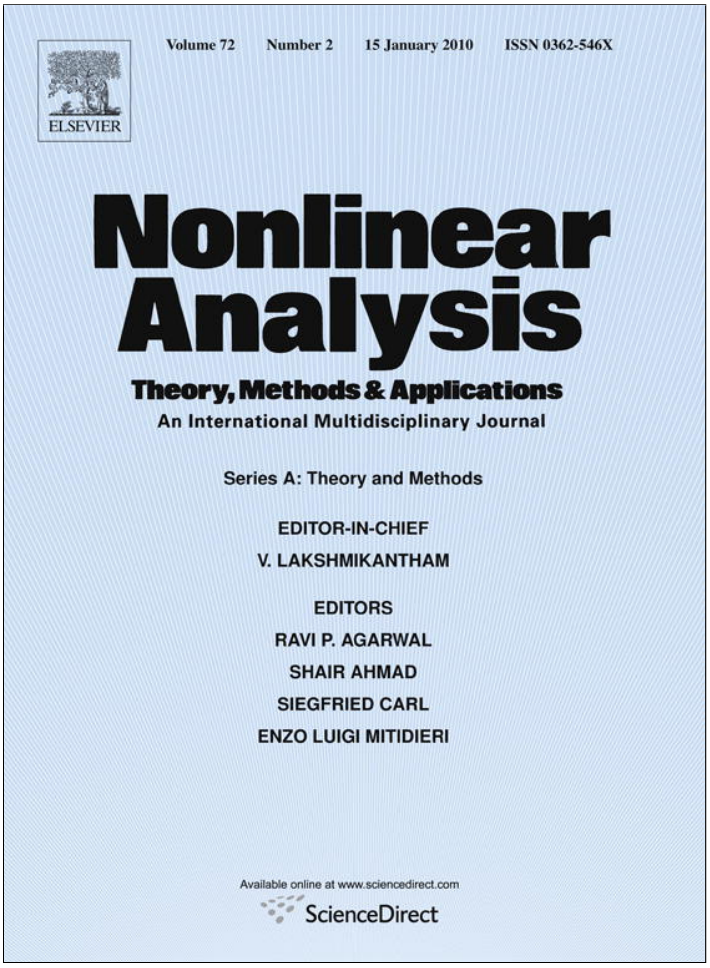
Volume 72 Number 2 15 January 2010 ISSN 0362-546X
ELSEVIER
Nonlinear Analysis
Theory, Methods & Applications
An International Multidisciplinary Journal
Series A: Theory and Methods
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Nonlinear Analysis 72 (2010) 689–703

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Best constant in critical Sobolev inequalities of second-order in the presence of symmetries

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ARTICLE INFO

Article history:
Received 12 May 2009
Accepted 6 July 2009

MSC:
35B40
35J20
35B33

Keywords:
BiLaplacian
Paneitz-type operator
Invariance under isometries
Best constant

ABSTRACT

Let (M, g) be a smooth compact Riemannian manifold. We first give the value of the best first constant for the critical embedding $H^2(M) \hookrightarrow L^{2^\sharp}(M)$ for second-order Sobolev spaces of functions invariant by some subgroup of the isometry group of (M, g). We also prove that we can take $\epsilon = 0$ in the corresponding inequality under some geometric assumptions. As an application we give a sufficient condition for the existence of a smooth positive symmetric solution to a critical equation with a symmetric Paneitz–Branson-type operator. A sufficient condition for the existence of a nodal solution to such an equation is also derived. We eventually prove a multiplicity result for such an equation.

Let (M, g) be a smooth compact Riemannian manifold of dimension n, and G a closed subgroup of the group of isometries $\mathrm{Isom}_g(M)$ of (M, g) such that $k := \min_{x\in M} \dim Gx$ and $\bar{n} := n - k \geq 5$, where Gx denotes the orbit of a point $x \in M$ under the action of G. We say that a function $f : M \to \mathbb{R}$ is G-invariant if $f(gx) = f(x)$ for any $x \in M$ and $g \in G$. Let $H^1(M)$ (resp. $H^2(M)$) be the Sobolev space of the functions $u \in L^2(M)$ such that $\nabla u \in L^2(M)$ (resp. and $\nabla^2 u \in L^2(M)$), and $H^l_G(M)$ be the subspace of $H^l(M)$ of G-invariant functions, $l = 1, 2$. It follows from an argument similar to Hebey–Vaugon [1], who dealt with $H^1_G(M)$, that $H^2_G(M)$ is continuously embedded into $L^p(M)$, $p \leq 2^\sharp := 2\bar{n}/(\bar{n} - 4)$, and that this embedding is compact when $p < 2^\sharp$. Hence the exponent $2^\sharp$ is critical from the Sobolev viewpoint. Let $K_0(n)$ be the best Sobolev constant for the embedding of $D^2_2(\mathbb{R}^n)$, the completion of the space $C^\infty_c(\mathbb{R}^n)$ of smooth functions with compact support for the norm $\|u\| = \|\Delta u\|_2$, into $L^{2^\sharp}(\mathbb{R}^n)$, namely

$$K_0(n)^{-1} = \inf_{u\in C^\infty_c(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} (\Delta_\xi u)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} dx\right)^{2/2^\sharp}} > 0, \tag{1}$$

where ξ denotes the Euclidean metric. The value of $K_0(n)$ is explicitly known (see Edmunds–Fortunato–Janelli [2], Lieb [3], Lions [4]). When G is reduced to the identity, Djadli–Hebey–Ledoux [5] (see also Hebey [6], Caraffa [7]) proved that $K_0(n)$ is the best first constant in the Sobolev inequality corresponding to the embedding of $H^2(M)$ into $L^{2^\sharp}(M)$ in the sense that for any $\epsilon > 0$ there exists $B_\epsilon > 0$ such that

$$\left(\int_M |u|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}} \leq (K_0(n) + \epsilon) \int_M (\Delta_g u)^2 dv_g + B_\epsilon \|u\|^2_{H^1} \tag{2}$$

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0362-546X/$ – see front matter
doi:10.1016/j.na.2009.07.010

for any $u \in H^2(M)$, where $\|u\|^2_{H^1} = \|u\|^2_2 + \|\nabla u\|^2_2$. Moreover $K_0(n)$ is the lowest constant such that such an inequality holds for any $\epsilon > 0$ and $u \in H^2(M)$. As a remark on this inequality, it follows from the Bochner–Lichnerowicz–Weitzenböck formula that $H^2(M)$ can be equipped with the norm $\|u\|^2_{H^2} = \|\Delta_g u\|^2_2 + \|u\|^2_{H^1}$ which is equivalent to the standard one (see [5]). We will always use this last norm in what follows. Hebey [6] then proved that we can take $\epsilon = 0$ in (2) in the sense that there exists $B > 0$ such that

$$\left(\int_M |u|^{2^\sharp}\,\mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \leq K_0(n)\int_M (\Delta_g u)^2\,\mathrm{d}v_g + B\|u\|^2_{H^1} \tag{3}$$

for any $u \in H^2(M)$.

The main purpose of this paper is to extend both results to the case of symmetric functions, first by giving the value of the best first constant for the Sobolev inequality corresponding to the embedding of $H^2_G(M)$ into $L^{2^\sharp}(M)$ for an arbitrary subgroup of isometries G, and then by proving that the corresponding optimal inequality (3) for G-invariant functions holds under some additional hypothesis on G.

Since a G-invariant function is also $\bar{G}$-invariant, we will always assume in what follows without restriction that G is closed.

Concerning the value of the best first constant, we prove that

Theorem 0.1. *Let (M, g) be a smooth compact Riemannian n-manifold and G a closed subgroup of $\mathrm{Isom}(M, g)$ such that $\bar{n} := n - k \geq 5$ where $k = \min_{x\in M} \dim Gx$. Then for any $\epsilon > 0$, there exists $B_\epsilon > 0$ such that for every $u \in H^2_G(M)$,*

$$\left(\int_M |u|^{2^\sharp}\,\mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \leq \left(\frac{K_0(\bar{n})}{A^{\frac{4}{\bar{n}}}} + \epsilon\right)\int_M (\Delta_g u)^2\mathrm{d}v_g + B_\epsilon\|u\|^2_2, \tag{4}$$

where A is the minimum volume of the k-dimensional orbits and $K_0(\bar{n})$ is given by (1). Moreover $K_0(\bar{n})A^{-\frac{4}{\bar{n}}}$ is the lowest constant such that such an inequality holds for any $\epsilon > 0$ and any $u \in H^2_G(M)$.

We now turn our attention to the problem of taking $\epsilon = 0$ in (4). Before stating our assumption, we recall that, given a closed subgroup G' of $\mathrm{Isom}_g(M)$, an orbit $G'x$ is said principal if its stabilizer $S_x := \{g \in G', gx = x\}$ is minimal up to conjugacy i.e. for all $y \in M$, S_y contains a subgroup conjugate to S_x. In particular, the principal orbits are of maximal dimension (but the converse is false). The union denoted by Ω of all the principal orbits is then a dense open subset of M, and Ω/G' is a smooth connected manifold which can be equipped with a Riemannian metric $\bar{g}$ in such a way that the canonical surjection $\Pi : x \in \Omega \to \bar{x} \in \Omega/G'$ is a Riemannian submersion. We define a metric $\tilde{g}$ belonging to the conformal class of $\bar{g}$ by

$$\tilde{g} = \bar{v}^{\frac{2}{\bar{n}-4}}\bar{g}, \tag{5}$$

where $\bar{v}(\bar{x}) = |\Pi^{-1}(\bar{x})| = |G'x|$ denotes the volume of $G'x$ for the metric induced by g. We refer to Bredon [8] for more details (see also Hebey–Vaugon [1] and Faget [9]).

Let A be the minimum volume of a k-dimensional orbit. We consider the two following sets of assumption (H) and (H′) on the G-orbits of dimension k and minimal volume A:

(H) for each G-orbit Gx_0 of minimal dimension k and minimal volume A, there exist $\delta > 0$ and a closed subgroup G' of $\mathrm{Isom}_g(M)$ such that
(H1) $G'x_0 = Gx_0$ and, for all $x \in B_{Gx_0}(\delta) := \{y \in M, d_g(y, Gx_0) < \delta\}$,
(H2) $G'x$ is principal and $G'x \subset Gx$,
(H3) $\bar{x}_0$ is a minimum of $\bar{v} : \bar{x} \in B_{Gx_0}(\delta)/G' \to |G'x|$.
(H′) *for each G-orbit Gx_0 of minimal dimension k and minimal volume A, there exist $\delta > 0$ and a closed normal subgroup G' of G such that* (H1) *and* (H2) *of* (H) *hold*,
(H′3) $\dim Gx > \dim Gx_0 = k$, for any $\bar{x} \neq \bar{x}_0$,
and
(H′4) $\bar{x}_0$is a critical point of $\bar{v}$.

In particular, under (H) or (H′), $\dim G'x = \dim Gx_0 = k$ for all $x \in B_{Gx_0}(\delta)$, and we can consider the Riemannian quotient $\bar{n}$-manifold $N := B_{Gx_0}(\delta)/G'$, where $\bar{n} = n - k$. Examples of manifolds and isometries subgroups satisfying these hypotheses are given in [10].

Our result is the following:

Theorem 0.2. *Let (M, g) be a smooth compact Riemannian n-manifold and G a closed subgroup of $\mathrm{Isom}(M, g)$ satisfying the assumption* (H) *or* (H′) *and $\bar{n} \geq 5$, where $k = \min_{x\in M} \dim Gx$. Then there exists $B > 0$ such that*

$$\left(\int_M |u|^{2^\sharp}\,\mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \leq \frac{K_0(\bar{n})}{A^{\frac{4}{\bar{n}}}}\int_M (\Delta_g u)^2\mathrm{d}v_g + B\|u\|^2_{H^1} \tag{6}$$

for any $u \in H^2_G(M)$.

As an application of Theorem 0.1, we provide a sufficient condition for the existence of a G-invariant solution for a critical equation with a symmetric Paneitz–Branson-type operator P_g like

$$P_g u := \Delta_g^2 u - \mathrm{div}_g(b^\sharp du) + au = f|u|^{2^\sharp-1}, \tag{7}$$

where b is a smooth $(2,0)$-tensor field, i.e. $b = b_{ij}dx^i \otimes dx^j$ in a chart, that we suppose symmetric in the sense that $b_{ij} = b_{ji}$, and G-invariant (i.e. $\phi^* b = b$ for any $\phi \in G$), and $a, f \in C(M)$ are G-invariant. We assume that P_g is coercive in the sense that there exists a constant $C > 0$ such that $\int_M (P_g u)u\, dv_g \geq C\|u\|_{H^2}$ for all $u \in H_G^2(M)$. A necessary condition for (7) to admit a solution is then $\max_M f > 0$, what we assume from now on. We refer to [5,11,12] and references therein for an introduction to Paneitz–Branson-type operators.

Using Lions' concentration-compactness principle as in [13] (see also [5] when G is reduced to the identity, or [9,14] for equations involving the p-Laplacian in the presence of symmetry), and regularity results as developed in [5,12], we have that if

$$\inf_{u\in H_G^2(M), u\not\equiv 0} \frac{\int_M (P_g u)u\, dv_g}{\left(\int_M f|u|^{2^\sharp} dv_g\right)^{2/2^\sharp}} < \left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}\right)^{-1} \|f\|_\infty^{-2/2^\sharp}, \tag{8}$$

then (7) has a non-trivial G-invariant solution of class $C^{4,\eta}(M)$. Note that the large inequality always holds (see the proof of Theorem 0.3). Moreover if b and a are positive real numbers (i.e. b has the form bg, $b \in (0, +\infty)$) with $0 < a \leq b^2/4$ and $f > 0$, then this solution can be chosen smooth and positive (see again [5,12] for such an assertion). We are such left with the problem of finding conditions ensuring (8). Taking the constant function equal to 1 as test-function we obtain that if

$$\frac{\int_M a\, dv_g}{\left(\|f\|_\infty^{-1}\int_M f\, dv_g\right)^{2/2^\sharp}} < \left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}\right)^{-1}$$

then (8) holds. We now look for a sufficient local condition for (8) to hold. Since b is G-invariant it defines on the quotient N, when it exists, a smooth $(2,0)$-tensor field $\bar b$ defined by $\bar b_{\bar x}(\bar X_{\bar x}, \bar Y_{\bar x}) = b_x(X_x, Y_x)$ where $x \in \Pi^{-1}(\bar x)$ and X_x, Y_x are the unique vectors at x normal to $T_x(\Pi^{-1}(\bar x))$ (i.e. horizontal) and such that $d\Pi(x)X_x = \bar X_{\bar x}$ and $d\Pi(x)Y_x = \bar Y_{\bar x}$. The result is the following:

Theorem 0.3. *Let (M,g) be a smooth compact Riemannian n-manifold, G a closed subgroup of $\mathrm{Isom}(M,g)$ such that $\bar n := n-k \geq 6$ where $k = \min_{x\in M} \dim Gx$, and b, a, f as above. If there exists a k-dimensional orbit Gx_0 of volume A with $f(x_0) = \max f$ such that conditions (H1) and (H2) stated above hold and*

$$(\bar n-6)(\bar n-4)(\bar n+2)\frac{\Delta_{\bar g}\bar f(\bar x_0)}{\|f\|_\infty} + 8(\bar n-1)Tr_{\bar g}\bar b(\bar x_0) - 4(\bar n^2-2\bar n-4)S_{\bar g}(\bar x_0) - 4(\bar n^2-16)\frac{\Delta_{\bar g}\bar v(\bar x_0)}{A} < 0, \tag{9}$$

then (8) holds.

For example if M is the product of a compact Riemannian m-manifold (M', g') with the standard sphere $(S^{n-m}(r), h) \subset \mathbb{R}^{n-m+1}$ of radius $r > 0$, and G is the product of the identity on M' with some finite group $G' \subset O(n-m+1)$ acting freely on $S^{n-m}(r)$, then all the G-orbit are principal, and in particular have same dimension $k = 0$, and have same cardinal. We can globally quotient M by G and the canonical submersion $\Pi : M \to N = M/G = M' \times S^{n-m}(r)/G'$ is a local isometry. In particular the scalar curvature of N at a point $\bar x_0 = (y_0, \bar z_0) \in M' \times S^{n-m}(r)/G'$ is equal to $S_{g'}(y_0) + S_h(z_0) = S_{g'}(y_0) + \frac{(n-m)(n-m-1)}{r^2}$. Hence (9) writes

$$(n-6)(n-4)(n+2)\frac{\Delta_g f(x_0)}{\|f\|_\infty} + 8(n-1)Tr_g b(x_0) < 4(n^2-2n-4)\left(S_{g'}(y_0) + \frac{(n-m)(n-m-1)}{r^2}\right).$$

As another example taking now $r = 1$ and $G = \{Id_{M'}\} \times O(r_2) \times O(r_1)$ with $r_1 + r_2 = n-m+1$, $r_2 \geq r_1$, we see that the G-orbit of a point $x_0 = (y_0, 0, z_0) \in M' \times \mathbb{R}^{r_2} \times \mathbb{R}^{r_1}$, $\|z_0\| = 1$, has minimal dimension $k = r_1 - 1$ and minimal volume. Let $G' = \{Id_{M'}\} \times \{Id_{\mathbb{R}^{r_2}}\} \times O(r_1)$. Then G' satisfies $H1$ and $H2$, and $\bar v : \bar x \in N \to Vol(G'x)$ has a maximum at $\bar x_0$ so that $\Delta_{\bar g}\bar v(\bar x_0) \geq 0$. Moreover Dellinger [15, prop. 3.1] showed that $S_{\bar g}(\bar x_0) \geq S_{g'}(y_0) + r_2(r_2-1)$. Hence if

$$(\bar n-6)(\bar n-4)(\bar n+2)\frac{\Delta_g f(x_0)}{\|f\|_\infty} + 8(\bar n-1)Tr_{\bar g}\bar b(\bar x_0) < 4(\bar n^2-2\bar n-4)(S_{g'}(y_0) + r_2(r_2-1))$$

then (9) holds. Other examples can be found in [15].

We denote by $B_0(g)$ the infimum of the B such that (6) holds for any $u \in H_G^2(M)$. Then under the assumptions of Theorem 0.2,

$$\left(\int_M |u|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}} \leq \frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}\int_M (\Delta_g u)^2 dv_g + B_0(g)\|u\|_{H^1}^2$$

for any $u \in H_G^2(M)$. This inequality is optimal with respect to both constants. Using on one hand the constant function equal to one, and on the other hand the u_ϵ used in the proof of Theorem 0.3, we obtain that

$$B_0(g) \geq \max\{Vol_g(M)^{-\frac{4}{\bar n}}, B_0(g)_{extr}\},$$

with

$$B_0(g)_{extr} = \frac{K_0(\bar{n})}{A^{4/\bar{n}}} \frac{\bar{n}-4}{2\bar{n}(\bar{n}-1)} \max\left(\frac{\bar{n}^2-2\bar{n}-4}{\bar{n}-4} S_{\bar{g}}(\bar{x}_0) + (\bar{n}+4)\frac{\Delta_{\bar{g}}\bar{v}(\bar{x}_0)}{A}\right)$$

where the maximum is taken over all the points x_0 such that Gx_0 satisfies assumption or (H) or (H′).

As another application we prove the following multiplicity result for the critical equation (7) whose proof follows the line of [16]. Indeed the proof there deals with the case where (M, g) is the standard unit sphere and the coefficients of (7) are constants.

Theorem 0.4. *If the coefficients b, a, and f of Eq. (7) are G-invariant for some subgroup* $G \subset \mathrm{Isom}_g(M)$ *such that* $k = \min_{x\in M} \dim Gx \geq 1$*, and if* $b \geq 0$, $a > 0$, $f > 0$*, then (7) has an infinite number of distinct solutions* (u_m) *such that* $\int_M |u_m|^{2^\sharp} \, dv_g \to +\infty$.

We eventually turn to the problem of finding a nodal G-invariant solution for Eq. (7). Following Hebey and Vaugon [17], who dealt with this question for critical equations involving the Laplacian, we suppose that there exists $\tau \in \mathrm{Isom}_g(M)$ such that $\tau^2 = Id$ and $G\tau = \tau G$. We say that a function u is τ-antisymmetric if $(\tau u)(x) := u(\tau x) = -u(x)$. We denote by $\langle G, \tau\rangle$ the subgroup generated by τ and G, and by $H^2_{G,\tau}(M)$ the subspace of $H^2_G(M)$ of τ-antisymmetric functions. We assume that the coefficients b, a, f of (7) are $\langle G, \tau\rangle$-invariant, and that the set $\{x \in M, \tau x = x\}$ divides M in two smooth G-invariant submanifolds M_+ and M_- with $M_- = \tau(M_+)$. In particular the minimum volume of a k-dimensional $\langle G, \tau\rangle$-orbit is $2A$.

Theorem 0.5. *If*

$$\inf_{u\in H^2_{\langle G,\tau\rangle}(M)\setminus\{0\}, u\geq 0 \text{ in } M_+} \frac{\int_M (P_g u)u \, dv_g}{\left(\int_M f|u|^{2^\sharp} dv_g\right)^{2/2^\sharp}} < \left(\frac{K_0(\bar{n})}{(2A)^{\frac{4}{\bar{n}}}}\right)^{-1} \|f\|_\infty^{-2/2^\sharp}, \tag{10}$$

then (7) has a τ*-antisymmetric G-invariant solution* $u \in C^{4,\eta}(M)$ *which is nonnegative in* M_+ *and nonpositive in* M_-*. Moreover if b, a are positive real numbers with* $0 < a \leq b^2/4$ *and* $f > 0$*, then* $u > 0$ *in* M_+ *and* $u < 0$ *in* M_-*. In particular u is a nodal solution of (7). Eventually, under the same hypothesis on G as in Theorem 0.3, we have that if (9) holds then (10) holds.*

Except the last two, these results are analogous for the second-order Sobolev spaces of Faget's results [9,10] which concerns first-order Sobolev spaces. Our proof will follow the line of Faget's proof combined with the result of [13] to deal with the difficulties specific to the fourth order.

1. Proof of Theorem 0.1

We will first prove a local version of the inequality (4) following the lines of Faget [9] and then deduce (4) using a standard gluing argument. The optimality of $K_0(\bar{n})A^{-\frac{4}{\bar{n}}}$ will be proved in the last step.

We first prove that

Step 1.1. *Let* $x \in M$ *such that* $m := \dim Gx < n$*. For any* $\epsilon > 0$*, there exists* $\delta > 0$ *and* $C_\epsilon > 0$ *such that (4) holds for every G-invariant function* $u \in C_c^\infty(B_{Gx}(\delta))$*, where* $B_{Gx}(\delta) = \{y \in M, \ d_g(y, Gx) < \delta\}$*,* d_g *being the distance induced by g.*

Proof. According to [9, lemma 2], there exists a normal chart (Ω, ψ) around x such that $\psi(\Omega) = U_1 \times U_2 \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$, $\psi(Gx \cap \Omega) \subset U_1$, $u \circ \psi^{-1}$ depends only on the U_2 variable and $|g_{ij} - \delta_{ij}|, |\Gamma^l_{ij}| \leq \epsilon$ for every $1 \leq i, j, l \leq n$, where (g_{ij}) and Γ^l_{ij} denote respectively the metric g and the associated Christoffel symbols read in the chart (Ω, ψ). Then $(\sigma(\Omega), \psi \circ \sigma^{-1})$ is a chart around $\sigma(x)$ that is isometric to (Ω, ψ). Since Gx is compact, it can be covered by a finite number of such charts $(\sigma_k(\Omega), \psi \circ \sigma_k^{-1})$, $1 \leq k \leq l$. We let $\Omega_k = \sigma_k(\Omega)$ and $\psi_k = \psi \circ \sigma_k^{-1}$. Let $\beta \in C_c^\infty(U_1)$, $\beta \geq 0$, that we see as a function defined on $U_1 \times U_2$, and $\alpha_k = (\sum_{j=1}^l \beta \circ \psi_j)^{-1} \beta \circ \psi_k$, $1 \leq k \leq l$. Then (α_k) is a partition of unity relative to the covering $\{\Omega_k, \ 1 \leq k \leq l\}$ such that $\alpha_k \circ \psi_k^{-1}$ only depends on the U_1-variable. We will thus consider as well α_k as a function defined on U_1 only. We choose $\delta > 0$ such that $B_x(\delta) \subset \Omega$. Then $B_{Gx}(\delta) \subset \bigcup_{k=1}^l \Omega_k$.

Let $v \in C_c^\infty(B_{Gx}(\delta))$ be G-invariant. Then

$$\begin{aligned}\int_M v \, dv_g &= \sum_{k=1}^l \int_{\Omega_k} \alpha_k v \, dv_g = \sum_{k=1}^l \int_\Omega (\alpha_k \circ \sigma_k) v \, dv_g \\ &= \sum_{k=1}^l \int_{U_1\times U_2} (\alpha_k \circ \psi_k^{-1})(v \circ \psi^{-1})\sqrt{\det(g_{ij})} \, dxdy \\ &= (1 + O(\epsilon)) \sum_{k=1}^l \int_{U_1} \alpha_k \circ \psi_k^{-1} \, dx \int_{U_2} v \circ \psi^{-1} \, dy.\end{aligned}$$

On the other hand, if σ_g denotes the metric induced by g on Gx, we have

$$|Gx| = \sum_{k=1}^{l} \int_{Gx\cap\Omega_k} \alpha_k \, \mathrm{d}v_{\sigma_g} = \sum_{k=1}^{l} \int_{Gx\cap\Omega} \alpha_k \circ \sigma_k \, \mathrm{d}v_{\sigma_g}$$
$$= (1+O(\epsilon)) \sum_{k=1}^{l} \int_{U_1} \alpha_k \circ \psi_k^{-1} \, \mathrm{d}x.$$

Hence

$$\int_M v \, \mathrm{d}v_g = (1+O(\epsilon))|Gx| \int_{U_2} v \circ \psi^{-1} \, \mathrm{d}y.$$

In particular, if $u \in C_c^\infty(B_{Gx}(\delta))$ is G-invariant we get

$$\int_M |u|^{2^\sharp} \, \mathrm{d}v_g = (1+O(\epsilon))|Gx| \int_{U_2} |u_2|^{2^\sharp} \, \mathrm{d}y, \tag{11}$$

and

$$\int_M (\Delta_\xi u)^2 \, \mathrm{d}v_g = (1+O(\epsilon))|Gx| \int_{U_2} (\Delta_\xi u_2)^2 \, \mathrm{d}y,$$

where $u_2 = u \circ \psi^{-1}$ and $U_2 \subset \mathbb{R}^{n-m}$. Assume that $m = k$. Using (1) we obtain

$$\left(\int_M |u|^{2^\sharp} \, \mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \le (1+O(\epsilon))|Gx|^{\frac{2}{2^\sharp}} K_0(\bar{n}) \int_{U_2} (\Delta_\xi u_2)^2 \, \mathrm{d}y$$
$$= (1+O(\epsilon))|Gx|^{\frac{2}{2^\sharp}-1} K_0(\bar{n}) \int_M (\Delta_\xi u)^2 \, \mathrm{d}v_g.$$

Since $\Delta_g u = -g^{ij}(\partial_{ij}u - \Gamma_{ij}^l \partial_l u) = \Delta_\xi u + O(\epsilon)(|\nabla u|_g + |\nabla^2 u|_g)$, we have

$$\int_M (\Delta_\xi u)^2 \, \mathrm{d}v_g = (1+O(\epsilon)) \int_M (\Delta_g u)^2 \, \mathrm{d}v_g + O(\epsilon)\|u\|_{H^1}^2.$$

Hence

$$\left(\int_M |u|^{2^\sharp} \, \mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \le (1+O(\epsilon))|Gx|^{\frac{2}{2^\sharp}-1} K_0(\bar{n}) \int_M (\Delta_g u)^2 \, \mathrm{d}v_g + O(\epsilon)\|u\|_{H^1}^2$$
$$\le (1+O(\epsilon)) K_0(\bar{n}) A^{-\frac{4}{\bar{n}}} \int_M (\Delta_g u)^2 \, \mathrm{d}v_g + O(\epsilon)\|u\|_{H^1}^2, \tag{12}$$

where A is the minimum volume of the k-dimensional orbits. Eventually, since the embedding of $H^2(M)$ into $H^1(M)$ is compact, it is easily seen that for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$\|\nabla u\|_2^2 \le \delta \|\Delta\|_2^2 + C_\delta \|u\|_2^2 \tag{13}$$

for any $u \in H^2(M)$. We can now deduce the claim in the case $m = k$ from this inequality and (12).

Now if $m > k$, then $2^\sharp = \frac{2(\bar{n})}{\bar{n}-4} < \frac{2(n-m)}{n-m-4}$ and thus the embedding $H^2(U_2) \hookrightarrow L^{2^\sharp}(U_2)$ is compact. Given $\eta > 0$, it easily follows that there exists B_η such that for every $v \in H^2(U_2)$,

$$\left(\int_{U_2} |v|^{2^\sharp} \, \mathrm{d}y\right)^{\frac{2}{2^\sharp}} \le \eta \int_{U_2} (\Delta_\xi v)^2 \, \mathrm{d}y + B_\eta \|v\|_{H^1}^2.$$

In particular, (11) becomes

$$\left(\int_M |u|^{2^\sharp} \, \mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \le (1+O(\epsilon))|Gx|^{\frac{2}{2^\sharp}} \eta \int_{U_2} (\Delta_\xi u_2)^2 \, \mathrm{d}y + B'_\eta \|u_2\|_{H^1}^2$$

and then as before

$$\left(\int_M |u|^{2^\sharp} \, \mathrm{d}v_g\right)^{\frac{2}{2^\sharp}} \le (1+O(\epsilon))|Gx|^{\frac{2}{2^\sharp}-1} \eta \int_M (\Delta_g u)^2 \, \mathrm{d}v_g + B'_\eta \|u\|_{H^1}^2.$$

We now take a $\eta > 0$ small enough and use (13) to deduce the claim. □

We now prove the global inequality by using a partition of unity:

Step 1.2. *For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that (4) holds for every $u \in H_G^2(M)$.*

Proof. Since the space of smooth G-invariant functions $C_G^\infty(M)$ is dense in $H_G^2(M)$, it suffices to prove the claim for a function $u \in C_G^\infty(M)$. Given ϵ we choose δ as in the previous step. As M is compact, we can extract from the covering $\{B_{Gx}(\delta), x \in M\}$ a finite covering $\{B_{Gx_i}(\delta), 1 \le i \le l\}$. Let (η_i) be a partition of unity relative to this covering. According to the previous step, we can write

$$\begin{aligned}\|u\|_{2^\sharp}^2 &= \left\| \sum_{i=1}^l (\sqrt{\eta_i} u)^2 \right\|_{2^\sharp/2} \le \sum_{i=1}^l \left(\int_M |\sqrt{\eta_i} u|^{2^\sharp} \mathrm{d}v_g \right)^{2/2^\sharp} \\ &\le \left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}} + \epsilon \right) \sum_{i=1}^l \int_M (\Delta_g(\sqrt{\eta_i} u))^2 \mathrm{d}v_g + B_\epsilon \sum_{i=1}^l \|\sqrt{\eta_i} u\|_2^2 \\ &= \left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}} + \epsilon \right) \sum_{i=1}^l \int_M (\Delta_g(\sqrt{\eta_i} u))^2 \mathrm{d}v_g + B_\epsilon \|u\|_2^2.\end{aligned}$$

Using the inequality $(x+y)^2 \le (1+\epsilon)x^2 + C_\epsilon y^2$ we have

$$\begin{aligned}\sum_{i=1}^l \int_M (\Delta_g(\sqrt{\eta_i} u))^2 \mathrm{d}v_g &= \sum_{i=1}^l \int_M (\sqrt{\eta_i} \Delta_g u + u \Delta_g \sqrt{\eta_i} - 2(\nabla u, \nabla \sqrt{\eta_i})_g)^2 \mathrm{d}v_g \\ &\le (1+\epsilon) \int_M (\Delta_g u)^2 \mathrm{d}v_g + C_\epsilon \|u\|_{H^1}^2.\end{aligned}$$

We thus get

$$\|u\|_{2^\sharp}^2 \le \left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}} + O(\epsilon) \right) \int_M (\Delta_g u)^2 \mathrm{d}v_g + C_\epsilon \|u\|_{H^1}^2,$$

from which we deduce the claim using (13). □

It remains to prove that $K_0(\bar n) A^{-\frac{4}{\bar n}}$ is the optimal constant in the inequality (4):

Step 1.3. *$K_0(\bar n) A^{-\frac{4}{\bar n}}$ is optimal.*

Proof. Clearly, it suffices to prove that for any $\epsilon > 0$ and any $C > 0$,

$$\inf_{u \in H^2(M)} \frac{\int_M (\Delta_g u)^2 \mathrm{d}v_g + C\|u\|_2^2}{\left(\int_M |u|^{2^\sharp} \mathrm{d}v_g\right)^{2/2^\sharp}} \le A^{\frac{4}{\bar n}} K_0(\bar n)^{-1} + \epsilon.$$

Let $\delta > 0$ be small as in the first step and $\eta \in C^\infty([0, +\infty), [0, 1])$ with compact support in $[0, 2\delta)$ be such that $\eta = 1$ in $[0, \delta]$. Then the functions $\bar u_\epsilon \in C_c^\infty(\mathbb{R}^{\bar n})$ defined by

$$\bar u_\epsilon(x) = \frac{\eta(\|x\|)}{(\epsilon^2 + |x|^2)^{\frac{n-4}{2}}}, \quad x \in \mathbb{R}^{\bar n}$$

satisfy

$$\lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}^{\bar n}} (\Delta_g \bar u_\epsilon)^2 \mathrm{d}x + C\|\bar u_\epsilon\|_2^2}{\left(\int_{\mathbb{R}^{\bar n}} \bar u_\epsilon^{2^\sharp} \mathrm{d}x\right)^{2/2^\sharp}} = K_0(\bar n)^{-1}. \tag{14}$$

We refer for example to Esposito-Robert [12] for this result.

Let Gx_0 be a k-dimensional orbit of minimum volume A (such an orbit exists according to Faget [10, lemma 4]) and $u_\epsilon \in C_{c,G}^\infty(B_{Gx}(2\delta))$ be defined by

$$u_\epsilon(x) = \frac{\eta(d_g(x, Gx_0))}{(\epsilon^2 + d_g(x, Gx_0)^2)^{\frac{n-4}{2}}}. \tag{15}$$

With similar computations (and the same notations) as in the first step, we can prove that

$$\frac{\int_M (\Delta_g u_\epsilon)^2 \mathrm{d}v_g + C\|u_\epsilon\|_2^2}{\left(\int_M u_\epsilon^{2^\sharp} \mathrm{d}v_g\right)^{2/2^\sharp}} \le (1 + O(\epsilon)) A^{\frac{4}{\bar n}} \frac{\int_{U_2} (\Delta_\xi (u_\epsilon \circ \psi^{-1}))^2 \,\mathrm{d}x + C\|u_\epsilon \circ \psi^{-1}\|_2^2}{\left(\int_{U_2} |u_\epsilon \circ \psi^{-1}|^{2^\sharp} \,\mathrm{d}x\right)^{2/2^\sharp}}, \tag{16}$$

where $U_2 \subset \mathbb{R}^{\bar n}$. Since $u_\epsilon \circ \psi^{-1} = \bar u_\epsilon$, we deduce the claim by plugging (14) into (16). □

2. Proof of Theorem 0.2

We proceed by contradiction and assume that (6) does not hold. In particular for any $\alpha > 0$,

$$\lambda_\alpha := \inf_{u\in H^2_G(M),u\not\equiv 0} \frac{\int_M \left((\Delta_g u)^2 + \alpha|\nabla u|_g^2 + \frac{\alpha^2}{4}u^2\right) dv_g}{\left(\int_M |u|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}}} < \left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}\right)^{-1}. \tag{17}$$

Since (8) holds, λ_α is attained by some positive $u_\alpha \in C^\infty_G(M)$ normalized by $\int_M u_\alpha^{2^\sharp} dv_g = 1$ which satisfies the equation

$$\left(\Delta_g + \frac{\alpha}{2}\right)^2 u_\alpha = \Delta_g^2 u_\alpha + \alpha\Delta_g u_\alpha + \frac{\alpha^2}{4}u_\alpha = \lambda_\alpha u_\alpha^{2^\sharp-1}. \tag{E_α}$$

The proof of Theorem 0.2 will rely on the study of the asymptotic behaviour of the u_α's. We will show that they concentrate around a k-dimensional orbit Gx_0 of minimum volume A. Passing to the quotient manifold $B_{Gx_0}(\delta)/G'$ using assumption (H) or (H′), we will deduce a contradiction.

Multiplying (E_α) by u_α and integrating over M, we see that (u_α) is bounded in $H^2(M)$ so that, up to a subsequence, $u_\alpha \to 0$ weakly in $H^2(M)$ and strongly in $H^1(M)$. Using then the inequality (4) for some $\epsilon > 0$, we get

$$\begin{aligned}\int_M (\Delta_g u_\alpha)^2 dv_g + \alpha\|\nabla u_\alpha\|_2^2 + \frac{\alpha^2}{4}\|u_\alpha\|_2^2 &= \lambda_\alpha\left(\int_M |u_\alpha|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}}\\ &\le \lambda_\alpha\left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}+\epsilon\right)\int_M(\Delta_g u_\alpha)^2 dv_g + C_\epsilon\lambda_\alpha\|u_\alpha\|_{H^1}^2.\end{aligned} \tag{18}$$

Since $u_\alpha \to 0$ in $H^1(M)$, $\liminf_{\alpha\to+\infty}\int_M(\Delta_g u_\alpha)^2 dv_g > 0$ (otherwise, using (4), we would get a contradiction with the normalization condition) and ϵ is arbitrary, we get $\liminf_{\alpha\to+\infty}\lambda_\alpha \ge K_0(\bar n)^{-1}A^{4/\bar n}$. Hence, with (17), we obtain

$$\lim_{\alpha\to+\infty}\lambda_\alpha = \left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}\right)^{-1}.$$

We then deduce from (18) that

$$\lim_{\alpha\to+\infty}\alpha\|\nabla u_\alpha\|_2^2 + \frac{\alpha^2}{4}\|u_\alpha\|_2^2 = 0. \tag{19}$$

In view of (17) and (19), we easily check that the same argument as the one used in [13] gives the existence of a k-dimensional orbit Gx_0 such that

$$\begin{aligned}&u_\alpha^{2^\sharp} dv_g \rightharpoonup \delta_{Gx_0}\\ &(\Delta_g u_\alpha)^2 dv_g \rightharpoonup \left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}\right)^{-1}\delta_{Gx_0}\end{aligned} \tag{20}$$

weakly in the sense of measure, where δ_{Gx_0} is the Dirac measure on Gx_0 defined by defined by $\delta_{Gx_0}(\phi) = \int_G \phi(\sigma x_0)\, dm(\sigma)$ for $\phi \in C(M)$, m being the Haar measure of G normalized by $m(G) = 1$. Let $x_\alpha \in M$ be such that $u(x_\alpha) = \|u_\alpha\|_\infty \to +\infty$. Then the x_α converge to some point of Gx_0, say x_0 to simplify the notation. For future use, let us also note that a slight adaptation of [13] yields the pointwise inequality

$$d_g(Gx_\alpha, Gx)^{\frac{\bar n-4}{2}} u_\alpha(x) \le C \tag{21}$$

which holds for any $x \in M$ and any α, the constant C being independent of α.

To prove that $|Gx_0| = A$, we proceed by contradiction assuming that $|Gx_0| > A$. Then, according to Faget [10, lemma 3], $|Gx| \ge B > A$ for every point $x \in M$ in a neighborhood $B_{Gx_0}(2\delta)$ of Gx_0. We fix a smooth cut-off function $\tilde\eta : [0, +\infty) \to [0, 1]$ with compact support in $[0, 2]$ and such that $\tilde\eta = 1$ in $[0, 1)$, and let $\eta = \tilde\eta(d_g(., x_0)/\delta) \in C^\infty_{c,G}(B_{Gx_0}(2\delta))$. Multiplying ($E_\alpha$) by $\eta^2 u_\alpha$ and integrating by parts, we get, using (19), that

$$\begin{aligned}\int_M \Delta_g u_\alpha \Delta_g(\eta^2 u_\alpha)\, dv_g + o(1) &= \lambda_\alpha\int_M (\eta u_\alpha)^2 u_\alpha^{2^\sharp-2} dv_g\\ &\le \lambda_\alpha\|\eta u_\alpha\|_{2^\sharp}^2\|u_\alpha\|_{2^\sharp}^{2^\sharp-2}\\ &\le \lambda_\alpha\|\eta u_\alpha\|_{2^\sharp}^2.\end{aligned}$$

Independently, by the Hölder inequality and in view of (19), we have

$$\begin{aligned}\int_M \Delta_g u_\alpha \Delta_g(\eta^2 u_\alpha)\,dv_g &= \int_M \left\{\eta^2(\Delta_g u_\alpha)^2 + u_\alpha \Delta_g u_\alpha \Delta_g(\eta^2) - 2\Delta_g u_\alpha (\nabla u_\alpha, \nabla \eta^2)_g\right\} dv_g + o(1)\\ &= \int_M \eta^2 (\Delta_g u_\alpha)^2\,dv_g + o(1)\\ &= \int_M (\Delta_g(\eta u_\alpha))^2\,dv_g + o(1).\end{aligned}$$

Concerning the second equality just note that

$$\begin{aligned}\int_M \Delta_g u_\alpha (\nabla u_\alpha, \nabla\eta^2)_g\,dv_g &= \int_M (\nabla u_\alpha, \nabla(\nabla u_\alpha, \nabla \eta^2)_g)_g\,dv_g\\ &= \int_M \nabla_i u_\alpha \nabla^i \nabla_j u_\alpha \nabla^j \eta^2\,dv_g + \int_M \nabla_i u_\alpha \nabla_j u_\alpha \nabla^i\nabla^j \eta^2\,dv_g\\ &= -\int_M \nabla_i u_\alpha \nabla^i u_\alpha \nabla_j \nabla^j \eta^2\,dv_g - \int_M \nabla_j \nabla_i u_\alpha \nabla^i u_\alpha \nabla^j \eta^2\,dv_g + O(\|u_\alpha\|_{H^1}^2)\end{aligned}$$

so that $\int_M \nabla^2 u_\alpha (\nabla u_\alpha, \nabla\eta^2)\,dv_g = O(\|u_\alpha\|_{H^1}^2)$ and then

$$\int_M \Delta_g u_\alpha (\nabla u_\alpha, \nabla \eta^2)_g\,dv_g = O(\|u_\alpha\|_{H^1}^2).$$

Hence

$$\int_M (\Delta_g(\eta u_\alpha))^2\,dv_g + o(1) \le \lambda_\alpha \|\eta u_\alpha\|_{2^\sharp}^2.$$

According to Theorem 0.1 and (17), we can write that, given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for any α,

$$\int_M (\Delta_g(\eta u_\alpha))^2\,dv_g + o(1) \le \left(\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}\right)^{-1}\left(\frac{K_0(\bar n)}{B^{\frac{4}{\bar n}}} + \epsilon\right)\int_M (\Delta_g(\eta u_\alpha))^2 dv_g + C_\epsilon o(1),$$

i.e.

$$\left(1 - \left(AB^{-1}\right)^{\frac{4}{\bar n}}\right)\int_M (\Delta_g(\eta u_\alpha))^2\,dv_g \le o(1) + O(\epsilon) + C_\epsilon o(1).$$

Since $B > A$ and $\epsilon > 0$ is arbitrary, we deduce that

$$\lim_{\alpha\to+\infty}\int_{B_{Gx_0}(\delta)} (\Delta_g u_\alpha)^2\,dv_g \le \lim_{\alpha\to+\infty}\int_M (\Delta_g(\eta u_\alpha))^2\,dv_g = 0$$

which contradicts (20).

2.1. *Proof under assumption* (H)

Since Gx_0 is a k-dimensional orbit of minimum volume A, we can consider, according to assumption (H), the quotient $\bar n$-manifold $N = B_{Gx_0}(3\delta)/G'$, where the positive number $\delta > 0$ and the closed subgroup G' of $\mathrm{Isom}_g(M)$ are given by (H). Using (H3) and (3), recalling the definition (5) of the metric $\tilde g$, we have

$$\begin{aligned}\left(\int_{B_{Gx_0}(3\delta)} (\eta u_\alpha)^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}} &= \left(\int_N (\bar\eta \bar u_\alpha)^{2^\sharp}\,\bar v\,dv_{\tilde g}\right)^{\frac{2}{2^\sharp}}\\ &= \left(\int_N (\bar\eta\bar u_\alpha)^{2^\sharp}\,\bar v^{-\frac{4}{\bar n-4}}\,dv_{\tilde g}\right)^{\frac{2}{2^\sharp}} \le A^{-\frac{4}{\bar n}}\left(\int_N (\bar\eta \bar u_\alpha)^{2^\sharp}\,dv_{\tilde g}\right)^{\frac{2}{2^\sharp}}\\ &\le \frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}\int_N (\Delta_{\tilde g}(\bar\eta\bar u_\alpha))^2\,dv_{\tilde g} + C\|\bar\eta\bar u_\alpha\|_{H^1(N)}^2, \qquad (22)\end{aligned}$$

where, as before, $\eta = \tilde\eta(d_g(.,x_0)/\delta) \in C^\infty_{c,G}(B_{Gx_0}(2\delta))$, and $\bar\eta \in C_c^\infty(B_{\bar x_0}(2\delta))$ is defined by the relation $\bar\eta \circ \Pi = \eta$, Π being the canonical surjection from $B_{Gx_0}(3\delta)$ into N. According to [11],

$$\Delta_{\tilde g}\bar u = \bar v^{\frac{2}{\bar n-4}}\Delta_{\tilde g}\bar u + \frac{\bar n-2}{\bar n-4}\bar v^{-\frac{\bar n-6}{\bar n-4}}(\nabla\bar u, \nabla\bar v)_{\tilde g}$$

for any $\bar{u} \in C^2(N)$. Hence

$$\int_N (\Delta_{\tilde{g}}(\bar{\eta}\bar{u}_\alpha))^2 \, dv_{\tilde{g}} = \int_N (\Delta_{\tilde{g}}(\bar{\eta}\bar{u}_\alpha))^2 \bar{v} \, dv_{\tilde{g}} + I_1 + I_2$$
$$= \int_{B_{Gx_0}(3\delta)} (\Delta_g(\eta u_\alpha))^2 \, dv_g + I_1 + I_2 \tag{23}$$

with

$$|I_1| = \left(\frac{\bar{n}-2}{\bar{n}-4}\right)^2 \int_N (\nabla(\bar{\eta}\bar{u}_\alpha), \nabla\bar{v})^2_{\tilde{g}} \, \bar{v}^{-2} \, dv_{\tilde{g}} = O\left(\int_N |\nabla(\bar{\eta}\bar{u}_\alpha)|^2_{\tilde{g}} \, dv_{\tilde{g}}\right), \tag{24}$$

and

$$I_2 = -2\frac{\bar{n}-2}{\bar{n}-4} \int_N \Delta_{\tilde{g}}(\bar{\eta}\bar{u}_\alpha)(\nabla(\bar{\eta}\bar{u}_\alpha), \nabla\bar{v})_{\tilde{g}} \bar{v}^{-\frac{\bar{n}-2}{\bar{n}-4}} \, dv_{\tilde{g}}$$
$$= -2\frac{\bar{n}-2}{\bar{n}-4} \int_N \Delta_{\tilde{g}}(\bar{\eta}\bar{u}_\alpha)(\nabla(\bar{\eta}\bar{u}_\alpha), \nabla\bar{v})_{\tilde{g}} \, dv_{\tilde{g}}.$$
$$= -2\frac{\bar{n}-2}{\bar{n}-4} \int_N (\nabla(\bar{\eta}\bar{u}_\alpha), \nabla(\nabla(\bar{\eta}\bar{u}_\alpha), \nabla\bar{v})_{\tilde{g}})_{\tilde{g}} \, dv_{\tilde{g}}$$
$$= -2\frac{\bar{n}-2}{\bar{n}-4} \int_N \nabla^2(\bar{\eta}\bar{u}_\alpha)(\nabla(\bar{\eta}\bar{u}_\alpha), \nabla\bar{v}) \, dv_{\tilde{g}} + O\left(\int_N |\nabla(\bar{\eta}\bar{u}_\alpha)|^2 \, dv_{\tilde{g}}\right).$$

We have

$$\int_N \nabla^2(\bar{\eta}\bar{u}_\alpha)(\nabla(\bar{\eta}\bar{u}_\alpha), \nabla\bar{v}) \, dv_{\tilde{g}} = \int_N \nabla^i\nabla^j(\bar{\eta}\bar{u}_\alpha)\nabla_j(\bar{\eta}\bar{u}_\alpha)\nabla_i\bar{v} \, dv_{\tilde{g}}$$
$$= -\int_N \nabla^j(\bar{\eta}\bar{u}_\alpha)\nabla^i\nabla_j(\bar{\eta}\bar{u}_\alpha)\nabla_i\bar{v} \, dv_{\tilde{g}} - \int_N \nabla^j(\bar{\eta}\bar{u}_\alpha)\nabla_j(\bar{\eta}\bar{u}_\alpha)\nabla^i\nabla_i\bar{v} \, dv_{\tilde{g}}$$
$$= -\int_N \nabla^2(\bar{\eta}\bar{u}_\alpha)(\nabla(\bar{\eta}\bar{u}_\alpha), \nabla\bar{v}) \, dv_{\tilde{g}} + O\left(\int_N |\nabla(\bar{\eta}\bar{u}_\alpha)|^2_{\tilde{g}} \, dv_{\tilde{g}}\right).$$

Hence

$$\int_N \nabla^2(\bar{\eta}\bar{u}_\alpha)(\nabla(\bar{\eta}\bar{u}_\alpha), \nabla\bar{v}) \, dv_{\tilde{g}} = O\left(\int_N |\nabla(\bar{\eta}\bar{u}_\alpha)|^2_{\tilde{g}} \, dv_{\tilde{g}}\right),$$

and then

$$I_2 = O\left(\int_N |\nabla(\bar{\eta}\bar{u}_\alpha)|^2_{\tilde{g}} \, dv_{\tilde{g}}\right). \tag{25}$$

Inserting (23), (24), (25) into (22) we obtain

$$\left(\int_{B_{Gx_0}(\delta)} u_\alpha^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}} \le \left(\int_{B_{Gx_0}(3\delta)} (\eta u_\alpha)^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}}$$
$$\le \frac{K_0(\bar{n})}{A^{\frac{4}{\bar{n}}}} \int_{B_{Gx_0}(3\delta)} (\Delta_g(\eta u_\alpha))^2 \, dv_g + C\|\bar{\eta}\bar{u}_\alpha\|^2_{H^1(N)}$$
$$= \frac{K_0(\bar{n})}{A^{\frac{4}{\bar{n}}}} \int_{B_{Gx_0}(3\delta)} (\Delta_g(\eta u_\alpha))^2 \, dv_g + O\left(\|u_\alpha\|^2_{H^1}\right). \tag{26}$$

We have

$$\int_{B_{Gx_0}(3\delta)} (\Delta_g(\eta u_\alpha))^2 \, dv_g = \int_{B_{Gx_0}(3\delta)} \left(\eta\Delta_g u_\alpha + u_\alpha\Delta_g\eta - 2(\nabla u_\alpha, \nabla\eta)_g\right)^2 \, dv_g$$
$$= \int_M \eta^2(\Delta_g u_\alpha)^2 \, dv_g + O\left(\|u_\alpha\|^2_{H^1}\right) + \int_M \eta u_\alpha\Delta_g u_\alpha\Delta_g\eta \, dv_g - 2\int_M \eta\Delta_g u_\alpha(\nabla u_\alpha, \nabla\eta)_g \, dv_g$$
$$\le \int_M (\Delta_g u_\alpha)^2 \, dv_g + O\left(\|u_\alpha\|^2_{H^1}\right) + \int_M (\nabla u_\alpha, \nabla(\eta\Delta_g\eta u_\alpha))_g \, dv_g - \int_M \Delta_g u_\alpha(\nabla u_\alpha, \nabla\eta^2)_g \, dv_g. \tag{27}$$

We easily see that the second integral is $O\left(\|u_\alpha\|_{H^1}^2\right)$. Independently, in the same way as we treat I_2 above, we also get that the last integral is $O\left(\|u_\alpha\|_{H^1}^2\right)$. Hence

$$\int_{B_{Gx_0}(3\delta)} (\Delta_g(\eta u_\alpha))^2 \, dv_g \le \int_M (\Delta_g u_\alpha)^2 \, dv_g + O\left(\|u_\alpha\|_{H^1}^2\right).$$

Now, using (E_α) and (17), we obtain

$$\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}} \int_{B_{Gx_0}(3\delta)} (\Delta_g(\eta u_\alpha))^2 \, dv_g \le 1 - \frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}} \alpha \|u_\alpha\|_{H^1}^2 + O\left(\|u_\alpha\|_{H^1}^2\right). \tag{28}$$

Plugging (28) into (26) we get that for α large,

$$\frac{K_0(\bar n)}{A^{\frac{4}{\bar n}}}\alpha \le \frac{1 - \left(\int_{B_{Gx_0}(\delta)} u_\alpha^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}}}{\|u_\alpha\|_{H^1}^2} + O(1).$$

We claim that the quotient of the right-hand side is bounded, from which we get a contradiction. Since $2/2^\sharp \le 1$ and $\int_{B_{Gx_0}(\delta)} u_\alpha^{2^\sharp} \, dv_g \le 1$, we first write that

$$\begin{aligned} 1 - \left(\int_{B_{Gx_0}(\delta)} u_\alpha^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}} &\le \int_{M \setminus B_{Gx_0}(\delta)} u_\alpha^{2^\sharp} \, dv_g \\ &\le \sup_{M \setminus B_{Gx_0}(\delta)} u_\alpha^{2^\sharp - 2} \int_M u_\alpha^2 \, dv_g. \end{aligned} \tag{29}$$

We now prove that

$$u_\alpha \to 0 \quad \text{in } C^0_{loc}(M \setminus B_{Gx_0}(\delta)), \tag{30}$$

which in particular implies our claim. Letting $v_\alpha = \left(\Delta_g + \frac{\alpha}{2}\right) u_\alpha \in C^{2,\eta}(M)$, we can rewrite (E_α) as $\left(\Delta_g + \frac{\alpha}{2}\right) v_\alpha = \lambda_\alpha u_\alpha^{2^*-1} \ge 0$. Multiplying this inequality by $v_\alpha^- := \max\{-v_\alpha, 0\}$, we get that $v_\alpha \ge 0$. Hence $\Delta_g v_\alpha \le \lambda_\alpha u_\alpha^{2^*-1}$. It follows from (21) that the sequence (u_α) is bounded in $C^0_{loc}(M \setminus B_{Gx_0}(\delta))$. In particular the right-member of the last inequality is bounded in some $L^q(M)$ with $q > 2^*$ (in fact in any $L^q(M)$). The convergence in (30) then follows from the De Giorgi–Nash–Moser iteration scheme (see e.g. [18]) and the convergence of the u_α's to 0 in $L^2(M)$ (see (19)). As said above, this proves the claim and ends the proof of the theorem under assumption (H).

2.2. *Proof under assumption* (H′)

According to (3),

$$\left(\int_N (\bar\eta \bar u_\alpha)^{2^\sharp} \, dv_{\bar g}\right)^{\frac{2}{2^*}} \le K_0(\bar n) \int_N (\Delta_{\bar g}(\bar\eta \bar u_\alpha))^2 \, dv_{\bar g} + O\left(\|\bar\eta \bar u_\alpha\|_{H^1(N)}^2\right). \tag{31}$$

Since we assume that $\bar x_0$ is a critical point of $\bar v$, with $\bar v(\bar x_0) = A$, we have $|\bar v(\bar x) - A| = O(d_{\bar g}(\bar x, \bar x_0)^2)$, which can be written as

$$A^{-1}\bar v(\bar x) - C d_{\bar g}(\bar x, \bar x_0)^2 \le 1 \le A^{-1}\bar v(\bar x) + C d_{\bar g}(\bar x, \bar x_0)^2.$$

Since $\int_M (\eta u_\alpha)^{2^\sharp} \, dv_{\bar g} \to 1 > 0$ and $2/2^\sharp < 1$, so that $(1-x)^{\frac{2}{2^\sharp}} \ge 1 - x$ for all $0 \le x \le 1$, we have for δ small enough that

$$\begin{aligned} \left(\int_N (\bar\eta \bar u_\alpha)^{2^\sharp} \, dv_{\bar g}\right)^{\frac{2}{2^\sharp}} &\ge \left(A^{-1} \int_{B_{Gx_0}(3\delta)} (\eta u_\alpha)^{2^\sharp} \, dv_g - C \int_N (\bar\eta \bar u_\alpha)^{2^\sharp} d_{\bar g}(\bar x, \bar x_0)^2 \, dv_{\bar g}\right)^{\frac{2}{2^\sharp}} \\ &= A^{-\frac{2}{2^\sharp}} \left(\int_{B_{Gx_0}(\delta)} u_\alpha^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}} \left(1 - C \int_N (\bar\eta \bar u_\alpha)^{2^\sharp} d_{\bar g}(\bar x, \bar x_0)^2 \, dv_{\bar g}\right)^{\frac{2}{2^\sharp}} \\ &\ge A^{-\frac{2}{2^\sharp}} \left(\int_{B_{Gx_0}(\delta)} u_\alpha^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}} - C \int_N (\bar\eta \bar u_\alpha)^{2^\sharp} d_{\bar g}(\bar x, \bar x_0)^2 \, dv_{\bar g}. \end{aligned} \tag{32}$$

On the other hand, in view of (28) and doing as in (27),

$$K_0(\bar n)\int_N(\Delta_{\bar g}(\bar\eta\bar u_\alpha))^2\,dv_{\bar g}\le K_0(\bar n)A^{-1}\int_{B_{Gx_0}(3\delta)}(\Delta_g(\eta u_\alpha))^2\,dv_g+C\int_N(\Delta_{\bar g}(\bar\eta\bar u_\alpha))^2 d_{\bar g}(\bar x,\bar x_0)^2\,dv_{\bar g}$$
$$\le A^{-\frac{2}{2^\sharp}}+(O(1)-K_0(\bar n)A^{-1}\alpha)\|u_\alpha\|_{H^1}^2+C\int_N(\Delta_{\bar g}\bar u_\alpha)^2\bar\eta^2 d_{\bar g}(\bar x,\bar x_0)^2\,dv_{\bar g}. \tag{33}$$

Inserting (32) and (33) into (31) yields

$$K_0(\bar n)A^{-1}\alpha\le O(1)+A^{-\frac{2}{2^\sharp}}\frac{1-\left(\int_{B_{Gx_0}(\delta)}u_\alpha^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}}}{\|u_\alpha\|_{H^1}^2}+C\frac{\int_N(\bar\eta\bar u_\alpha)^{2^\sharp}d_{\bar g}(\bar x,\bar x_0)^2\,dv_{\bar g}}{\|u_\alpha\|_{H^1}^2}+C\frac{\int_N(\Delta_{\bar g}\bar u_\alpha)^2 d_{\bar g}(\bar x,\bar x_0)^2\,dv_{\bar g}}{\|u_\alpha\|_{H^1}^2}.$$

In view of (29) and (30), the first integral in the right-member is bounded. Independently, writing $d_{\bar g}(\bar x,\bar x_0)^2\le 2d_{\bar g}(\bar x,\bar x_\alpha)^2+2d_{\bar g}(\bar x_0,\bar x_\alpha)^2$, we obtain

$$K_0(\bar n)A^{-1}\alpha\le O(1)+C\frac{\int_N\bar\eta^2\bar u_\alpha^{2^\sharp}d_{\bar g}(\bar x,\bar x_\alpha)^2\,dv_{\bar g}}{\|u_\alpha\|_{H^1}^2}+C\frac{\int_N\bar\eta^2\bar u_\alpha^{2^\sharp}d_{\bar g}(\bar x_0,\bar x_\alpha)^2\,dv_{\bar g}}{\|u_\alpha\|_{H^1}^2}$$
$$+C\frac{\int_N(\Delta_{\bar g}\bar u_\alpha)^2\bar\eta^2 d_{\bar g}(\bar x,\bar x_\alpha)^2\,dv_{\bar g}}{\|u_\alpha\|_{H^1}^2}+C\frac{\int_N(\Delta_{\bar g}\bar u_\alpha)^2\bar\eta^2 d_{\bar g}(\bar x_0,\bar x_\alpha)^2\,dv_{\bar g}}{\|u_\alpha\|_{H^1}^2}. \tag{34}$$

To deal with the first integral, we write using (21) that

$$\bar\eta^2\bar u_\alpha^{2^\sharp}d_{\bar g}(\bar x,\bar x_\alpha)^2=(d_{\bar g}(\bar x,\bar x_\alpha)\bar u_\alpha^{\frac{2}{\bar n-4}})(\bar\eta\bar u_\alpha)(\bar\eta d_{\bar g}(\bar x,\bar x_\alpha)\bar u_\alpha^{\frac{\bar n}{\bar n-4}})\bar u_\alpha^{\frac{2}{\bar n-4}}$$
$$\le C(\bar\eta\bar u_\alpha)(\bar\eta d_{\bar g}(\bar x,\bar x_\alpha)\bar u_\alpha^{\frac{\bar n}{\bar n-4}})\bar u_\alpha^{\frac{2}{\bar n-4}}$$

and apply the Hölder inequality with $\frac{\bar n-2}{2(\bar n)}+\frac12+\frac1{\bar n}=1$ to get

$$\int_N\bar\eta^2\bar u_\alpha^{2^\sharp}d_{\bar g}(\bar x,\bar x_\alpha)^2\,dv_{\bar g}\le C\|\bar\eta\bar u_\alpha\|_{2^*}\|\bar u_\alpha\|_{2^\sharp}^{\frac{2}{\bar n-4}}\left(\int_N\bar\eta^2\bar u_\alpha^{2^\sharp}d_{\bar g}(\bar x,\bar x_\alpha)^2\,dv_{\bar g}\right)^{\frac12}$$
$$\le 0(\|u_\alpha\|_{H^1})\left(\int_N\bar\eta^2\bar u_\alpha^{2^\sharp}d_{\bar g}(\bar x,\bar x_\alpha)^2\,dv_{\bar g}\right)^{\frac12},$$

where $2^*=2(\bar n)/(\bar n-2)$, and we used the fact the embedding $H^1(N)\hookrightarrow L^{2^*}(N)$ is continuous. It follows that the first integral in (34) is bounded. We treat the second one in a similar way using, instead of (21), the inequality

$$\|\bar u_\alpha\|_\infty d_{\bar g}(\bar x_\alpha,\bar x_0)^{\frac{\bar n-4}{2}}\le C$$

proved in [13] following Faget's idea.

Concerning the third integral, we put $u_\alpha\eta^2 d_g(Gx_\alpha,Gx)^2$ as a test-function in (E_α) and pass to the quotient to get

$$\int_N(\Delta_{\bar g}\bar u_\alpha)^2\bar\eta^2 d_{\bar g}(\bar x,\bar x_\alpha)^2\bar v\,dv_{\bar g}=\lambda_\alpha\int_N\bar u_\alpha^{2^\sharp}\bar r_\alpha\bar v\,dv_{\bar g}-\frac{\alpha^2}{4}\int_N\bar u_\alpha^2\bar r_\alpha\bar v\,dv_{\bar g}-\alpha\int_N(\nabla\bar u_\alpha,\nabla(\bar u_\alpha\bar r_\alpha))_{\bar g}\bar v\,dv_{\bar g}$$
$$-\int_N\bar u_\alpha\Delta_{\bar g}\bar u_\alpha\Delta_{\bar g}\bar r_\alpha\bar v\,dv_{\bar g}+2\int_N\Delta_{\bar g}\bar u_\alpha(\nabla\bar u_\alpha,\nabla\bar r_\alpha)_{\bar g}\bar v\,dv_{\bar g},$$

where we let $\bar r_\alpha(\bar x)=\bar\eta^2 d_{\bar g}(\bar x,\bar x_\alpha)^2$. The first integral on the right-hand side is exactly the first one in (34) that we just bounded by $O(\|u_\alpha\|_{H^1}^2)$. Independently, integrating by parts in the fourth integral and then applying the Hölder inequality, we get

$$\int_N\bar u_\alpha\Delta_{\bar g}\bar u_\alpha\Delta_{\bar g}\bar r_\alpha\bar v\,dv_{\bar g}=\int_N|\nabla\bar u_\alpha|_{\bar g}^2\Delta_{\bar g}\bar r_\alpha\bar v\,dv_{\bar g}+\int_N\bar u_\alpha(\nabla\bar u_\alpha,\nabla(\Delta_{\bar g}\bar r_\alpha\bar v))_{\bar g}\,dv_{\bar g}$$
$$=O(\|u_\alpha\|_{H^1}^2).$$

Eventually, integrating by parts in the last integral we get

$$\int_N\Delta_{\bar g}\bar u_\alpha(\nabla\bar u_\alpha,\nabla\bar r_\alpha)_{\bar g}\bar v\,dv_{\bar g}=\int_N\nabla^2\bar u_\alpha(\nabla\bar u_\alpha,\nabla\bar r_\alpha)_{\bar g}\bar v\,dv_{\bar g}+\int_N(\nabla\bar u_\alpha,\nabla\bar v)_{\bar g}(\nabla\bar u_\alpha,\nabla\bar r_\alpha)_{\bar g}\,dv_{\bar g}$$
$$+\int_N\nabla^2\bar r_\alpha(\nabla\bar u_\alpha,\nabla\bar u_\alpha)_{\bar g}\bar v\,dv_{\bar g}$$
$$=\int_N\nabla^2\bar u_\alpha(\nabla\bar u_\alpha,\nabla\bar r_\alpha)_{\bar g}\bar v\,dv_{\bar g}+O(\|u_\alpha\|_{H^1}^2).$$

Integrating again by parts in the last integral we obtain

$$\begin{aligned}\int_N \nabla^2 \bar{u}_\alpha (\nabla \bar{u}_\alpha, \nabla \bar{r}_\alpha)_{\bar{g}} \bar{v}\, dv_{\bar{g}} &= -\int_N \nabla^2 \bar{u}_\alpha (\nabla \bar{u}_\alpha, \nabla \bar{r}_\alpha)_{\bar{g}} \bar{v}\, dv_{\bar{g}} - \int_N \nabla^2 \bar{r}_\alpha (\nabla \bar{u}_\alpha, \nabla \bar{u}_\alpha)_{\bar{g}} \bar{v}\, dv_{\bar{g}} \\ &\quad - \int_N (\nabla \bar{u}_\alpha, \nabla \bar{v})_{\bar{g}} (\nabla \bar{u}_\alpha, \nabla \bar{r}_\alpha)_{\bar{g}}\, dv_{\bar{g}} \\ &= -\int_N \nabla^2 \bar{u}_\alpha (\nabla \bar{u}_\alpha, \nabla \bar{r}_\alpha)_{\bar{g}} \bar{v}\, dv_{\bar{g}} + O(\|u_\alpha\|^2_{H^1}),\end{aligned}$$

so that

$$\int_N \Delta_{\bar{g}} \bar{u}_\alpha (\nabla \bar{u}_\alpha, \nabla \bar{v}_\alpha 2)_{\bar{g}} \bar{v}\, dv_{\bar{g}} = O(\|u_\alpha\|^2_{H^1}).$$

Hence

$$\int_N (\Delta_{\bar{g}} \bar{u}_\alpha)^2 \bar{\eta}^2 d_{\bar{g}}(\bar{x}, \bar{x}_\alpha)^2 \bar{v}\, dv_{\bar{g}} \le -\alpha \int_N \bar{u}_\alpha (\nabla \bar{u}_\alpha, \nabla(\bar{\eta}^2 d_{\bar{g}}(\bar{x}, \bar{x}_\alpha)^2))_{\bar{g}} \bar{v}\, dv_{\bar{g}} + O(\|u_\alpha\|^2_{H^1}).$$

As $d_{\bar{g}}(\bar{x}, \bar{x}_\alpha) \le 6\delta$ for any $\bar{x} \in N$ and any α, we have

$$\begin{aligned}\int_N \bar{u}_\alpha (\nabla \bar{u}_\alpha, \nabla(\bar{\eta}^2 d_{\bar{g}}(\bar{x}, \bar{x}_\alpha)^2))_{\bar{g}} \bar{v}\, dv_{\bar{g}} &= \int_N \bar{u}_\alpha (\nabla \bar{u}_\alpha, \nabla \bar{\eta}^2)_{\bar{g}} d_{\bar{g}}(\bar{x}, \bar{x}_\alpha)^2 \bar{v}\, dv_{\bar{g}} \\ &\quad + 2\int_N d_{\bar{g}}(\bar{x}, \bar{x}_\alpha) \bar{u}_\alpha (\nabla \bar{u}_\alpha, \nabla d_{\bar{g}}(\bar{x}, \bar{x}_\alpha))_{\bar{g}} \bar{\eta}^2 \bar{v}\, dv_{\bar{g}} \\ &= \delta O(\|u_\alpha\|^2_{H^1}),\end{aligned}$$

so that

$$\int_N (\Delta_{\bar{g}} \bar{u}_\alpha)^2 \bar{\eta}^2 d_{\bar{g}}(\bar{x}, \bar{x}_\alpha)^2 \bar{v}\, dv_{\bar{g}} \le (\alpha\delta + 1) O(\|u_\alpha\|^2_{H^1}).$$

We deal in a similar way with the fourth integral.

Choosing δ small enough we can thus rewrite (34) as $\alpha \le O(1)$ which is the desired contradiction.

3. Proof of Theorem 0.3

We take as test-functions to estimate the left-hand side in (8) the functions u_ϵ defined by (15). Since we assumed (H1) and (H2) we can consider the quotient manifold $N = B_{Gx_0}/G'$. We also let

$$\theta_\epsilon = \begin{cases} \epsilon^{8-\bar{n}} & \text{if } \bar{n} \ge 9 \\ |\ln \epsilon| & \text{if } \bar{n} = 8 \\ 1 & \text{if } \bar{n} = 6, 7. \end{cases}$$

According to [12], we have that

$$\int_M a u_\epsilon^2\, dv_g = \int_N \bar{a} \bar{u}_\epsilon^2 \bar{v}\, dv_{\bar{g}} = O(\theta_\epsilon)$$

when $\bar{n} \ge 6$,

$$\begin{aligned}\int_M b^\sharp(du_\epsilon, du_\epsilon)\, dv_g &= \int_N \bar{b}^\sharp(d\bar{u}_\epsilon, d\bar{u}_\epsilon) \bar{v}\, dv_{\bar{g}} \\ &= \begin{cases} \dfrac{4(\bar{n}-1)(\bar{n}-4)\omega_{\bar{n}}}{2^{\bar{n}}(\bar{n}-6)} \bar{v}(\bar{x}_0) Tr_{\bar{g}} \bar{b}(\bar{x}_0) \epsilon^{6-\bar{n}} & \text{if } \bar{n} \ge 7 \\ \dfrac{(\bar{n}-4)^2 \omega_{\bar{n}-1}}{\bar{n}} \bar{v}(\bar{x}_0) Tr_{\bar{g}} \bar{b}(\bar{x}_0) |\ln \epsilon| + O(1) & \text{if } \bar{n} \ge 6, \end{cases}\end{aligned}$$

and, for $\bar{n} \ge 5$,

$$\begin{aligned}\int_M f u_\epsilon^{2^\sharp}\, dv_g &= \int_N \bar{v} \bar{f} \bar{u}_\epsilon^{2^\sharp}\, dv_{\bar{g}} \\ &= \frac{\bar{v}(\bar{x}_0) \bar{f}(\bar{x}_0) \omega_{\bar{n}}}{2^{\bar{n}}} \epsilon^{-\bar{n}} - \frac{\omega_{\bar{n}}}{6(\bar{n}-2)2^{\bar{n}}} (S_{\bar{g}}(\bar{x}_0) \bar{v}(\bar{x}_0) \bar{f}(\bar{x}_0) + 3\Delta_{\bar{g}}(\bar{v}\bar{f})(\bar{x}_0)) \epsilon^{2-\bar{n}} + O(\epsilon^{4-\bar{n}})\end{aligned}$$

with

$$\begin{aligned}\Delta_{\bar g}(\bar v\bar f)(\bar x_0) &= \bar f(\bar x_0)\Delta_{\bar g}\bar v(\bar x_0)+\bar v(\bar x_0)(\bar x_0)\Delta_{\bar g}\bar f(\bar x_0)\\ &= f(x_0)\Delta_{\bar g}\bar v(\bar x_0)+A\Delta_g f(x_0)\end{aligned}$$

since $\Delta_g f(\bar x_0)=\Delta_{\bar g}\bar f(\bar x_0)-\frac{(\nabla\bar v,\nabla\bar f)_{\bar g}(\bar x_0)}{\bar v(x_0)}=\Delta_{\bar g}\bar f(\bar x_0)$ as it follows by integrating by parts and then passing to the quotient in $\int_M\Delta_g f\phi\,dv_g$ where ϕ is a smooth G-invariant function with compact support in a small neighborhood of Gx_0.

It remains to estimate $\int_M(\Delta_g u_\epsilon)^2\,dv_g$. Mimicking [12] we get

$$\int_M(\Delta_g u_\epsilon)^2\,dv_g=\int_N(\Delta_{\bar g}\bar u_\epsilon)^2\bar v\,dv_{\bar g}=\frac{\bar n(\bar n-4)(\bar n^2-4)\omega_{\bar n}}{2^{\bar n}}A\epsilon^{4-\bar n}$$
$$-\begin{cases}\epsilon^{6-\bar n}\left(\dfrac{\bar n(\bar n^2+4\bar n-20)(\bar n-4)\omega_{\bar n}}{6(\bar n-6)2^{\bar n}}AS_{\bar g}(\bar x_0)+\dfrac{(\bar n-4)^2\omega_{\bar n-1}}{2\bar n}\Delta_{\bar g}\bar v(\bar x_0)I\right)+O(\theta_\epsilon), & \bar n\ge 7\\ |\ln\epsilon|\dfrac{2(\bar n-4)^2\omega_{\bar n-1}}{\bar n}(AS_{\bar g}(\bar x_0)+\Delta_{\bar g}\bar v(\bar x_0))+o(|\ln\epsilon|), & \bar n=6,\end{cases}$$

where $I=\int_0^\infty\frac{s^{\bar n+1}(\bar n+2s^2)^2}{(1+s^2)^{\bar n}}\,ds$. Since

$$\int_0^\infty\frac{s^\alpha}{(1+s^2)^\beta}\,ds=\frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{2\beta-\alpha-1}{2}\right)}{2\Gamma(\beta)}$$

when $2\beta-\alpha>1$, and $2\Gamma(\bar n)=2^{\bar n}\pi^{-1/2}\Gamma(\bar n/2)\Gamma\left(\frac{\bar n+1}{2}\right)$ (a particular case of the duplication formula), we have

$$I=\frac{\bar n(\bar n^2+4)}{2^{\bar n}(\bar n-6)}\frac{\omega_{\bar n}}{\omega_{\bar n-1}}.$$

Hence

$$\int_M(\Delta_g u_\epsilon)^2\,dv_g=\frac{\bar n(\bar n-4)(\bar n^2-4)\omega_{\bar n}}{2^{\bar n}}A\epsilon^{4-\bar n}$$
$$-\begin{cases}\epsilon^{6-\bar n}\dfrac{(\bar n-4)\omega_{\bar n}}{(\bar n-6)2^{\bar n}}\left(\dfrac16\bar n(\bar n^2+4\bar n-20)AS_{\bar g}(\bar x_0)+\dfrac12(\bar n-4)(\bar n^2+4)\Delta_{\bar g}\bar v(\bar x_0)\right)+O(\theta_\epsilon), & \bar n\ge 7\\ |\ln\epsilon|\dfrac{2(\bar n-4)^2\omega_{\bar n-1}}{\bar n}(AS_{\bar g}(\bar x_0)+\Delta_{\bar g}\bar v(\bar x_0))+o(|\ln\epsilon|), & \bar n=6.\end{cases}$$

We eventually obtain

$$\frac{\int_M(P_g u_\epsilon)u_\epsilon\,dv_g}{\left(\int_M fu_\epsilon^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}}}=\left(\frac{K_0(\bar n)}{A^{\frac4{\bar n}}}\right)^{-1}\|f\|_\infty^{-2/2^\sharp}$$
$$\times\begin{cases}\left(1+\dfrac{F}{2\bar n(\bar n^2-4)(\bar n-6)}\epsilon^2+o(\epsilon^2)\right) & \text{if } \bar n\ge 7,\\ \left(1+\dfrac{2^{\bar n}\omega_{\bar n-1}(\bar n-4)}{\omega_{\bar n}\bar n^2(\bar n^2-4)}\left(Tr_{\bar g}\bar b(\bar x_0)-2S_{\bar g}(\bar x_0)-2\dfrac{\Delta_{\bar g}\bar v(\bar x_0)}{A}+o(1)\right)\epsilon^2|\ln\epsilon|\right), & \text{if } \bar n=6,\end{cases}$$

where F is the left-hand side in (9). This proves the theorem.

As a final remark we note that if (8) holds then the infimum on the right-hand side of (8) is attained by some nonnegative $u\in H_G^2(M)$ which is a solution of (7) in the sense that

$$\int_M\left(\Delta_g u\Delta_g\phi+b^\sharp(du,d\phi)_g+au\phi\right)dv_g=\int_M fu^{2^\sharp-1}\phi\,dv_g \tag{35}$$

for any $\phi\in H_G^2(M)$. Now if $\phi\in H^2(M)$, then the function ϕ_G defined by

$$\phi_G(x)=\int_G\phi(\sigma(x))\,dm(\sigma),$$

where m is the Haar measure of G normalized by $m(G)=1$, belongs to $H_G^2(M)$. Writing (35) with ϕ_G as test-function, we get that (35) holds with ϕ. Indeed

$$\begin{aligned}\int_M\Delta_g u\Delta_g\phi_G\,dv_g &= \int_G\int_M\Delta_g u(x)\Delta_g\phi(\sigma(x))\,dv_g dm(\sigma)\\ &= \int_G\int_M\Delta_g u(\sigma^{-1}(x))\Delta_g\phi(x)\,dv_g dm(\sigma)\end{aligned}$$

$$= m(G) \int_M \Delta_g u \Delta_g \phi \, dv_g$$
$$= \int_M \Delta_g u \Delta_g \phi \, dv_g.$$

The other terms are treated similarly. We thus get that u is a solution of (7) in $H^2(M)$.

4. Proof of Theorem 0.4

The proof follows closely the line of [16]. We briefly sketch it for the reader's convenience and refer to [16] for the details. Since $a > 0$, $b \geq 0$, $\|u\|^2 := \int_M (P_g u) u \, dv_g = \int_M (\Delta_g u)^2 + b|\nabla u|_g^2 + au^2 \, dv_g$ is a norm in $H^2(M)$. Moreover as $f > 0$, $\|u\|_{2^\sharp}^{2^\sharp} := \int_M f|u|^{2^\sharp} \, dv_g$ is a norm on $L^{2^\sharp}(M)$. We let J be the functional associated to (7) given by

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^\sharp} \int_M f|u|^{2^\sharp} \, dv_g.$$

As $k = \min_{x \in M} \dim Gx \geq 1$ by hypothesis, the embedding of $H_G^2(M)$ into $L^{2^\sharp}(M)$ is compact, so that J restricted to $H_G^2(M)$ satisfies the Palais–Smale condition. We can then apply Theorem 2.13 of Ambrosetti and Rabinowitz [19] (the other hypotheses of this theorem are easily seen to be satisfied by J) to get the existence of an increasing sequence $(\alpha_m)_m$ of critical values for J restricted to $H_G^2(M)$ given by a minimax formulation. Using once again the compactness of $H_G^2(M)$ into $L^{2^\sharp}(M)$, we can prove that $\lim_{m \to +\infty} \alpha_m = +\infty$. Let u_m be a critical point of J restricted to $H_G^2(M)$ corresponding to α_m. Then the u_m are distinct, $\int_M |u_m|^{2^\sharp} \, dv_g \to +\infty$, and the u_m's are solutions of (7) in the sense that (35) holds for any $\phi \in H_G^2(M)$. As in the proof of Theorem 0.3, we get that (35) holds indeed for any $\phi \in H^2(M)$. This ends the proof of the theorem.

5. Proof of Theorem 0.5

The first part of the theorem can be proved using Lions' concentration-compactness principle as before. Now to prove that u is positive (resp. negative) in M_+ (resp. M_-) we write, according to the hypothesis made on b, a, f, that

$$P_g u = (\Delta_g + \beta_1)(\Delta_g + \beta_2) u = f|u|^{2^\sharp - 2} u \quad \text{in } M$$

for some $\beta_1, \beta_2 > 0$. We let $v = (\Delta_g + \beta_2) u$. Let $x \in S_\tau$ and $x_n \in M_+$ such that $x_n \to x$. Then, since $\tau \in \mathrm{Isom}_g(M)$ and $\tau^2 = Id$,

$$d_g(\tau x_n, x) = d_g(x_n, \tau x) = d_g(x_n, x) \to 0.$$

As a consequence $u(x) = 0$ for any $x \in S_\tau$. Moreover since u is τ-antisymmetric,

$$(\Delta_g u) \circ \tau = \Delta_g (u \circ \tau) = -\Delta_g u,$$

so that

$$\Delta_g u(x) = \lim_{n \to +\infty} (\Delta_g u)(\tau x_n) = -\lim_{n \to +\infty} \Delta_g u(x_n) = -\Delta_g u(x),$$

and thus $\Delta_g u = 0$ on S_τ. We thus get that $v = 0$ on S_τ. Hence

$$(\Delta_g + \beta_1) v = f|u|^{2^\sharp - 2} u \geq 0 \quad \text{in } M_+,$$
$$v = 0 \quad \text{on } S_\tau = \partial M_+,$$

which implies that $v > 0$ in M_+. Then from

$$(\Delta_g + \beta_2) u = v > 0 \quad \text{in } M_+,$$
$$u = 0 \quad \text{on } S_\tau = \partial M_+,$$

we get that $u > 0$ in M_+. Arguing in the same way in M_-, we obtain that $u < 0$ in M_-.

Let Gx_0 be a k-dimensional orbit of minimum volume A. We can assume without loss of generality that $x_0 \in M_+$. Then $Gx_0 \subset M_+$. Consider $u_\epsilon \in C_{c,G}^\infty(B_{Gx}(2\delta))$ be defined by (15) with 2δ less than the injectivity radius of M and less than $d_g(Gx_0, S_\tau)$. We τ-antisymmetrize u_ϵ by considering $u_{\tau,\epsilon} \in H_{G,\tau}^2(M)$ defined by

$$u_{\tau,\epsilon} = \begin{cases} u_\epsilon & \text{in } M_+ \\ -u_\epsilon \circ \tau & \text{in } M_-. \end{cases}$$

Then in view of the computations made before,

$$\begin{aligned}\frac{\int_M (P_g u_\epsilon)u_\epsilon \, dv_g}{\left(\int_M f|u_{\tau,\epsilon}|^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}}} &= \frac{\int_{M_+} (P_g u_\epsilon)u_\epsilon \, dv_g + \int_{M_-} (P_g(u_\epsilon \circ \tau))(u_\epsilon \circ \tau) \, dv_g}{\left(\int_{M_+} f|u_\epsilon|^{2^\sharp} \, dv_g + \int_{M_-} f|u_\epsilon \circ \tau|^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}}} \\ &= 2^{\frac{4}{\bar{n}}} \frac{\int_{M_+} (P_g u_\epsilon)u_\epsilon \, dv_g}{\left(\int_{M_+} f u_\epsilon^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}}} \\ &= \left(\frac{K_0(\bar{n})}{(2A)^{\frac{4}{\bar{n}}}}\right)^{-1} \|f\|_\infty^{-2/2^\sharp} \begin{cases} \left(1 + \dfrac{F}{2\bar{n}(\bar{n}^2-4)(\bar{n}-6)}\epsilon^2 + o(\epsilon^2)\right) & \text{if } \bar{n} \geq 7, \\ \left(1 + \dfrac{2^{\bar{n}}\omega_{\bar{n}-1}(\bar{n}-4)}{\omega_{\bar{n}}\bar{n}^2(\bar{n}^2-4)} \left(Tr_{\bar{g}}\bar{A}(\bar{x}_0) - 2S_{\bar{g}}(\bar{x}_0) \right.\right. & \\ \left.\left. \quad - 2\dfrac{\Delta_{\bar{g}}\bar{v}(\bar{x}_0)}{A} + o(1)\right)\epsilon^2|\ln\epsilon|\right), & \text{if } \bar{n} = 6, \end{cases}\end{aligned}$$

where F is the left-hand side in (9). This proves the theorem.

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