\sigma_2, t\nu_1 + (1-t)\nu_2) \\ &\le t\phi(\sigma_1, \nu_1) + (1-t)\phi(\sigma_2, \nu_2). \end{aligned}$$

The result follows taking the infimum in ν_1, ν_2.

Since Φ is convex, it attains its maximum at an extreme point of the convex compact $P(\bar{U})$ i.e. at some Dirac mass δ_x, $x \in \bar{U}$:

$$\frac{1}{\lambda_\infty} = \max_{x\in\bar{U}} \inf_{\nu\in P(\partial U)} W_1(\delta_x, \nu) + \frac{1}{\alpha}\nu(\Gamma_2).$$

It is well-known that $W_1(\delta_x, \nu) = \int_{\bar{U}} d(x, y)\, d\nu(y)$ for any $x \in \bar{U}$. This follows from the fact that the unique $\pi \in P(\bar{U} \times \bar{U})$ with marginals δ_x and ν is $\pi = \delta_x \otimes \nu$. Indeed such a π must have support in $\{x\} \times \operatorname{supp} \nu$ so that for any $A, B \subset \bar{U}$, $\pi(A \times B) = 0 = (\delta_x \otimes \nu)(A \times B)$ if $x \notin A$, and if $x \in A$,

$$\pi(A \times B) = \pi(\{x\} \times B) = \pi(X \times B) = \nu(B) = (\delta_x \otimes \nu)(A \times B).$$

Given $x \in \bar{U}$, we consider $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$ such that $d(x, \Gamma_i) = d(x, x_i),\ i = 1, 2$. We write $\nu \in P(\partial U)$ as $\nu = \nu_1 + \nu_2$ where $\nu_i = \nu_{|\Gamma_i},\ i = 1, 2$. Then

$$\begin{aligned} W_1(\delta_x, \nu) &= \int_{\partial U} d(x, y)\, d\nu(y) = \int_{\Gamma_1} d(x, y)\, d\nu_1(y) + \int_{\Gamma_2} d(x, y)\, d\nu_2(y) \\ &\geq d(x, \Gamma_1)\nu_1(\Gamma_1) + d(x, \Gamma_2)\nu_2(\Gamma_2) \\ &= W_1(\delta_{x_1}, \beta\delta_{x_1} + (1-\beta)\delta_{x_2}), \end{aligned}$$

where $\beta = \nu_1(\Gamma_1)$. We thus have

$$\frac{1}{\lambda_\infty} = \max_{x\in\bar{U}} \inf_{0\leq\beta\leq 1} \beta d(x, \Gamma_1) + (1-\beta) d(x, \Gamma_2) + \frac{1-\beta}{\alpha}.$$

We deduce Theorem 2 noticing that for any $x \in \bar{U}$, the inf in β is

$$\begin{cases} d(x, \Gamma_2) + \dfrac{1}{\alpha} & \text{if } d(x, \Gamma_1) - d(x, \Gamma_2) - \dfrac{1}{\alpha} \geq 0 \text{ i.e. } x \in \mathcal{A} \\ d(x, \Gamma_1) & \text{otherwise.} \end{cases}$$

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