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Existence of solution to a critical trace equation with variable exponent

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Abstract. In this paper we study sufficient local conditions for the existence of non-trivial solution to a critical equation for the $p(x)$-Laplacian where the critical term is placed as a source through the boundary of the domain. The proof relies on a suitable generalization of the concentration–compactness principle for the trace embedding for variable exponent Sobolev spaces and the classical mountain pass theorem.

Keywords: Sobolev embedding, variable exponents, critical exponents, concentration compactness

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set. The purpose of this article is the study of the existence of a nontrivial solution to the critical trace equation

$$\begin{cases} -\Delta_{p(x)}u + h|u|^{p(x)-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\,\partial_\nu u = |u|^{r(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian corresponding to some given function $p : \bar{\Omega} \to (1, +\infty)$ (notice that when p is constant we recover the usual p-Laplacian), ∂_ν is the outer normal derivative, and h is a smooth function satisfying some coercivity assumption (see the definition of the norm in (3.4)). The exponents $p : \bar{\Omega} \to (1, +\infty)$ and $r : \partial\Omega \to [1, +\infty)$ are continuous functions that verify

$$1 < p^- := \inf_{x\in\Omega} p(x) \leqslant p^+ := \sup_{x\in\Omega} p(x) < N \quad \text{and} \quad r(x) \leqslant p_*(x) = \frac{(N-1)p(x)}{N-p(x)}. \tag{1.2}$$

The exponent p_* is critical from the point of view of the Sobolev trace embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ (see Theorems 2.4 and 2.5 in Section 2 for a precise statement).

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We focus in this paper on the critical problem for (1.1) in the sense that we will assume from now on that

$$\mathcal{A}_T := \{x \in \partial\Omega : r(x) = p_*(x)\} \neq \emptyset. \tag{1.3}$$

Under this assumption the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ is generally not compact so that the existence of a non-trivial solution to (1.1) is a non-trivial problem. Our main purpose is to find conditions on p, r and Ω in the spirit of [1,12] and [18], where this kind of problem has been considered in the constant exponent case, ensuring the existence of a non-trivial solution to (1.1).

Observe that problem (1.1) is variational in the sense that weak solutions are critical points of the associated functional

$$\mathcal{F}(u) := \int_\Omega \frac{1}{p(x)}\left[|\nabla u|^{p(x)} + h|u|^{p(x)}\right] \mathrm{d}x - \int_{\partial\Omega} \frac{1}{r(x)}|u|^{r(x)} \,\mathrm{d}S, \tag{1.4}$$

where $\mathrm{d}S$ denotes the boundary measure. This functional $\mathcal{F}$ is well defined in $W^{1,p(x)}(\Omega)$ thanks to (1.2) (see Theorem 2.4 in Section 2). The main tool available in order to find critical points for C^1 functionals in Banach spaces is the well known Mountain Pass Theorem (MPT). The MPT has two types of hypotheses, geometrical and topological.

For the functional $\mathcal{F}$ it is fairly easy to see that when $p^+ < r^-$ the geometrical hypotheses of the MPT are satisfied. The topological hypothesis is the so-called Palais–Smale condition that requires for a sequence of approximate critical points to be precompact. When $r(x)$ is *uniformly subcritical*, i.e.

$$\inf_{x\in\partial\Omega}\left(p_*(x) - r(x)\right) > 0, \tag{1.5}$$

the immersion $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ is compact. It is then straightforward to check that the Palais–Smale condition is satisfied for every energy level c.

Notice that there are some cases where the subcriticality is violated but still the immersion is compact. In fact, in [21] the authors find conditions on the exponents p and r such that $\mathcal{A}_T \neq \emptyset$ but the immersion remains compact. This type of conditions were first discovered in [28] where the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, $q(x) \leqslant p^*(x) := Np(x)/(N - p(x))$ was analyzed. The result in [21] shows that if the criticality set $\mathcal{A}_T$ is "small" and we have a control on how the exponent r reaches p_* at the criticality set, then the immersion $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ remains compact, and so the existence of solutions to (1.1) follows as in the subcritical case.

However, in the general case $\mathcal{A}_T \neq \emptyset$, the present paper is, up to our knowledge, the first work regarding the existence of solutions for (1.1).

Recently, in [21], the authors analyzed the problem of the existence of extremals for the immersion $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$, that is functions realizing the infimum in

$$0 < T\left(p(\cdot), r(\cdot), \Omega\right) := \inf_{v\in W^{1,p(x)}(\Omega)} \frac{\|v\|_{W^{1,p(x)}(\Omega)}}{\|v\|_{L^{r(x)}(\partial\Omega)}}.$$

In [21] the main tool used to deal with the existence of extremals problem is the extension of the celebrated Concentration–Compactness Principle (CCP) of P.L. Lions to the variable exponent case. In the case of the immersion $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ this was done independently by [22] and [23] (see also

[20] where a refinement of the result was obtained). For the trace immersion, this result was proved in the above mentioned paper [21].

In order to state our main results we need to introduce some notation. Given some nonempty, closed subset $\Gamma \subset \partial\Omega$ (possibly empty), we consider the space $W_\Gamma^{1,p(x)}(\Omega)$ defined by

$$W_\Gamma^{1,p(x)}(\Omega) := \overline{\left\{u \in C^1(\bar{\Omega}) \colon u = 0 \text{ in a neighborhood of } \Gamma\right\}},$$

the closure being taken in the $\|\cdot\|_{1,p(x)}$-norm. This is the space of functions vanishing on Γ. Observe that $W_\emptyset^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega)$ and, more generally, that $W_\Gamma^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega)$ if and only if Γ has $p(x)$-capacity zero. See [25]. Given a critical point $x \in \mathcal{A}_T$, we define the localized best Sobolev trace constant $\bar{T}_x$ around x by

$$\bar{T}_x = \sup_{\varepsilon>0} T\big(p(\cdot), r(\cdot), \Omega_\varepsilon, \Gamma_\varepsilon\big), \tag{1.6}$$

where

$$\begin{aligned} &T\big(p(\cdot), r(\cdot), \Omega_\varepsilon, \Gamma_\varepsilon\big) = \inf_{v \in W_{\Gamma_\varepsilon}^{1,p(x)}(\Omega_\varepsilon)} \frac{\|v\|_{W^{1,p(x)}(\Omega_\varepsilon)}}{\|v\|_{L^{r(x)}(\partial\Omega_\varepsilon)}} \quad \text{and} \\ &\Omega_\varepsilon = \Omega \cap B_\varepsilon(x), \qquad \Gamma_\varepsilon = \Omega \cap \partial B_\varepsilon(x). \end{aligned} \tag{1.7}$$

Our first result states that the functional $\mathcal{F}$ defined in (1.4) verifies the Palais–Smale condition for any energy level c below a *critical energy level* c^* given by

$$c^* := \inf_{x \in \mathcal{A}_T} \left(\frac{1}{p(x)} - \frac{1}{p_*(x)}\right) \bar{T}_x^{\frac{p(x)p_*(x)}{p_*(x)-p(x)}}.$$

As an immediate corollary of this result, we obtain applying the MPT the existence of a solution to (1.1) provided there exists a function $v \in W^{1,p(x)}(\Omega)$ such that

$$\sup_{t>0} \mathcal{F}(tv) < c^*. \tag{1.8}$$

The rest of the paper is devoted to find conditions on p, r and Ω that allow us to construct a function v that satisfies (1.8). The idea used in the construction of such v is to rescale and truncate an extremal for the Sobolev trace immersion

$$\bar{K}(N,p)^{-1} = \inf_{f \in C_c^\infty(\mathbb{R}^N)} \frac{(\int_{\mathbb{R}_+^N} |\nabla f|^p \,\mathrm{d}x)^{1/p}}{(\int_{\mathbb{R}^{N-1}} |f|^{p_*} \,\mathrm{d}y)^{1/p_*}}.$$

These extremals were found by Nazaret in [29] by means of mass transportation methods extending the well known result of Escobar in [12] where the case $p = 2$ was studied. These extremals are of the form

$$V_{\lambda,y_0}(y,t) = \lambda^{-\frac{N-p}{p-1}} V\left(\frac{y-y_0}{\lambda}, \frac{t}{\lambda}\right), \quad y \in \mathbb{R}^{N-1},\ t > 0, \tag{1.9}$$

where

$$V(y,t) = r^{-\frac{N-p}{p-1}}, \quad r = \sqrt{(1+t)^2 + |y|^2}. \tag{1.10}$$

Similar ideas were used recently in [21] were the existence problem for extremals in the critical Sobolev trace immersion was studied. These ideas were also previously used for (1.1) in the constant exponent case by Adimurthi and Yadava [1], Escobar [12], and Fernandez Bonder and Saintier in [18]. Let us mention that these ideas are classical when dealing with critical equations. They go back to the seminal paper of Aubin [2] and Brezis and Nirenberg [6] and have been widely used since then in the constant exponent case (see e.g. [4,9–13,15,18,26,31–33] and references therein). In the variable setting we refer to the recent paper [19] where analogous results for the critical problem with Dirichlet boundary conditions have been obtained.

1.1. Organization of the paper

The rest of the paper is organized as follows. In Section 2, we collect some preliminaries on variable exponent spaces that will be used throughout the paper. In Section 3 we give an existence criteria for solutions, namely condition (1.8). In Section 4 we give conditions that ensure the validity of such criteria. We leave for the Appendix some asymptotic expansions needed in the proof of our results.

2. Preliminaries on variable exponent Sobolev spaces

In this section we review some preliminary results regarding Lebesgue and Sobolev spaces with variable exponent. All of these results and a comprehensive study of these spaces can be found in [8].

We denote by $\mathcal{P}(\Omega)$ the set of Lebesgue measurable functions $p : \Omega \to [1,\infty)$. Given $p \in \mathcal{P}(\Omega)$ we consider the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ defined by

$$L^{p(x)}(\Omega) = \left\{u \in L^1_{\text{loc}}(\Omega)\colon \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\right\}.$$

This space is endowed with the (Luxembourg) norm

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x)} := \inf\left\{\lambda > 0\colon \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}\mathrm{d}x \leqslant 1\right\}.$$

The following Hölder-type inequality is proved in [17,27] (see also [8], pp. 79, Lemma 3.2.20 (3.2.23)).

Proposition 2.1 (Hölder-type inequality)**.** *Let $f \in L^{p(x)}(\Omega)$ and $g \in L^{q(x)}(\Omega)$. Then the following inequality holds*

$$\|fg\|_{L^{s(x)}(\Omega)} \leqslant \left(\left(\frac{s}{p}\right)^+ + \left(\frac{s}{q}\right)^+\right)\|f\|_{L^{p(x)}(\Omega)}\|g\|_{L^{q(x)}(\Omega)},$$

where

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}.$$

The following proposition, also proved in [27], will be most useful (see also [8], Chapter 2, Section 1).

Proposition 2.2. *Set* $\rho(u) := \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x$. *For* $u \in L^{p(x)}(\Omega)$ *and* $\{u_k\}_{k\in\mathbb{N}} \subset L^{p(x)}(\Omega)$, *we have*

$$u \neq 0 \quad \Longrightarrow \quad \left(\|u\|_{L^{p(x)}(\Omega)} = \lambda \Longleftrightarrow \rho\left(\frac{u}{\lambda}\right) = 1\right), \tag{2.1}$$

$$\|u\|_{L^{p(x)}(\Omega)} < 1(=1; >1) \quad \Longleftrightarrow \quad \rho(u) < 1(=1; >1), \tag{2.2}$$

$$\|u\|_{L^{p(x)}(\Omega)} > 1 \quad \Longrightarrow \quad \|u\|^{p^-}_{L^{p(x)}(\Omega)} \leqslant \rho(u) \leqslant \|u\|^{p^+}_{L^{p(x)}(\Omega)}, \tag{2.3}$$

$$\|u\|_{L^{p(x)}(\Omega)} < 1 \quad \Longrightarrow \quad \|u\|^{p^+}_{L^{p(x)}(\Omega)} \leqslant \rho(u) \leqslant \|u\|^{p^-}_{L^{p(x)}(\Omega)}, \tag{2.4}$$

$$\lim_{k\to\infty} \|u_k\|_{L^{p(x)}(\Omega)} = 0 \quad \Longleftrightarrow \quad \lim_{k\to\infty} \rho(u_k) = 0, \tag{2.5}$$

$$\lim_{k\to\infty} \|u_k\|_{L^{p(x)}(\Omega)} = \infty \quad \Longleftrightarrow \quad \lim_{k\to\infty} \rho(u_k) = \infty. \tag{2.6}$$

The following lemma is the extension to variable exponents of the well-known Brezis–Lieb lemma (see [5]). The proof is analogous to that of [5]. See Lemma 3.4 in [22].

Lemma 2.3. *Let* $f_n \to f$ *a.e. and* $f_n \rightharpoonup f$ *in* $L^{p(x)}(\Omega)$ *then*

$$\lim_{n\to\infty}\left(\int_\Omega |f_n|^{p(x)}\,\mathrm{d}x - \int_\Omega |f - f_n|^{p(x)}\,\mathrm{d}x\right) = \int_\Omega |f|^{p(x)}\,\mathrm{d}x.$$

We now define the variable exponent Lebesgue spaces on $\partial\Omega$. First we denote by $\mathcal{P}(\partial\Omega)$ the set of $\mathcal{H}^{N-1}$-measurable functions $r : \partial\Omega \to [1, \infty)$. We then assume that Ω is C^1 so that $\partial\Omega$ is a $(N-1)$-dimensional C^1 immersed manifold on $\mathbb{R}^N$ (although the trace theorem require less regularity on $\partial\Omega$, the C^1 regularity will be enough for our purposes). Therefore the boundary measure agrees with the $(N-1)$-Hausdorff measure restricted to $\partial\Omega$. We denote this measure by $\mathrm{d}S$. Then, the Lebesgue spaces on $\partial\Omega$ are defined as

$$L^{r(x)}(\partial\Omega) := \left\{u \in L^1_{\mathrm{loc}}(\partial\Omega, \mathrm{d}S)\colon \int_{\partial\Omega} \left|u(x)\right|^{r(x)}\mathrm{d}S < \infty\right\},$$

and the corresponding (Luxembourg) norm is given by

$$\|u\|_{L^{r(x)}(\partial\Omega)} = \|u\|_{r(x),\partial\Omega} := \inf\left\{\lambda > 0\colon \int_{\partial\Omega}\left|\frac{u(x)}{\lambda}\right|^{r(x)}\mathrm{d}S \leqslant 1\right\}.$$

We can define in a similar way the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{u \in W^{1,1}_{\mathrm{loc}}(\Omega)\colon \partial_i u \in L^{p(x)}(\Omega) \text{ for } i = 1, \ldots, N\right\},$$

where $\partial_i u = \frac{\partial u}{\partial x_i}$ is the ith-distributional partial derivative of u. This space has a corresponding modular given by

$$\rho_{1,p(x)}(u) := \int_\Omega |u|^{p(x)} + |\nabla u|^{p(x)}\,\mathrm{d}x,$$

and so the corresponding norm for this space is

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{1,p(x)} := \inf\left\{\lambda > 0\colon \rho_{1,p(x)}\left(\frac{u}{\lambda}\right) \leqslant 1\right\}. \tag{2.7}$$

The $W^{1,p(x)}(\Omega)$ norm can also be defined as $\|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. Both norms turn out to be equivalent but we use the first one for convenience.

The following Sobolev trace theorems are proved in [16].

Theorem 2.4 ([16, Theorem 2.1])**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary and let $p \in \mathcal{P}(\Omega)$ be such that $p \in W^{1,\gamma}(\Omega)$ with $1 \leqslant p^- \leqslant p^+ < N < \gamma$. Then there is a continuous boundary trace embedding $W^{1,p(x)}(\Omega) \subset L^{p_*(x)}(\partial\Omega)$.*

We used the following notation: for a μ-measurable function f we denote $f^+ := \sup f$ and $f^- := \inf f$, where by sup and inf we denote the essential supremum and essential infimum respectively with respect to the measure μ.

The regularity assumption on p can be relaxed when the exponent r is uniformly subcritical in the sense of (1.5). It holds the following theorem.

Theorem 2.5 ([16, Theorem 2.2])**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Suppose that $p \in C^0(\bar{\Omega})$ and $1 < p^- \leqslant p^+ < N$. If $r \in \mathcal{P}(\partial\Omega)$ is uniformly subcritical then the boundary trace embedding $W^{1,p(x)}(\Omega) \to L^{r(x)}(\partial\Omega)$ is compact.*

Corollary 2.6 ([16, Corollary 2.2])**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Suppose that $p \in C^0(\bar{\Omega})$ and $1 < p^- \leqslant p^+ < N$. If $r \in C^0(\partial\Omega)$ satisfies the condition*

$$1 \leqslant r(x) < p_*(x) \quad \textit{for every } x \in \partial\Omega,$$

then there is a compact boundary trace embedding $W^{1,p(x)}(\Omega) \to L^{r(x)}(\partial\Omega)$.

For much more on these spaces, we refer to [8].

3. Existence criteria for solutions

We consider the equation

$$\begin{cases} -\Delta_{p(x)} u + h(x)|u|^{p(x)-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\,\partial_\nu u = |u|^{r(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p \in \mathcal{P}(\Omega)$, $1 < p^- \leqslant p^+ < N$, and $r \in \mathcal{P}(\partial\Omega)$ is critical in the sense that $\mathcal{A}_T \neq \emptyset$ where $\mathcal{A}_T$ is defined in (1.3). In order to study (3.1) by means of variational methods, we need to consider the functional $\mathcal{F} : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) := \int_\Omega \frac{1}{p(x)}\bigl(|\nabla u|^{p(x)} + h(x)|u|^{p(x)}\bigr)\,\mathrm{d}x - \int_{\partial\Omega} \frac{1}{r(x)}|u|^{r(x)}\,\mathrm{d}S. \tag{3.2}$$

Then $u \in W^{1,p(x)}(\Omega)$ is a weak solution of (3.1) if and only if u is a critical point of $\mathcal{F}$. We need to assume that the smooth function h is such that the functional

$$\mathcal{J}(u) := \int_\Omega |\nabla u|^{p(x)} + h(x)|u|^{p(x)}\,\mathrm{d}x \tag{3.3}$$

is coercive in the sense that the norm

$$\|u\| := \inf\biggl\{\lambda > 0 \int_\Omega \frac{|\nabla u|^{p(x)} + h(x)|u(x)|^{p(x)}}{\lambda^{p(x)}}\,\mathrm{d}x \leqslant 1\biggr\}, \tag{3.4}$$

is equivalent to the usual norm $\|\cdot\|_{1,p(x)}$ of $W^{1,p(x)}(\Omega)$ defined in (2.7).

It is not difficult to prove that $\mathcal{F}$ verifies the geometrical assumptions of the Mountain Pass Theorem (cf. the proof of Theorem 3.2). The first non-trivial result needed to apply the Mountain Pass Theorem is to check that the Palais–Smale condition holds below some critical energy level c^* that can be computed explicitly in terms of the Sobolev trace constant $T(p(\cdot), r(\cdot), \Omega)$. Once this fact is proved, the main difficulty is to exhibit some Palais–Smale sequence with energy below the critical level c^*.

This approach has been used with success by several authors for treating critical elliptic problems, starting with the seminal papers of [2,3,6]. See, for instance [4,9–13,15,18,26,31–33] and references therein.

Our first result gives an explicit value of the energy below which the functional $\mathcal{F}$ satisfy the Palais–Smale condition.

Theorem 3.1. *Assume that h is such that $\mathcal{J}$ is coercive (see (3.4) above) and $p^+ < r^-$. The functional $\mathcal{F}$ satisfies the Palais–Smale condition at level*

$$0 < c < \inf_{x \in \mathcal{A}_T} \left(\frac{1}{p(x)} - \frac{1}{p_*(x)}\right) \bar{T}_x^{\frac{p(x)p_*(x)}{p_*(x)-p(x)}}.$$

Proof. Let $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p(x)}(\Omega)$ be a Palais–Smale sequence for $\mathcal{F}$. Recall that this means that the sequence $\{\mathcal{F}(u_k)\}_{k\in\mathbb{N}}$ is bounded, and that $\mathcal{F}'(u_k) \to 0$ strongly in the dual space $(W^{1,p(x)}(\Omega))'$. Recalling that the functional $\mathcal{J}$ defined by (3.3) is assumed to be coercive (see the norm (3.4) above), it then follows that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p(x)}(\Omega)$. In fact, for k large, we have that

$$\begin{aligned}
c + 1 + \mathrm{o}(1)\|u_k\| &\geqslant \mathcal{F}(u_k) - \frac{1}{r^-}\bigl\langle \mathcal{F}'(u_k), u_k\bigr\rangle \\
&\geqslant \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\int_\Omega |\nabla u_k|^{p(x)} + h(x)|u_k|^{p(x)}\,\mathrm{d}x + \int_{\partial\Omega}\left(\frac{1}{r^-} - \frac{1}{r(x)}\right)|u_k|^{r(x)}\,\mathrm{d}S \\
&\geqslant \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\int_\Omega |\nabla u_k|^{p(x)} + h(x)|u_k|^{p(x)}\,\mathrm{d}x = \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\mathcal{J}(u_k).
\end{aligned}$$

We may thus assume that $u_k \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$. We claim that u turns out to be a weak solution to (3.1). The proof of this fact follows closely the one in [30] and this argument is taken from [7,14], where the constant exponent case is treated.

In fact, since $\{u_k\}_{k\in\mathbb{N}}$ is a Palais–Smale sequence, we have that

$$\langle \mathcal{F}'(u_k), v\rangle = \int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k \nabla v\,\mathrm{d}x + \int_\Omega h|u_k|^{p(x)-2}u_k v\,\mathrm{d}x - \int_{\partial\Omega} |u_k|^{r(x)-2}u_k v\,\mathrm{d}S = \mathrm{o}(1)$$

for any $v \in C^1(\bar{\Omega})$. Without loss of generality, we can assume that $u_k \to u$ a.e. in Ω, $\mathcal{H}^{N-1}$-a.e. in $\partial\Omega$, and in $L^{p(x)}(\Omega)$. It is easy to see, from standard integration theory, that

$$\int_\Omega h|u_k|^{p(x)-2}u_k v\,\mathrm{d}x \to \int_\Omega h|u|^{p(x)-2}uv\,\mathrm{d}x$$

and

$$\int_{\partial\Omega} |u_k|^{r(x)-2}u_k v\,\mathrm{d}S \to \int_{\partial\Omega} |u|^{r(x)-2}uv\,\mathrm{d}S,$$

so the claim will follow if we show that

$$\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k\nabla v\,\mathrm{d}x \to \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,\mathrm{d}x.$$

This is a consequence of the monotonicity of the $p(x)$-Laplacian. We can assume that there exist $\xi \in (L^{p'(x)}(\Omega))^N$ such that

$$|\nabla u_k|^{p(x)-2}\nabla u_k \rightharpoonup \xi \quad \text{weakly in } \left(L^{p'(x)}(\Omega)\right)^N.$$

The idea is to show that $\nabla u_k \to \nabla u$ a.e. in Ω, then this will imply that $\xi = |\nabla u|^{p(x)-2}\nabla u$ and thus, the claim.

Let $\delta > 0$ then, by Egoroff's theorem, there exists $E_\delta \subset \Omega$ such that $|\Omega \setminus E_\delta| < \delta$ and $u_k \to u$ uniformly in E_δ. As a consequence, given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $|u_k(x) - u(x)| < \varepsilon/2$ for $x \in E_\delta$ and for any $k \geqslant k_0$.

Define the truncation β_ε as

$$\beta_\varepsilon(t) = \begin{cases} -\varepsilon & \text{if } t \leqslant -\varepsilon,\\ t & \text{if } -\varepsilon < t < \varepsilon,\\ \varepsilon & \text{if } t \geqslant \varepsilon.\end{cases}$$

Now we make use of the following well known monotonicity inequality

$$\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y) \geqslant 0 \tag{3.5}$$

which is valid for any $x, y \in \mathbb{R}^N$ and $p \geqslant 1$ and we obtain

$$\left(|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u\right)\nabla\left(\beta_\varepsilon(u_k - u)\right) \geqslant 0,$$

since $\nabla\beta_\varepsilon(u_k-u)=\nabla u_k-\nabla u$ in E_δ and $\nabla\beta_\varepsilon(u_k-u)=0$ in $\Omega\setminus E_\delta$. Therefore, we obtain

$$\int_{E_\delta}\big(|\nabla u_k|^{p(x)-2}\nabla u_k-|\nabla u|^{p(x)-2}\nabla u\big)(\nabla u_k-\nabla u)\,\mathrm{d}x$$
$$\leqslant\int_\Omega\big(|\nabla u_k|^{p(x)-2}\nabla u_k-|\nabla u|^{p(x)-2}\nabla u\big)\nabla\beta_\varepsilon(u_k-u)\,\mathrm{d}x.$$

Now, observe that $\beta_\varepsilon(u_k-u)\rightharpoonup 0$ weakly in $W_0^{1,p(x)}(\Omega)$ and so

$$\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla\beta_\varepsilon(u_k-u)\,\mathrm{d}x\to 0.$$

Now, for k sufficiently large, we obtain that

$$\int_\Omega|\nabla u_k|^{p(x)-2}\nabla u_k\nabla\beta_\varepsilon(u_k-u)\,\mathrm{d}x\leqslant C\varepsilon$$

for some constant $C>0$. In fact, since $\beta_\varepsilon(u_k-u)$ is bounded in $W^{1,p(x)}(\Omega)$,

$$\big\langle\mathcal{F}'(u_k),\beta_\varepsilon(u_k-u)\big\rangle=\mathrm{o}(1),$$

so that

$$\int_\Omega|\nabla u_k|^{p(x)-2}\nabla u_k\nabla\beta_\varepsilon(u_k-u)\,\mathrm{d}x=\mathrm{o}(1)+I_1+I_2,$$

where

$$|I_1|=\left|\int_{\partial\Omega}|u_k|^{r(x)-2}u_k\beta_\varepsilon(u_k-u)\,\mathrm{d}S\right|\leqslant\varepsilon\int_{\partial\Omega}|u_k|^{r(x)-1}\,\mathrm{d}S\leqslant C\varepsilon$$

and

$$|I_2|=\left|\int_\Omega h|u_k|^{p(x)-2}u_k\beta_\varepsilon(u_k-u)\,\mathrm{d}x\right|\leqslant\varepsilon\|h\|_\infty\int_\Omega|u_k|^{p(x)-1}\,\mathrm{d}x\leqslant C\varepsilon.$$

As a consequence, we get that

$$0\leqslant\limsup_{k\to\infty}\int_{E_\delta}\big(|\nabla u_k|^{p(x)-2}\nabla u_k-|\nabla u|^{p(x)-2}\nabla u\big)(\nabla u_k-\nabla u)\,\mathrm{d}x\leqslant C\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, it follows that $(|\nabla u_k|^{p(x)-2}\nabla u_k-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_k-\nabla u)\to 0$ strongly in $L^1(E_\delta)$ and thus, up to a subsequence, also a.e. in E_δ. By a standard diagonal argument, we can assume that $(|\nabla u_k|^{p(x)-2}\nabla u_k-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_k-\nabla u)\to 0$ a.e. in E_δ for every $\delta>0$ and so the convergence holds a.e. in Ω.

Finally, it is easy to see that $(|x_k|^{p-2}x_k - |x|^{p-2}x)(x_k - x) \to 0$ for $x_k, x \in \mathbb{R}^N$ and $p \geqslant 1$ imply that $x_k \to x$, so we get that $\nabla u_k \to \nabla u$ a.e. in Ω. This concludes the proof of the claim.

By the Concentration–Compactness Principle for variable exponents in the trace case, see [21], it holds that

$$|u_k|^{r(x)}\,\mathrm{d}S \rightharpoonup \nu = |u|^{r(x)}\,\mathrm{d}S + \sum_{i\in I}\nu_i\delta_{x_i} \quad \text{weakly}^* \text{ in the sense of measures,} \tag{3.6}$$

$$|\nabla u_k|^{p(x)}\,\mathrm{d}x \rightharpoonup \mu \geqslant |\nabla u|^{p(x)}\,\mathrm{d}x + \sum_{i\in I}\mu_i\delta_{x_i} \quad \text{weakly}^* \text{ in the sense of measures,} \tag{3.7}$$

$$\bar{T}_{x_i}\nu_i^{1/p_*(x_i)} \leqslant \mu_i^{1/p(x_i)}, \tag{3.8}$$

where I is a countable set, $\{\nu_i\}_{i\in I}$ and $\{\mu_i\}_{i\in I}$ are positive numbers, the points $\{x_i\}_{i\in I}$ belong to the critical set $\mathcal{A}_T \subset \partial\Omega$, and $\bar{T}_{x_i}$ is the localized best Sobolev constant around x_i defined by (1.7).

It is not difficult to check that $v_k := u_k - u$ is a PS-sequence for the functional $\tilde{\mathcal{F}}$ defined by

$$\tilde{\mathcal{F}}(v) := \mathcal{F}(v) - \int_\Omega \frac{1}{p(x)} h|v|^{p(x)}\,\mathrm{d}x.$$

Now, by the Brezis–Lieb Lemma 2.3 we get

$$\begin{aligned}
\mathcal{F}(u_k) - \mathcal{F}(u) &= \int_\Omega \frac{1}{p(x)}\big[|\nabla v_k|^{p(x)} + h|v_k|^{p(x)}\big]\,\mathrm{d}x - \int_{\partial\Omega}\frac{1}{r(x)}|v_k|^{r(x)}\,\mathrm{d}S + \mathrm{o}(1)\\
&= \tilde{\mathcal{F}}(v_k) + \int_\Omega \frac{1}{p(x)} h|v_k|^{p(x)}\,\mathrm{d}x + \mathrm{o}(1)\\
&= \tilde{\mathcal{F}}(v_k) + \mathrm{o}(1).
\end{aligned}$$

Independently since u is a weak solution of (3.1), and recalling that $p^+ < r^-$, we have

$$\begin{aligned}
\mathcal{F}(u) &\geqslant \frac{1}{p^+}\int_\Omega \big(|\nabla u|^{p(x)} + h(x)|u|^{p(x)}\big)\,\mathrm{d}x - \frac{1}{r^-}\int_{\partial\Omega}|u|^{r(x)}\,\mathrm{d}S\\
&= \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\int_{\partial\Omega}|u|^{r(x)}\,\mathrm{d}S\\
&\geqslant 0.
\end{aligned}$$

Therefore, $\mathcal{F}(u_k) \geqslant \tilde{\mathcal{F}}(v_k) + \mathrm{o}(1)$. Let $\phi \in C^1(\bar{\Omega})$. As $\tilde{\mathcal{F}}'(v_k) \to 0$, we have

$$\begin{aligned}
\mathrm{o}(1) &= \langle \tilde{\mathcal{F}}'(v_k), v_k\phi\rangle\\
&= \int_\Omega |\nabla v_k|^{p(x)}\phi\,\mathrm{d}x - \int_{\partial\Omega}|v_k|^{r(x)}\phi\,\mathrm{d}S + \int_\Omega |\nabla v_k|^{p(x)-2}\nabla v_k\nabla\phi v_k\,\mathrm{d}x\\
&= A - B + C.
\end{aligned}$$

Since (v_k) is bounded in $W^{1,p(x)}(\Omega)$ and converges to 0 in $L^{p(x)}(\Omega)$, it is easy to see, using Hölder inequality as stated in Proposition 2.1, that $C \to 0$ as $k \to \infty$. Moreover, by means of Lemma 2.3, (3.6) and (3.7), there holds

$$A \to \int_{\Omega} \phi \, \mathrm{d}\tilde{\mu} \quad \text{and} \quad B \to \int_{\partial\Omega} \phi \, \mathrm{d}\tilde{\nu},$$

where $\tilde{\mu} = \mu - |\nabla u|^{p(x)} \, \mathrm{d}x$ and $\tilde{\nu} = \nu - |u|^{r(x)} \, \mathrm{d}S$. So we conclude that $\tilde{\mu} = \tilde{\nu}$. In particular $\nu_i \geqslant \mu_i$ ($i \in I$) from where we obtain with (3.8) that $\nu_i \geqslant \bar{T}_{x_i}^{\frac{(N-1)p(x_i)}{p(x_i)-1}}$. Hence

$$\begin{aligned}
c &= \lim_{k\to\infty} \mathcal{F}(u_k) \geqslant \lim_{k\to\infty} \tilde{\mathcal{F}}(v_k) = \int \frac{1}{p(x)} \, \mathrm{d}\tilde{\mu} - \int \frac{1}{r(x)} \, \mathrm{d}\tilde{\nu} \\
&= \int \left(\frac{1}{p(x)} - \frac{1}{r(x)} \right) \mathrm{d}\tilde{\nu} = \sum_{i\in I} \left(\frac{1}{p(x_i)} - \frac{1}{p_*(x_i)} \right) \nu_i \\
&\geqslant \#(I) \inf_{i\in I} \frac{p(x_i)-1}{(N-1)p(x_i)} \bar{T}_{x_i}^{\frac{(N-1)p(x_i)}{p(x_i)-1}}.
\end{aligned}$$

We deduce that if $c < \inf_{i\in I} \frac{p(x_i)-1}{(N-1)p(x_i)} \bar{T}_{x_i}^{\frac{(N-1)p(x_i)}{p(x_i)-1}}$ then I must be empty implying that $u_k \to u$ strongly in $W^{1,p(x)}(\Omega)$. $\square$

As a corollary, we can apply the Mountain Pass Theorem to obtain the following necessary existence condition.

Theorem 3.2. *Assume that $r^- > p^+$ and that h is such that $\mathcal{J}$ is coercive (see (3.4)). If there exists $v \in W^{1,p(x)}(\Omega)$ such that*

$$\sup_{s>0} \mathcal{F}(sv) < \inf_{x\in\mathcal{A}_T} \left(\frac{1}{p(x)} - \frac{1}{p_*(x)} \right) \bar{T}_x^{\frac{p(x)p_*(x)}{p_*(x)-p(x)}} \tag{3.9}$$

then (3.1) has a non-trivial nonnegative solution.

Proof. The proof is an immediate consequence of the Mountain Pass Theorem, Theorem 3.1 and assumption (3.9). In fact, it suffices to verify that $\mathcal{F}$ has the Mountain Pass geometry and that $\mathcal{F}(su) < 0$ for some $s > 0$. Concerning the latter condition notice that for $s > 1$,

$$\begin{aligned}
\mathcal{F}(su) &= \int_{\Omega} \frac{s^{p(x)}}{p(x)} \big(|\nabla u|^{p(x)} + h(x)|u|^{p(x)} \big) \, \mathrm{d}x - \int_{\partial\Omega} \frac{s^{r(x)}}{r(x)} |u|^{r(x)} \, \mathrm{d}S \\
&\leqslant s^{p^+} \int_{\Omega} \frac{1}{p(x)} \big(|\nabla u|^{p(x)} + h(x)|u|^{p(x)} \big) \, \mathrm{d}x - s^{r^-} \int_{\partial\Omega} \frac{1}{r(x)} |u|^{r(x)} \, \mathrm{d}S,
\end{aligned}$$

which tends to $-\infty$ as $s \to +\infty$ since $r^- > p^+$.

It remains to see that $\mathcal{F}$ has the Mountain Pass geometry. Clearly $\mathcal{F}(0) = 0$ and, if $\|v\|_{1,p(x)} = s$ is small enough, then

$$\int_\Omega |\nabla v|^{p(x)} + h|v|^{p(x)}\,\mathrm{d}x \geqslant c_1\|v\|_{1,p(x)}^{p^+} = c_1 s^{p^+}$$

since $\mathcal{J}$ is coercive, and on the other hand

$$\|v\|_{r(x),\partial\Omega} \leqslant C\|v\|_{1,p(x)} = Cs < 1$$

for s small, so that

$$\int_{\partial\Omega} |v|^{r(x)}\,\mathrm{d}S \leqslant c_2\|v\|_{1,p(x)}^{r^-} = c_2 s^{r^-}.$$

Therefore

$$\mathcal{F}(v) \geqslant \frac{c_1}{p^+}s^{p^+} - \frac{c_2}{r^-}s^{r^-} > 0$$

since $p^+ < r^-$. This completes the proof. $\square$

4. Local conditions for (3.9)

In this section we provide local conditions for (3.9) to hold. These conditions are analogous to the ones found in [19] where the critical problem for the $p(x)$-Laplacian with Dirichlet boundary condition was studied.

The idea is to evaluate $\mathcal{F}(sz_\varepsilon)$ for a suitable test function z_ε constructed by a scaled and truncated version of the extremal for $\bar{K}(N, p(x))^{-1}$ for a critical point $x \in \mathcal{A}_T$. Then, a refined asymptotic analysis will yield the desired result.

In order to construct the test function we need to recall the *Fermi coordinates* from differential geometry. Briefly speaking, the Fermi coordinates describe a neighborhood of a point $x_0 \in \partial\Omega$ with variables (y, t) where $y \in \mathbb{R}^{N-1}$ are the coordinates in a local chart of $\partial\Omega$ such that $y = 0$ corresponds to x_0, and $t > 0$ is the distance to $\partial\Omega$ along the unit inward normal vector.

Definition 4.1 (Fermi coordinates)**.** We consider the following change of variables around a point $x_0 \in \partial\Omega$.

We assume that $x_0 = 0$ and that $\partial\Omega$ has the following representation in a neighborhood V of 0:

$$\partial\Omega \cap V = \{x \in V : x_n = \psi(x'),\ x' \in U \subset \mathbb{R}^{N-1}\},$$
$$\Omega \cap V = \{x \in V : x_n > \psi(x'),\ x' \in U \subset \mathbb{R}^{N-1}\}.$$

The function $\psi : U \subset \mathbb{R}^{N-1} \to \mathbb{R}$ is assumed to be at least of class C^2 and that $\psi(0) = 0$, $\nabla\psi(0) = 0$.

The change of variables is then defined as $\Phi : U \times (0, \delta) \to \Omega \cap V$

$$\Phi(y, t) = (y, \psi(y)) + t\nu(y),$$

where $\nu(y)$ is the unit inward normal vector, i.e.

$$\nu(y) = \frac{(-\nabla\psi(y), 1)}{\sqrt{1 + |\nabla\psi(y)|^2}}.$$

It is well known that for $\delta > 0$ small Φ defines a smooth diffeomorphism (see [12]). For a general construction of the Fermi coordinates in differential manifolds, we refer to the book [24].

Now, we are in position to construct the test functions needed in order to satisfy (3.9). Assume that $0 \in \mathcal{A}_T \subset \partial\Omega$. Then, the test-functions we consider are defined in the Fermi coordinates by

$$v_\varepsilon(x) = \eta(y, t)V_{\varepsilon,0}(y, t), \quad x = \Phi(y, t),$$

where $V_{\varepsilon,0}$ is defined in (1.9) by rescaling an extremal V of $\bar{K}(N, p(0))^{-1}$, and $\eta \in C_c^\infty(B_{2\delta} \times [0, 2\delta), [0, 1])$ is a smooth cut-off function. We normalize v_ε by considering the function z_ε defined by

$$z_\varepsilon = Cv_\varepsilon, \quad C = \bar{K}\big(N, p(0)\big)^{-\frac{p(0)}{p(0)_* - p(0)}} \|V\|^{-1}_{p(0)_*, \partial\mathbb{R}^N_+}.$$

With this choice of C, the function $Z(y, t) := CV(y, t)$ satisfies

$$\int_{\mathbb{R}^N_+} |\nabla Z|^{p(0)}\,\mathrm{d}y\,\mathrm{d}t = \int_{\partial\mathbb{R}^N_+} |Z|^{p(0)_*}\,\mathrm{d}y = \bar{K}\big(N, p(0)\big)^{-\frac{p(0)p(0)_*}{p(0)_* - p(0)}}.$$

From now on, we assume that $p \in \mathcal{P}(\Omega)$ and $r \in \mathcal{P}(\partial\Omega)$ are of class C^2, $0 \in \partial\Omega$ and we let $p := p(0)$ and $r := r(0)$.

In Propositions A.2–A.4 in the Appendix we compute some asymptotic expansions needed in order to properly evaluate $\mathcal{F}(sz_\varepsilon)$. These propositions are fundamental in the proof of our next result. We choose to postpone their proofs to the appendix because they are technical and long.

Eventually the following result provides a sufficient local condition for (3.9) to hold.

Theorem 4.2. *Assume that $r^- > p^+$, and that h is such that $\mathcal{J}$ is coercive. Assume moreover that there exists a point $x_0 \in \mathcal{A}_T$ such that $\bar{T} = \bar{T}_{x_0}$ and such that x_0 is a local minimum of $p(x)$ and a local maximum of $r(x)$ and $p(x_0) < \min\{\sqrt{N}, \frac{N^2}{3N-2}\}$. Let H the mean curvature of $\partial\Omega$. Assume eventually that one of the following conditions hold*

(1) $\frac{\partial p}{\partial t}(x_0) > 0$,
(2) $\frac{\partial p}{\partial t}(x_0) = 0$ *and* $H(x_0) > 0$ *or*
(3) $\frac{\partial p}{\partial t}(x_0) = 0$, $H(x_0) = 0$, $1 < p(x_0) < 2$ *and* $h(x_0) < 0$ *or*
(4) $\frac{\partial p}{\partial t}(x_0) = 0$, $H(x_0) = 0$, $p(x_0) \geqslant 2$ *and* $\Delta p(x_0) > 0$ *or* $\Delta_y r(x_0) < 0$.

Then there exists a nontrivial solution to (3.1). Here $\frac{\partial p}{\partial t}(x_0) = -\partial_\nu p(x_0)$ (with ν the unit exterior normal vector), $\Delta_y r(x_0) := \Delta(r \circ \Phi(\cdot, 0))(0)$, and $\Delta p(x_0) := \Delta(p \circ \Phi)(0)$.

Notice that, as a consequence of the definition of the Fermi coordinates, we have that $\Delta_y r(x_0)$ coincides with the Laplacian of r at x_0 for the natural metric of $\partial\Omega$.

Proof. We assume, without loss of generality that $x_0 = 0$ and denote $p = p(0)$. Observe that $r(0) = p_*$.

We first consider the case where $\partial_t p(0) > 0$. In fact, from Propositions A.2–A.4, we have

$$\begin{aligned} f_\varepsilon(s) = \mathcal{F}(sz_\varepsilon) &= \bar{D}_0 + \bar{D}_1 \varepsilon \ln \varepsilon - \bar{A}_0 + \mathrm{o}(\varepsilon \ln \varepsilon) \\ &= f_0(s) + \varepsilon \ln \varepsilon f_1(s) + \mathrm{O}(\varepsilon). \end{aligned}$$

C^1-uniformly in $s \in [0, s_0]$, with

$$f_0(s) = \bar{K}(N, p)^{-\frac{p_* p}{p_* - p}} \left(\frac{s^p}{p} - \frac{s^{p_*}}{p_*} \right)$$

and

$$f_1(s) = -\frac{N}{p} \frac{s^p}{p} \partial_t p(0) \int_{\mathbb{R}^n_+} t |\nabla Z|^p \, \mathrm{d}y \, \mathrm{d}t.$$

Notice that f_0 reaches its maximum in $[0, s_0]$ at $s = 1$. Moreover, it is a nodegenerate maximum since $f_0''(1) = (p - p_*) \bar{K}(N, p)^{-\frac{p_* p}{p_* - p}} \neq 0$. It follows that f_ε reaches a maximum at $s_\varepsilon = 1 + a\varepsilon \ln \varepsilon + \mathrm{O}(\varepsilon)$ for $a = -\frac{f_1'(1)}{f_0''(1)}$. Hence

$$\sup_{s>0} \mathcal{F}(sz_\varepsilon) = \mathcal{F}(s_\varepsilon z_\varepsilon) = \left(\frac{1}{p} - \frac{1}{p_*} \right) \bar{K}(N, p)^{-\frac{p_* p}{p_* - p}} + f_1(1) \varepsilon \ln \varepsilon + \mathrm{O}(\varepsilon).$$

If $\partial_t p(0) > 0$ then $f_1(1) < 0$ and the result follows.

Assume now that $\partial_t p(0) = 0$ and $H(0) > 0$. Then we have

$$\begin{aligned} \mathcal{F}(sz_\varepsilon) &= \bar{D}_0 + \bar{D}_2 \varepsilon + \mathrm{o}(\varepsilon) - \bar{A}_0 \\ &= f_0(s) + f_2(s)\varepsilon + \mathrm{o}(\varepsilon). \end{aligned}$$

C^1-uniformly in $[0, s_0]$, with

$$f_2(s) = -H(0) \frac{s^p}{p} \int_{\mathbb{R}^N_+} t |\nabla Z|^p \, \mathrm{d}y \, \mathrm{d}t + \frac{H(0)}{N-1} s^p \int_{\mathbb{R}^N_+} \frac{t|y|^2}{r^2} |\nabla Z|^p \, \mathrm{d}y \, \mathrm{d}t.$$

As before f_ε reaches its maximum at $s_\varepsilon = 1 + a\varepsilon + \mathrm{o}(\varepsilon)$ with $a = \frac{f_2'(1)}{f_0''(1)}$. So,

$$\sup_{s>0} \mathcal{F}(sz_\varepsilon) = \mathcal{F}(s_\varepsilon z_\varepsilon) = \left(\frac{1}{p} - \frac{1}{p_*} \right) \bar{K}(N, p)^{-\frac{p_* p}{p_* - p}} + f_2(1)\varepsilon + \mathrm{o}(\varepsilon).$$

So, we need that $f_2(1) < 0$, i.e.

$$-H(0) \frac{1}{p} \int_{\mathbb{R}^N_+} t |\nabla Z|^p \, \mathrm{d}y \, \mathrm{d}t + \frac{H(0)}{N-1} \int_{\mathbb{R}^N_+} \frac{t|y|^2}{r^2} |\nabla Z|^p \, \mathrm{d}y \, \mathrm{d}t < 0.$$

But,

$$
\begin{aligned}
&-\frac{1}{p}\int_{\mathbb{R}^N_+} t|\nabla Z|^p\,\mathrm{d}y\,\mathrm{d}t+\frac{1}{N-1}\int_{\mathbb{R}^N_+}\frac{t|y|^2}{r^2}|\nabla Z|^p\,\mathrm{d}y\,\mathrm{d}t\\
&\leqslant\left(-\frac{1}{p}+\frac{1}{N-1}\right)\int_{\mathbb{R}^N_+} t|\nabla Z|^p\,\mathrm{d}y\,\mathrm{d}t\\
&<0
\end{aligned}
$$

if $p < N-1$. So, since $H(0) > 0$, the result follows.

Now suppose that $\partial_t p(0) = 0$ and $H(0) = 0$. Then

$$
\mathcal{F}(sz_\varepsilon)=\bar{D}_0+\bar{D}_4\varepsilon^2\ln\varepsilon+\mathrm{o}\big(\varepsilon^2\ln\varepsilon\big)+\bar{C}_0\varepsilon^p+\mathrm{o}\big(\varepsilon^p\big)-\bar{A}_0-\bar{A}_1\varepsilon^2\ln\varepsilon.
$$

If $1 < p < 2$

$$
\mathcal{F}(sz_\varepsilon)=(\bar{D}_0-\bar{A}_0)+\bar{C}_0\varepsilon^p+\mathrm{o}\big(\varepsilon^p\big)=f_0(s)+f_3(s)\varepsilon^p+\mathrm{o}\big(\varepsilon^p\big)
$$

with

$$
f_3(s)=h(0)\frac{s^p}{p}\int_{\mathbb{R}^N_+}|\nabla Z|^p\,\mathrm{d}y\,\mathrm{d}t.
$$

As before f_ε reaches its maximum at $s_\varepsilon = 1 + a\varepsilon^p + \mathrm{o}(\varepsilon^p)$ with $a = \frac{f_3'(1)}{f_0''(1)}$. Then,

$$
\sup_{s>0}\mathcal{F}(sz_\varepsilon)=\mathcal{F}(s_\varepsilon z_\varepsilon)=\left(\frac{1}{p}-\frac{1}{p_*}\right)\bar{K}(N,p)^{-\frac{p_*p}{p_*-p}}+f_3(1)\varepsilon^p+\mathrm{o}\big(\varepsilon^p\big).
$$

So, we need that $f_3(1) < 0$. But, this is equivalent to $h(0) < 0$.

If $p \geqslant 2$, we have

$$
\mathcal{F}(sz_\varepsilon)=(\bar{D}_0-\bar{A}_0)+(\bar{D}_4-\bar{A}_1)\varepsilon^2\ln\varepsilon+\mathrm{o}\big(\varepsilon^2\ln\varepsilon\big)=f_0(s)+f_4(s)\varepsilon^2\ln\varepsilon+\mathrm{o}\big(\varepsilon^2\ln\varepsilon\big),
$$

with

$$
\begin{aligned}
f_4(s)=&-\frac{s^p}{p}\frac{N}{2p}\left(\partial_{tt}p(0)\int_{\mathbb{R}^N_+}t^2|\nabla Z|^p\,\mathrm{d}y\,\mathrm{d}t+\Delta_y p(0)\int_{\mathbb{R}^N_+}|y|^2|\nabla Z|^p\,\mathrm{d}y\,\mathrm{d}z\right)\\
&+\frac{s^{p_*}}{p_*}\frac{1}{2p_*}\Delta_y r(0)\int_{\partial\mathbb{R}^N_+}|y|^2Z^{p_*}\,\mathrm{d}y.
\end{aligned}
$$

As before, we need that $f_4(1) < 0$. Since 0 is a local minimum of $p(x)$ and a local maximum of $r(x)$ and $\partial_t p(0) = 0$ it easily follows that $f_4(1) \leqslant 0$. Moreover if one of the following inequalities

$$
\Delta_y r(0)\leqslant 0\leqslant \Delta p(0)
$$

is strict, then $f_4(1) < 0$ and the result follows. $\square$

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Appendix. Asymptotic expansions

In this section we provide the asymptotic expansions needed in the proof of Theorem 4.2.

First we need the following asymptotic expansions for the Jacobian of the Fermi coordinates that are proved in [12].

Lemma A.1. *With the notation introduced in Definition 4.1, the following asymptotic expansions hold*

$$J\Phi(y,t) = 1 - Ht + \mathrm{O}\big(t^2 + |y|^2\big),$$

where H is the mean curvature of $\partial\Omega$.

Also, if we denote $v(y,t) = u(\Phi(y,t))$,

$$\big|\nabla u(x)\big|^2 = (\partial_t v)^2 + \sum_{i,j=1}^{N} \big(\delta^{ij} + 2h^{ij}t + \mathrm{O}\big(t^2 + |y|^2\big)\big)\, \partial_{y_i} v\, \partial_{y_j} v,$$

where h^{ij} is the second fundamental form of $\partial\Omega$.

The goal of this section is to prove the following propositions.

Proposition A.2. *There holds*

$$\int_\Omega f(x)|v_\varepsilon|^{p(x)}\,\mathrm{d}x = \bar{C}_0\varepsilon^p + \mathrm{o}\big(\varepsilon^p\big) \quad \textit{with } \bar{C}_0 = f(0)\int_{\mathbb{R}^N_+} V^p\,\mathrm{d}x. \tag{A.1}$$

Proposition A.3. *If $p < \frac{N-1}{2}$,*

$$\int_{\partial\Omega} f(x)|v_\varepsilon|^{r(x)}\,\mathrm{d}S_x = \bar{A}_0 + \bar{A}_1\varepsilon^2\ln\varepsilon + \mathrm{o}\big(\varepsilon^2\ln\varepsilon\big) \tag{A.2}$$

with

$$\bar{A}_0 = f(0)\int_{\mathbb{R}^{N-1}} V(y,0)^{p_*}\,\mathrm{d}y,$$

and

$$\begin{aligned}\bar{A}_1 &= -\frac{N-p}{2p} f(0)\int_{\mathbb{R}^{N-1}} \big(D^2 r(0)y, y\big)V(y,0)^{p_*}\,\mathrm{d}y \\ &= -\frac{1}{2p_*} f(0)\Delta r(0)\int_{\mathbb{R}^{N-1}} |y|^2 V(y,0)^{p_*}\,\mathrm{d}y.\end{aligned}$$

Proposition A.4. *Assume that $p < N^2/(3N-2)$. Then*

$$\int_\Omega f(x)\big|\nabla v_\varepsilon(x)\big|^{p(x)}\,\mathrm{d}x = \bar{D}_0 + \bar{D}_1\varepsilon\ln\varepsilon + \bar{D}_2\varepsilon + \bar{D}_3(\varepsilon\ln\varepsilon)^2 + \bar{D}_4\varepsilon^2\ln\varepsilon + \mathrm{O}\big(\varepsilon^2\big),$$

with

$$\bar{D}_0 = f(0)\int_{\mathbb{R}^N_+}|\nabla V|^p\,\mathrm{d}y\,\mathrm{d}t, \qquad \bar{D}_1 = -\frac{N}{p}f(0)\,\partial_t p(0)\int_{\mathbb{R}^N_+} t|\nabla V|^p\,\mathrm{d}y\,\mathrm{d}t,$$

and, assuming that $\partial_t p(0) = 0$,

$$\bar{D}_2 = \big(\partial_t f(0) - Hf(0)\big)\int_{\mathbb{R}^N_+} t|\nabla V|^p\,\mathrm{d}y\,\mathrm{d}t + p\bar{h}f(0)\int_{\mathbb{R}^N_+}\frac{t|y|^2}{r^2}|\nabla V|^p\,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{D}_3 = 0,$$

$$\bar{D}_4 = -\frac{N}{2p}f(0)\,\partial_{tt}p(0)\int_{\mathbb{R}^N_+} t^2|\nabla V|^p\,\mathrm{d}y\,\mathrm{d}t - \frac{N}{2(N-1)p}f(0)\Delta_y p(0)\int_{\mathbb{R}^N_+}|y|^2|\nabla V|^p\,\mathrm{d}y\,\mathrm{d}t.$$

Proof of Proposition A.2. We write

$$\int_\Omega f(x)|v_\varepsilon|^{p(x)}\,\mathrm{d}x = \int_{\mathbb{R}^N_+} f(y,t)\big|v_\varepsilon(y,t)\big|^{p(y,t)}\big(1+\mathrm{O}\big(|y|^2+|t|\big)\big)\,\mathrm{d}y\,\mathrm{d}t.$$

Now the result follows as in [19], Proposition 5.1. □

Proof of Proposition A.3. We have

$$\int_{\partial\Omega} f v_\varepsilon^{r(x)}\,\mathrm{d}S = \int_{\mathbb{R}^{N-1}} f\big(y,\psi(y)\big)v_\varepsilon\big(y,\psi(y)\big)^{r(y,\psi(y))}\big(1+\mathrm{O}\big(|y|^2\big)\big)\,\mathrm{d}y.$$

Now the proof follows as in [19], Proposition 5.1. □

To treat the gradient term, we need the following result.

Lemma A.5. *Assume $p < N^2/(3N-2)$ and that $p = p(y,t)$ has a local minimum at $(y,t) = (0,0)$. Given a bounded $g \in C^2(\Omega)$ and real numbers a^{ij}, $1 \leqslant i, j \leqslant N-1$, we have*

$$\sum_{i,j=1}^{N-1} a^{ij}\int_{\mathbb{R}^N_+} g(y,t)\eta(y,t)|\nabla V_\varepsilon|^{p(y,t)-2}\,\partial_i V_\varepsilon(y,t)\,\partial_j V_\varepsilon(y,t)\,\mathrm{d}y\,\mathrm{d}t$$
$$= \bar{B}_0 + \bar{B}_1\varepsilon\ln\varepsilon + \bar{B}_2\varepsilon + \bar{B}_3(\varepsilon\ln\varepsilon)^2 + \bar{B}_4\varepsilon^2\ln\varepsilon + \mathrm{O}\big(\varepsilon^2\big),$$

where $\partial_i = \frac{\partial}{\partial y_i}$, *and*

$$\bar{B}_0 = \bar{a} g(0) \int_{\mathbb{R}^N_+} \left|\nabla V(y,t)\right|^p \frac{|y|^2}{r^2}\,\mathrm{d}y\,\mathrm{d}t, \qquad \bar{B}_1 = -\frac{N}{p} g(0)\,\partial_t p(0) \bar{a} \int_{\mathbb{R}^N_+} \left|\nabla V(y,t)\right|^p \frac{|y|^2 t}{r^2}\,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{B}_2 = \bar{a} \int_{\mathbb{R}^N_+} \left|\nabla V(y,t)\right|^p \frac{t|y|^2}{r^2} \left\{ g(0)\,\partial_t p(0) \ln\left|\nabla V(y,t)\right| + \partial_t g(0)\right\}\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{B}_3 = \frac{N^2}{2p^2} g(0)\,\partial_t p(0)^2 \bar{a} \int_{\mathbb{R}^N_+} \left|\nabla V(y,t)\right|^p \frac{|y|^2 t^2}{r^2}\,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{B}_4 = -\frac{N}{p} \bar{a} \int_{\mathbb{R}^N_+} \left|\nabla V(y,t)\right|^p \frac{|y|^2 t^2}{r^2} \biggl(-\frac{g(0)}{2}\,\partial_{tt} p(0) + \partial_t p(0)\,\partial_t g(0) + \partial_t p(0)^2 g(0) \ln\left|\nabla V(y,t)\right| \biggr)\,\mathrm{d}y\,\mathrm{d}t$$
$$+ \sum_{i=1}^{N-1} \frac{N g(0)}{2p} a^{ii}\,\partial_{ii} p(0) \int_{\mathbb{R}^N_+} \left|\nabla V(y,t)\right|^p r^{-2}\left(y_1^4 - 3y_1^2 y_2^2\right)\mathrm{d}y\,\mathrm{d}t$$
$$+ \sum_{i,k=1}^{N-1} \frac{N g(0)}{2p}\left(a^{ii}\,\partial_{kk} p(0) + 2a^{ik}\,\partial_{ik} p(0)\right) \int_{\mathbb{R}^N_+} \left|\nabla V(y,t)\right|^p r^{-2} y_1^2 y_2^2\,\mathrm{d}y\,\mathrm{d}t,$$

where $\bar{a} = \frac{1}{N-1}\sum_{i=1}^{N-1} a^{ii}$ *and* $r = r(y,t) = \sqrt{(1+t)^2 + |y|^2}$.

Proof. Notice that

$$\left|\nabla V_\varepsilon(y,t)\right| = \frac{N-p}{p-1} \varepsilon^{\frac{N-p}{p(p-1)}} \left((\varepsilon+t)^2 + |y|^2\right)^{-\frac{N-1}{2(p-1)}}.$$

So, $|\nabla V_\varepsilon(y,t)| < 1$ if $|(y,t)| > C\varepsilon^{\frac{N-p}{p(N-1)}}$ where $C = (\frac{N-p}{p-1})^{\frac{p-1}{N-1}}$, and $\nabla = (\nabla_y, \partial_t)$. Moreover, since $p^-_{2\delta} = p := p(0,0)$,

$$\int_{B^+_{2\delta} \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(y,t)-2} |\nabla_y V_\varepsilon|^2\,\mathrm{d}y\,\mathrm{d}t$$
$$\leqslant \int_{B^+_{2\delta} \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(y,t)}\,\mathrm{d}y\,\mathrm{d}t$$
$$\leqslant \int_{B^+_{2\delta} \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p}\,\mathrm{d}y\,\mathrm{d}t$$
$$\leqslant C\varepsilon^{\frac{N-p}{p-1}} \int_{\mathbb{R}^N_+ \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} \left\{(\varepsilon+t)^2 + |y|^2\right\}^{-\frac{p(N-1)}{2(p-1)}}\,\mathrm{d}y\,\mathrm{d}t$$

$$\leqslant C\varepsilon^{\frac{N-p}{p-1}} \int_{\mathbb{R}^N \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} \left|(y,t)\right|^{-\frac{p(N-1)}{p-1}} \,\mathrm{d}y\,\mathrm{d}t$$

$$\leqslant C\varepsilon^{\frac{N-p}{p-1}} \int_{C\varepsilon^{\frac{N-p}{p(N-1)}}}^{+\infty} \rho^{N-1-\frac{p(N-1)}{p-1}} \,\mathrm{d}\rho.$$

Then, we obtain

$$\int_{B_{2\delta}^+ \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(y,t)-2} |\nabla_y V_\varepsilon|^2 \,\mathrm{d}y\,\mathrm{d}t \leqslant C\varepsilon^{\frac{N}{p_*}}.$$

Since $p \leqslant \frac{N^2}{3N-2}$, we get that $\frac{N}{p_*} \geqslant 2$, hence

$$\int_{B_{2\delta}^+ \setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(x,t)-2} |\nabla_y V_\varepsilon|^2 \,\mathrm{d}y\,\mathrm{d}t = \mathrm{O}(\varepsilon^2).$$

Hence

$$\begin{aligned}
&a^{ij} \int_{\mathbb{R}_+^N} g(y,t)\eta(y,t)|\nabla V_\varepsilon|^{p(y,t)-2}\, \partial_i V_\varepsilon(y,t)\, \partial_j V_\varepsilon(y,t) \,\mathrm{d}y\,\mathrm{d}t \\
&\quad = a^{ij} \int_{B^+_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} g(y,t)|\nabla V_\varepsilon|^{p(y,t)-2}\, \partial_i V_\varepsilon(y,t)\, \partial_j V_\varepsilon(y,t) \,\mathrm{d}y\,\mathrm{d}t + \mathrm{O}(\varepsilon^2) \\
&\quad = a^{ij} \int_{B^+_{C\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} g(\varepsilon y,\varepsilon t)\varepsilon^{N(1-\frac{p(\varepsilon y,\varepsilon t)}{p})} |\nabla V|^{p(\varepsilon y,\varepsilon t)-2}\, \partial_i V\, \partial_j V \,\mathrm{d}y\,\mathrm{d}t + \mathrm{O}(\varepsilon^2).
\end{aligned}$$

Letting

$$\phi_{ij} = |\nabla V|^{p-2}\, \partial_i V\, \partial_j V = \left|\nabla V(y,t)\right|^p \frac{y_i y_j}{r^2}, \quad \nabla = (\nabla_y, \partial_t),$$

we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{N-1} a^{ij} \int_{\mathbb{R}_+^n} g(y,t)\eta(y,t)|\nabla V_\varepsilon|^{p(y,t)-2}\, \partial_i V_\varepsilon\, \partial_j V_\varepsilon \,\mathrm{d}y\,\mathrm{d}t \\
&\quad = \bar{B}_0(\varepsilon) + \bar{B}_1(\varepsilon)\varepsilon \ln\varepsilon + \bar{B}_2(\varepsilon)\varepsilon + \bar{B}_3(\varepsilon)(\varepsilon\ln\varepsilon)^2 + \bar{B}_4(\varepsilon)\varepsilon^2\ln\varepsilon + \varepsilon^2 R(\varepsilon)
\end{aligned}$$

with coefficients $\bar{B}_i(\varepsilon)$, $i = 0, \ldots, 4$, defined as

$$\bar{B}_0 = \sum_{i,j=1}^{N-1} a^{ij} g(0) \int_{\mathbb{R}_+^N} \phi_{ij}(y,t) \,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{B}_1 = -\frac{N}{p} g(0)\, \partial_t p(0) \sum_{i,j=1}^{N-1} a^{ij} \int_{\mathbb{R}_+^N} t\phi_{ij}(y,t) \,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{B}_2 = \sum_{i,j=1}^{N-2} a^{ij} \int_{\mathbb{R}^N_+} \phi_{ij}(y,t)\big(g(0)t\,\partial_t p(0)\ln|\nabla V| + \nabla g(0)(y,t)\big)\,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{B}_3 = \frac{N^2}{2p^2} g(0)\,\partial_t p(0)^2 \sum_{i,j=1}^{N-1} a^{ij} \int_{\mathbb{R}^N_+} t^2 \phi_{ij}(y,t)\,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{B}_4 = -\frac{N}{p} \sum_{i,j=1}^{N-1} a^{ij} \int_{\mathbb{R}^N_+} \phi_{ij}(x,t)\bigg(\frac{g(0)}{2}\big(D^2 p(0)(y,t),(y,t)\big) + \partial_t p(0)t\big(\nabla g(0),(y,t)\big) + \partial_t p(0)^2 g(0) t^2 \ln|\nabla V|\bigg)\,\mathrm{d}y\,\mathrm{d}t,$$

but with integral over $B^+_{C\varepsilon^{\frac{N-p}{p(N-1)}-1}}$ instead of $\mathbb{R}^N_+$, and the error term $R(\varepsilon)$ satisfies

$$\begin{aligned}\big|R(\varepsilon)\big| &\leqslant C \int_{B^+_{C\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} r^2 |\nabla V|^p \ln|\nabla V|(1 + r\varepsilon\ln\varepsilon)\,\mathrm{d}y\,\mathrm{d}t \\ &\leqslant C \int_{B^+_{C\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} r^2 |\nabla V|^p \ln|\nabla V|\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

Clearly, this last integral is bounded by

$$C \int_1^{+\infty} \rho^{1-\frac{N-p}{p-1}} \ln\rho\,\mathrm{d}\rho,$$

which is finite since $p < \frac{N+2}{3}$. Moreover,

$$\big|\bar{B}_0 - \bar{B}_0(\varepsilon)\big| \leqslant C \int_{\mathbb{R}^N_+ \setminus B^+_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} |\nabla V|^p\,\mathrm{d}y\,\mathrm{d}t \leqslant C \int_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}^{\infty} r^{-1-\frac{N-p}{p-1}}\,\mathrm{d}r \leqslant C\varepsilon^{\frac{N(N-p)}{p(N-1)}} \leqslant C\varepsilon^2$$

since $p \leqslant \frac{N^2}{3N-2}$. Also for $i = 1, 2$,

$$\begin{aligned}\big|\bar{B}_i - \bar{B}_i(\varepsilon)\big| &\leqslant C \int_{\mathbb{R}^N_+ \setminus B^+_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} \big|(y,t)\big|\big(1 + \ln|\nabla V|\big)|\nabla V|^p\,\mathrm{d}y\,\mathrm{d}t \\ &\leqslant C \int_{\varepsilon^{\frac{N(1-p)}{p(N-1)}}}^{\infty} r^{1-\frac{N-p}{p-1}} \ln r\,\mathrm{d}r \\ &\leqslant C \int_{\varepsilon^{\frac{N(1-p)}{p(N-1)}}}^{\infty} r^{1-\frac{N-p}{p-1}+\alpha}\,\mathrm{d}r \quad \text{for any } \alpha > 0 \\ &\leqslant C\varepsilon^{\frac{N(N-2p+1)}{p(N-1)}-\beta} \quad \text{for any } \beta > 0 \text{ and if } p < \frac{N^2+N}{3N-1} \\ &= \mathrm{o}(\varepsilon).\end{aligned}$$

Eventually, for any $i = 3, 4$,

$$
\begin{aligned}
\left|\bar{B}_i - \bar{B}_i(\varepsilon)\right| &\leqslant C \int_{\mathbb{R}^N_+ \setminus B^+_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}} \left|(y,t)\right|^2 \big(1 + \ln|\nabla V|\big)|\nabla V|^p \,\mathrm{d}y\,\mathrm{d}t \\
&\leqslant C \int_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}^{\infty} r^{1-\frac{N-p}{p-1}} \ln r \,\mathrm{d}r \\
&\leqslant C \int_{\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}^{\infty} r^{1-\frac{N-p}{p-1}+\alpha} \,\mathrm{d}r \quad \text{for any } \alpha > 0 \\
&= \mathrm{o}(1),
\end{aligned}
$$

since $p < \frac{n+2}{3}$.

Hence if $p < N^2/(3N-2)$,

$$
\begin{aligned}
&\sum_{i,j=1}^{N-1} a^{ij} \int_{\mathbb{R}^N_+} g(y,t)\eta(y,t)|\nabla V_\varepsilon|^{p(y,t)-2}\, \partial_i V_\varepsilon(y,t)\, \partial_j U_\varepsilon(y,t)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad = \bar{B}_0 + \bar{B}_1 \varepsilon \ln\varepsilon + \bar{B}_2\varepsilon + \bar{B}_3((\varepsilon\ln\varepsilon)^2 + \bar{B}_4\varepsilon^2\ln\varepsilon + \mathrm{O}\big(\varepsilon^2\big).
\end{aligned}
$$

Finally, using the radial symmetry in the y variable, we can simplify the expressions for the $\bar{B}_i$'s. For $\bar{B}_4$, notice that

$$
\begin{aligned}
&\sum_{i,j=1}^{N-1} a^{ij}\, \partial_{kl} p(0) \int_{\mathbb{R}^N_+} |\nabla V|^p r^{-2} y_i y_j y^k y^l \,\mathrm{d}y\,\mathrm{d}t \\
&\quad = \sum_{i=1}^{N-1} a^{ii}\, \partial_{ii} p(0) \int_{\mathbb{R}^N_+} |\nabla V|^p r^{-2} y_1^4 \,\mathrm{d}y\,\mathrm{d}t \\
&\qquad + \left(\sum_{i\neq k} a^{ii}\, \partial_{kk} p(0) + 2a^{ik}\, \partial_{ik} p(0)\right) \int_{\mathbb{R}^N_+} |\nabla V|^p r^{-2} y_1^2 y_2^2 \,\mathrm{d}y\,\mathrm{d}t \\
&\quad = \sum_{i=1}^{N-1} a^{ii}\, \partial_{ii} p(0) \int_{\mathbb{R}^N_+} |\nabla V|^p r^{-2} \big(y_1^4 - 3y_1^2 y_2^2\big) \,\mathrm{d}y\,\mathrm{d}t \\
&\qquad + \sum_{i,k=1}^{N-1} \big(a^{ii}\, \partial_{kk} p(0) + 2a^{ik}\, \partial_{ik} p(0)\big) \int_{\mathbb{R}^N_+} |\nabla V|^p r^{-2} y_1^2 y_2^2 \,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}
$$

The other simplifications follow in the same manner. □

Lemma A.6. *Assume $p < N^2/(3N-2)$. There holds that*

$$
\int_{\mathbb{R}^N_+} f(y,t)\eta(y,t)|\nabla V_\varepsilon|^{p(y,t)} \,\mathrm{d}y\,\mathrm{d}t = \bar{C}_0 + \bar{C}_1\varepsilon\ln\varepsilon + \bar{C}_2\varepsilon + \bar{C}_3(\varepsilon\ln\varepsilon)^2 + \bar{C}_4\varepsilon^2\ln\varepsilon + \mathrm{O}\big(\varepsilon^2\big)
$$

with

$$\bar{C}_0 = f(0)\int_{\mathbb{R}^N_+} |\nabla V|^p \,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{C}_1 = -\frac{N}{p} f(0)\,\partial_t p(0)\int_{\mathbb{R}^N_+} t|\nabla V|^p \,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{C}_2 = \int_{\mathbb{R}^N_+} t|\nabla V|^p\big(f(0)\,\partial_t p(0)\ln|\nabla V| + \partial_t f(0)\big)\,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{C}_3 = \frac{N^2}{2p^2} f(0)\,\partial_t p(0)^2\int_{\mathbb{R}^N_+} t^2|\nabla V|^p \,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{C}_4 = -\frac{N}{p}\int_{\mathbb{R}^N_+} t^2|\nabla V|^p\left(\frac{f(0)}{2}\,\partial_{tt} p(0) + \partial_t p(0)\,\partial_t f(0) + \partial_t p(0)^2 f(0)\ln|\nabla V|\right)\mathrm{d}y\,\mathrm{d}t$$
$$- \frac{N}{2(N-1)p} f(0)\Delta_y p(0)\int_{\mathbb{R}^N_+} |y|^2|\nabla V|^p \,\mathrm{d}y\,\mathrm{d}t, \quad \Delta_y = \sum_{i=1}^{n-1}\partial_{ii}.$$

Proof. As before

$$\int_{\mathbb{R}^N_+\setminus B_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} |\nabla V_\varepsilon|^{p(y,t)}\,\mathrm{d}y\,\mathrm{d}t \leqslant C\varepsilon^{\frac{N}{p*}} = \mathrm{O}\big(\varepsilon^2\big),$$

so that

$$\int_{\mathbb{R}^N_+} f(y,t)\eta(y,t)|\nabla V_\varepsilon|^{p(y,t)}\,\mathrm{d}y\,\mathrm{d}t$$
$$= \int_{B^+_{C\varepsilon^{\frac{N-p}{p(N-1)}}}} f(y,t)|\nabla V_\varepsilon|^{p(y,t)}\,\mathrm{d}y\,\mathrm{d}t + \mathrm{O}\big(\varepsilon^2\big)$$
$$= \bar{C}_0(\varepsilon) + \bar{C}_1(\varepsilon)\varepsilon\ln\varepsilon + \bar{C}_2(\varepsilon)\varepsilon + \bar{C}_3(\varepsilon)(\varepsilon\ln\varepsilon)^2 + \bar{C}_4(\varepsilon)\varepsilon^2\ln\varepsilon + \mathrm{O}\big(\varepsilon^2\big),$$

where the constants $\bar{C}_i(\varepsilon)$ are the same as

$$\bar{C}_0 = f(0)\int_{\mathbb{R}^N_+} |\nabla V|^p \,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{C}_1 = -\frac{N}{p} f(0)\,\partial_t p(0)\int_{\mathbb{R}^N_+} t|\nabla V|^p \,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{C}_2 = \int_{\mathbb{R}^N_+} t|\nabla V|^p\big(f(0)\,\partial_t p(0)\ln|\nabla V| + \partial_t f(0)\big)\,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{C}_3 = \frac{N^2}{2p^2} f(0)\,\partial_t p(0)^2\int_{\mathbb{R}^N_+} t^2|\nabla V|^p \,\mathrm{d}y\,\mathrm{d}t,$$

$$\bar{C}_4 = -\frac{N}{p}\int_{\mathbb{R}^N_+} |\nabla V|^p \biggl(\frac{f(0)}{2}\bigl(D^2 p(0)(y,t),(y,t)\bigr) + \partial_t p(0)\,\partial_t f(0) t^2 + \partial_t p(0)^2 f(0) t^2 \ln|\nabla V|\biggr)\,\mathrm{d}y\,\mathrm{d}t$$

but with integral over $B^+_{C\varepsilon^{-\frac{N(p-1)}{p(N-1)}}}$ instead of $\mathbb{R}^N_+$. We can estimate $|\bar{C}_i(\varepsilon)-\bar{C}_i|$ as we estimated $|\bar{B}_i(\varepsilon)-\bar{B}_i|$ in the previous lemma.

Again, using the radial symmetry of V we can simplify the constants $\bar{C}_i$ as in the previous lemma. □

With the aid of the previous lemmas, we can now prove Proposition A.4.

Proof of Proposition A.4. First, by Lemma A.1,

$$\int_\Omega f(x)|\nabla v_\varepsilon|^{p(x)}\,\mathrm{d}x = \int_{\mathbb{R}^N_+} f(y,t)|\nabla v_\varepsilon|^{p(y,t)}\bigl(1 - Ht + \mathrm{O}\bigl(t^2+|y|^2\bigr)\bigr)\,\mathrm{d}y\,\mathrm{d}t,$$

where we denote $f(y,t) = f(\Phi(y,t))$ and $p(y,t) = p(\Phi(y,t))$.

Recall that, by Lemma A.1,

$$|\nabla v_\varepsilon|^2 = (\partial_t v_\varepsilon)^2 + \sum_{i,j=1}^{N-1}\bigl(\delta^{ij} + 2h^{ij}t + \mathrm{O}\bigl(t^2+|y|^2\bigr)\bigr)\,\partial_i v_\varepsilon\,\partial_j v_\varepsilon, \quad \partial_i = \frac{\partial}{\partial y_i}.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} f(y,t)|\nabla v_\varepsilon|^{p(y,t)}\bigl(1 - Ht + \mathrm{O}\bigl(t^2+|y|^2\bigr)\bigr)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad = \int_{\mathbb{R}^N_+} f(y,t)\bigl|\nabla(\eta V_\varepsilon)\bigr|^{p(y,t)}\bigl(1 - Ht + \mathrm{O}\bigl(t^2+|y|^2\bigr)\bigr)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad = \int_{\mathbb{R}^N_+} f(y,t)\eta(y,t)^{p(y,t)}|\nabla V_\varepsilon|^{p(y,t)}\bigl(1 - Ht + \mathrm{O}\bigl(t^2+|y|^2\bigr)\bigr)\,\mathrm{d}y\,\mathrm{d}t + R(\varepsilon),
\end{aligned}$$

where

$$\bigl|R(\varepsilon)\bigr| \leqslant C\int_{\mathbb{R}^N_+\setminus B_\delta} |V_\varepsilon|^{p(y,t)}\,\mathrm{d}y\,\mathrm{d}t \leqslant C\varepsilon^p \int_{\delta/\varepsilon}^\infty r^{-\frac{p(N-p)}{p-1}+N-1}\,\mathrm{d}r = \mathrm{O}\bigl(\varepsilon^2\bigr),$$

if $p \leqslant (n+2)/3$. Hence

$$\begin{aligned}
&\int_\Omega f(x)|\nabla v_\varepsilon|^{p(x)}\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N_+} f(y,t)\eta(y,t)^{p(y,t)}\Biggl[(\partial_t U_\varepsilon)^2 + \sum_{i,j=1}^{N-1}\bigl(\delta^{ij} + 2h^{ij}t + \mathrm{O}\bigl(t^2+|y|^2\bigr)\bigr)\,\partial_i V_\varepsilon\,\partial_j V_\varepsilon\Biggr]^{\frac{p(y,t)}{2}} \\
&\qquad \times \bigl(1 - Ht + \mathrm{O}\bigl(t^2+|y|^2\bigr)\bigr)\,\mathrm{d}y\,\mathrm{d}t + \mathrm{O}\bigl(\varepsilon^2\bigr)
\end{aligned}$$

with

$$
\begin{aligned}
&\left[(\partial_t V_\varepsilon)^2 + \sum_{i,j=1}^{N-1} \big(\delta^{ij} + 2h^{ij} t + \mathrm{O}\big(t^2 + |y|^2\big)\big)\, \partial_i V_\varepsilon\, \partial_j V_\varepsilon\right]^{\frac{p(y,t)}{2}} \\
&\quad = |\nabla V_\varepsilon|^{p(y,t)} \left[1 + \sum_{i,j=1}^{N-1} p(y,t) t h^{ij} |\nabla V_\varepsilon|^{-2}\, \partial_i V_\varepsilon\, \partial_j V_\varepsilon + \mathrm{O}\big(t^2 + |y|^2\big)\right] \\
&\quad = |\nabla V_\varepsilon|^{p(y,t)} + p(y,t) t h^{ij} |\nabla V_\varepsilon|^{p(y,t)-2}\, \partial_i V_\varepsilon\, \partial_j V_\varepsilon + |\nabla V_\varepsilon|^{p(y,t)} \mathrm{O}\big(t^2 + |y|^2\big).
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_\Omega f(x) |\nabla v_\varepsilon|^{p(x)}\, \mathrm{d}x &= \int_{\mathbb{R}^N_+} f(y,t) \eta(y,t)^{p(y,t)} |\nabla V_\varepsilon|^{p(y,t)}\, \mathrm{d}y\, \mathrm{d}t \\
&\quad + \sum_{i,j=1}^{N-1} h^{ij} \int_{\mathbb{R}^N_+} t f(y,t) p(y,t) \eta(y,t)^{p(y,t)} |\nabla V_\varepsilon|^{p(y,t)-2}\, \partial_i V_\varepsilon\, \partial_j V_\varepsilon\, \mathrm{d}y\, \mathrm{d}t \\
&\quad - H \int_{\mathbb{R}^N_+} t f(y,t) \eta(y,t)^{p(y,t)} |\nabla V_\varepsilon|^{p(y,t)}\, \mathrm{d}y\, \mathrm{d}t + \mathrm{O}\big(\varepsilon^2\big)
\end{aligned}
$$

since

$$
\begin{aligned}
\int_{\mathbb{R}^N_+} |\nabla V_\varepsilon|^{p(y,t)} \mathrm{O}\big(t^2 + |y|^2\big)\, \mathrm{d}y\, \mathrm{d}t &\leqslant C \int_{\mathbb{R}^N_+} \big|(y,t)\big|^2 |\nabla V_\varepsilon|^{p(y,t)}\, \mathrm{d}y\, \mathrm{d}t \\
&\leqslant C\varepsilon^2 \int_{\mathbb{R}^N_+} \big|(y,t)\big|^2 |\nabla V|^{p+\mathrm{O}(\varepsilon)}\, \mathrm{d}y\, \mathrm{d}t \\
&= C\varepsilon^2 \int_{\mathbb{R}^N_+} \big|(y,t)\big|^2 |\nabla V|^{p} \big(1 + \mathrm{O}(\varepsilon) \ln |\nabla V|\big)\, \mathrm{d}y\, \mathrm{d}t.
\end{aligned}
$$

As before this last integral is finite provided that $p < (N+2)/3$.

The proof now follows applying Lemmas A.5 and A.6. □

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