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Calculus of Variations

Asymptotic in Sobolev spaces for symmetric Paneitz-type equations on Riemannian manifolds

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Abstract We describe the asymptotic behaviour in Sobolev spaces of sequences of solutions of Paneitz-type equations [Eq. (E_α) below] on a compact Riemannian manifold (M, g) which are invariant by a subgroup of the group of isometries of (M, g). We also prove pointwise estimates.

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1 Introduction

Fourth-order equations of critical Sobolev growth have been an intensive target of investigations in the last years, particularly because of the applications of the fourth-order Paneitz operator to conformal geometry (see, e.g. [2] or [3] for a survey), and also because of the parallel that exists between fourth-order equations of critical growth and their second-order analogues. Independently, we know from the work of Hebey–Vaugon [12] that symmetry allows us to get better Sobolev embeddings, i.e. the critical Sobolev exponent increases when considering functions having some symmetry. This fact has already been used in [18] to prove the existence of an infinity of non-equivalent solutions to a fourth-order critical equations in $\mathbb{R}^n$. These two facts leads naturally to the study of the asymptotic behavior of symmetric solutions to such equations.

We now describe precisely the problem we are interested in. Let (M, g) be a smooth compact Riemannian n-manifold and G a closed subgroup of the group of isometries $\mathrm{Isom}_g(M)$ of (M, g) such $n - k \geq 5$, where $k = \min_{x\in M}\ dim\ Gx$, and Gx denotes the orbit of a

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point $x \in M$ under the action of G. We say that a function $f : M \to \mathbb{R}$ is G-invariant if $f(gx) = f(x)$ for any $x \in M$ and $g \in G$. Note that if f is invariant under the action of an arbitrary subgroup G' of $\mathrm{Isom}_g(M)$ then it is also $\overline{G'}$-invariant, so that the closedness assumption on G is not restrictive. We consider equations like

$$\Delta_g^2 u + k_\alpha \Delta_g u + h_\alpha u = f u^{2^\sharp - 1}, \quad u > 0, \quad u \text{ G-invariant}, \tag{E_α}$$

where $\Delta_g^2 u = \Delta_g(\Delta_g u)$ is the Bilaplacian of u for g, $\Delta_g u = -\mathrm{div}_g(\nabla u)$ being the Laplacian of u, $2^\sharp = \frac{2(n-k)}{n-k-4}$ is the critical exponent for the embedding of the Sobolev space $H^2_{2,G}(M)$, consisting of the G-invariant functions $u \in L^2(M)$ such that $\nabla u, \nabla^2 u \in L^2(M)$, into the Lebesgue spaces $L^q(M)$ (in particular, the space $H^2_{2,G}(M)$ is continuously embedded in $L^{2^\sharp}(M)$: this assertion is a consequence of Hebey–Vaugon [12]), f is a C^1 G-invariant function, (k_α) is a sequence of real numbers converging to some k_∞, and (h_α) is a sequence of continuous G-invariant functions uniformly converging to some h_∞. We assume that the operator $\Delta_g^2 + k_\infty \Delta_g + h_\infty$ is coercive in the sense that there exists some $\lambda > 0$ such that for any $u \in H^2_{2,G}(M)$,

$$\int_M \left((\Delta_g u)^2 + k_\infty |\nabla u|_g^2 + h_\infty |u|^2 \right) dv_g \geq \lambda \|u\|^2_{H_2^2}. \tag{1}$$

When k_α and h_α are constant independent of α, we refer to Hebey–Robert [11] for a necessary and sufficient condition for (1) to hold. It is easily seen that a necessary condition for (E_α) to admit a positive solution u is $\max_M f > 0$. Indeed, multiplying (E_α) by u, integrating by parts and using the coercivity assumption (1) yields

$$\int_M f u^{p^*} dv_g \geq \lambda \|u\|^2_{H_2^2} + o(1).$$

We then deduce that f must be positive somewhere, and then $\max_M f > 0$. From now on, we assume that $\max_M f > 0$. We also consider the limit equation obtained by letting formally $\alpha \to +\infty$ in (E_α), namely

$$\Delta_g^2 u + k_\infty \Delta_g u + h_\infty u = f u^{2^\sharp - 1}. \tag{E_∞}$$

For each α, let u_α be a G-invariant weak positive solution of (E_α) and assume that the sequence (u_α) is bounded in $H_2^2(M)$. The purpose of this note is to describe the asymptotic behavior in H_2^2 of the u_α's. In the case where g_α and h_α are constant independent of α, $f = 1$ and G is reduced to identity, Hebey–Robert [11] solved the problem by showing that the u_α can be written as the sum of a solution of the limit equation (E_∞) plus a finite sum of bubbles plus a rest strongly converging to 0 in H_2^2. A bubble is a sequence of functions obtained by rescaling a positive solution of the Euclidean critical equation $\Delta_\xi^2 u = u^{q-1}$ in $\mathbb{R}^n$, $q = 2n/(n-4)$, where ξ is the Euclidean metric. We prove here (cf. theorem below) that this decomposition still holds in the context of G-invariant functions under some assumptions on the orbits of G [assumption (H) below] and with an extended notion of bubble. The same technique can be used to deal with critical equations involving only the Laplacian, generalizing thus Clapp'result [4] who considered such equations in a smooth bounded open subset of $\mathbb{R}^n$, with the standard Euclidean metric, invariant under the action of some subgroup of $O(n)$.

We now recall some known facts and fix some notations. We refer to Bredon [1] for more details (see also [7,12]). Let G' be a closed subgroup of $\mathrm{Isom}_g(M)$. Then G' is a Lie group. For each $x \in M$, we let $\bar{x} = \Pi(x)$, where $\Pi : M \to M/G'$ is the canonical surjection, and denote by $G'x = \{gx,\ g \in G'\}$ (resp. $S_x = \{g \in G',\ gx = x\}$) the orbit (resp. the stabilizator) of x under the action of G'. Then $G'x$ is a compact submanifold of M naturally isomorphic to the quotient group G'/S_x. An orbit $G'x$ is said principal if its stabilizator is minimal up to conjugacy, i.e. for all $y \in M$, S_y contains a subgroup conjugate to S_x. In particular, the principal orbits are of maximal dimension (but the converse is false). If we denote by Ω the union of all the principal orbits, then Ω is a dense open subset of M and Ω/G' is a smooth connected manifold which can be equipped with a Riemaniann metric $\bar{g}$ in such a way that the canonical surjection from Ω to Ω/G' is a Riemannian submersion. We then consider the metric $\tilde{g}$ belonging to the conformal class of $\bar{g}$ defined by

$$\tilde{g} = \bar{v}^{\frac{2}{n-k-4}} \bar{g}, \tag{2}$$

where $\bar{v}(\bar{x}) = Vol(\Pi^{-1}(\bar{x})) = Vol(G'x)$ denotes the volume of $G'x$ computed with respect to the induced metric. We will denote by $B_x^{\bar{g}}(r)$ and $B_x^{\tilde{g}}(r)$ the geodesic balls centered at x of radius r for the metric $\bar{g}$ and $\tilde{g}$, respectively. We let $H_1^2(M)$ [resp. $H_2^2(M)$] be the usual Sobolev spaces of the functions $u \in L^2(M)$ such that $\nabla u \in L^2(M)$ [resp. and $\nabla^2 u \in L^2(M)$] with the norm $\|u\|_{H_1^2}^2 = \|u\|_2^2 + \|\nabla u\|_2^2$ (resp. $\|u\|_{H_2^2}^2 = \|u\|_{H_1^2}^2 + \|\nabla^2 u\|_2^2$). It follows from the Bochner–Lichnerowicz–Weitzenböck formula that $H_2^2(M)$ can also be equipped with the equivalent norm $\|u\|_{H_2^2}^2 = \|\Delta_g u\|_2^2 + \|u\|_{H_1^2}^2$ (see [5]). We will always use this last norm in the sequel. We also consider the closure of $C^\infty(M)$ for the norm $\|.\|_{H_2^2}$ that we denote by $\overset{\circ}{H}{}_2^2(M)$. We let $H_{l,G'}^2(M)$, $l = 0, 1, 2$, and $\overset{\circ}{H}{}_{2,G'}^2(M)$ be the space of G'-invariant functions in $H_l^2(M)$ and $\overset{\circ}{H}{}_2^2(M)$, respectively:

$$H_{l,G'}^2(M) = \left\{u \in H_l^2(M) s.t. \forall\, g \in G',\ u(gx) = u(x)\ a.e.\ in\ M\right\},$$
$$\overset{\circ}{H}{}_{2,G'}^2(M) = \left\{u \in \overset{\circ}{H}{}_2^2(M)\ s.t.\ \forall\, g \in G',\ u(gx) = u(x)\ a.e.\ in\ M\right\}.$$

We let $k := \min_{x \in M}\ dim\, Gx$, and make the following assumption on the G-orbits of minimal dimension k:

(H) *for each G - orbit Gx_0 of minimal dimension k, there exist $\delta > 0$ and a closed normal subgroup G' of G such that*

$$G'x_0 = Gx_0 \tag{H1}$$

and, for all $x \in B_{Gx_0}(\delta) := \{y \in M,\ d_g(y, Gx_0) < \delta\}$,

$$G'x \quad is\ principal\ and \quad G'x \subset Gx. \tag{H2}$$

We will also need the assumption (H3) defined later. We refer to Faget [7] for examples of groups satisfying (H). In particular, $dim\ G'x = dim\ Gx_0 = k$ for all $x \in B_{Gx_0}(\delta)$ and we can consider the Riemannian quotient $(n-k)$-manifold $N := B_{Gx_0}(\delta)/G'$. We fix a smooth cut-off function $\eta \in C_c^\infty(\mathbb{R}^{n-k})$ with support in $B_0(2)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ in $B_0(1)$. Given $\bar{x}_1 \in N$ and $\delta' \in (0, i_{\tilde{g}}(\bar{x}_1)/2)$, we let

$$\eta_{\bar{x}_1,\delta'}(\bar{x}) = \eta\left(\frac{d_{\tilde{g}}(\bar{x}_1, \bar{x})}{\delta'}\right)$$

for $\bar{x} \in N$. Here, $i_{\tilde{g}}(\bar{x}_1)$ denotes the injectivity radius of N at $\bar{x}_1$.

We define a bubble in this context. Let (x_α) be a sequence of points in M converging to some point $x_0 \in M$ such that Gx_0 is of dimension k and $f(x_0) > 0$. Then assumption (H) provides us with a subgroup G' of G and a $\delta > 0$ such that (H1) and (H2) hold. Let $2\delta' > 0$ be inferior to the injectivity radius of the quotient $(n-k)$-manifold $N := B_{Gx_0}(\delta)/G'$. Consider also a sequence $(R_\alpha) \subset [0, +\infty)$ such that $R_\alpha \to +\infty$. Given a (non-trivial non-necessarily) positive solution $u \in D_2^2(\mathbb{R}^{n-k})$ [where $D_2^2(\mathbb{R}^{n-k})$ is the closure of $C_c^\infty(\mathbb{R}^{n-k})$ for the norm $\|u\| = \|\Delta u\|_2$] of the Euclidean equation

$$\Delta_\xi^2 u = f(x_0) Vol(Gx_0)^{-\frac{4}{n-k-4}} |u|^{2^\sharp - 2} u, \tag{3}$$

we can define classically a bubble $\bar{B} = (\bar{B}_\alpha)$ by

$$\bar{B}_\alpha(\bar{x}) = \eta_{\bar{x}_\alpha, \delta'}(\bar{x}) R_\alpha^{\frac{n-k-4}{2}} u\left(R_\alpha exp_{\bar{x}_\alpha}^{-1}(\bar{x})\right), \quad \bar{x} \in N, \tag{4}$$

where exp is the exponential map of $(N, \tilde{g})$. Since G' is a normal subgroup of G, the quotient group $\bar{G} := G/G'$ acts on N by $\bar{g}\bar{x} := \overline{gx}$. This way $\bar{G} \subset \text{Isom}_{\bar{g}}(N)$ (see [8]). Note that $\bar{G}\bar{x}_0 = \bar{x}_0$ in view of (H1). We will assume that

$$\text{either (i) } dim \bar{G}\bar{x} \geq 1 \text{ or (ii) } \bar{G}\bar{x} \text{ is discrete for any } \bar{x} \in N \backslash \{\bar{x}_0\}. \tag{H3}$$

In case (ii), the orbit $\bar{G}\bar{x}_\alpha$ is discrete and we will prove later that its cardinal is bounded uniformly in α, so that, up to a subsequence, we can suppose it constant equal to $k(B)$. For notational convenience we also let $k(B) = 1$ in case (i). We let $\bar{m}$ be the Haar measure of $\bar{G}$ normalized by $\bar{m}(\bar{G}) = 1$, and consider, in both cases (i) and (ii), the symmetrized $\bar{B}_{\bar{G}} = (\bar{B}_{\bar{G},\alpha})$ of $\bar{B}$ under $\bar{G}$, namely

$$\bar{B}_{\bar{G},\alpha} := \int_{\bar{G}} \bar{B}_\alpha \circ \bar{\sigma} \, d\bar{m}(\bar{\sigma}) \tag{5}$$

Notice that $\bar{B}_{\bar{G},\alpha}$ is $\bar{G}$-invariant. See (39) for the explicit expression of $\bar{B}_{\bar{G},\alpha}$ in case (ii). A (generalized) bubble $B = (B_\alpha)$ of center (Gx_α) and weights (R_α) is then defined by the relation

$$B_\alpha = \bar{B}_{\bar{G},\alpha} \circ \Pi, \tag{6}$$

where $\Pi : B_{Gx_0}(\delta) \to N := B_{Gx_0}(\delta)/G'$ is the canonical surjection. Note that B_α is G-invariant.

This definition clearly extends the usual definition of a bubble to the case of G-invariant functions. We define the energy $E(\bar{B})$ of $\bar{B}$ by

$$E(\bar{B}) = \frac{1}{2} \int_{\mathbb{R}^{n-k}} (\Delta_\xi u)^2 dx - \frac{f(x_0) Vol(Gx_0)^{-\frac{4}{n-k-4}}}{2^\sharp} \int_{\mathbb{R}^{n-k}} |u|^{2^\sharp} dx, \tag{7}$$

and then the energy of the generalized bubble B by

$$E(B) = k(B) E(\bar{B}) \tag{8}$$

Arguing as in Hebey–Robert [11], we can prove the following minoration of the energy:

$$E(\bar{B}) \geq f(x_0)^{-\frac{n-k-4}{4}} Vol(Gx_0) \beta^\sharp,$$

where $\beta^\sharp = \frac{2}{n-k} K_0(n-k)^{-\frac{n-k}{4}}$, $K_0(n-k)$ being the best Sobolev constant for the injection $D_2^2(\mathbb{R}^{n-k}) \hookrightarrow L^{2^\sharp}(\mathbb{R}^{n-k})$ (see [10] or [5]), namely

$$\frac{1}{K_0(n-k)} = \inf_{u \in C_c^\infty(\mathbb{R}^{n-k}) \backslash \{0\}} \frac{\int_{\mathbb{R}^{n-k}} (\Delta_\xi u)^2 \, dx}{\left(\int_{\mathbb{R}^{n-k}} |u|^{2^\sharp} \, dx\right)^{2/2^\sharp}} > 0. \tag{9}$$

The value of $K_0(n)$ is explicitly known (see [6,13,15]). If we denote by A the minimum volume of G-orbit of dimension k, we then have the minoration

$$E(B) \geq k(B)\left(\max_M f\right)^{-\frac{n-k-4}{4}} A\beta^\sharp \tag{10}$$

which holds for any generalized bubble. Moreover, since a nonnegative nontrivial solution of (3) is of the form (see [11,14])

$$u(x) = f(x_0)^{-\frac{n-k-4}{8}} Vol(Gx_0)^{\frac{1}{2}} \alpha_{n-k}\left(\frac{\lambda}{1+\lambda^2|x-y|^2}\right)^{\frac{n-k-4}{2}},$$

where $\lambda > 0$, $y \in \mathbb{R}^{n-k}$, $\alpha_n = \big(n(n-4)(n^2-4)\big)^{(n-4)/8}$, the energy of a bubble is exactly

$$E(B) = k(B) f(x_0)^{-\frac{n-k-4}{4}} Vol(Gx_0)\beta^\sharp \tag{11}$$

Our result is the following:

Theorem *Let (M,g) be a Riemaniann manifold, G a closed subgroup of $Isom_g(M)$ satisfying (H1)–(H3) and (u_α) be a sequence of nonnegative G-invariant solutions of (E_α) bounded in $H_2^2(M)$. There exist a nonnegative solution $u^0 \in H_{2,G}^2(M)$ of (E_∞) and l bubbles $B^i = \big(B_\alpha^i\big)_\alpha$, $i = 1\ldots l$, such that, up to a subsequence,*

$$u_\alpha = u^0 + \sum_{i=1}^{l} B_\alpha^i + S_\alpha, \tag{12}$$

where the sequence $(S_\alpha) \subset H_2^2(M)$ converges strongly to 0 in H_2^2, and

$$J_g^\alpha(u_\alpha) = J_g^\infty(u^0) + \sum_{i=1}^{l} E(B^i) + o(1), \tag{13}$$

where J_g^α and J_g^∞ are the functional defined on $H_2^2(M)$ by (16) and (18), respectively, $x_i = \lim x_\alpha^i$, the (x_α^i) being the centers of the bubble B^i, and $E(B^i)$ is the energy of B^i defined by (11).

Moreover, if we assume that $f \geq 0$, $k_\infty > 0$ and the h_α's are real numbers with $0 < h_\infty \leq k_\infty^2/4$, then either $u^0 > 0$ or $u^0 = 0$, and there exists a constant $C > 0$ independent of α and $x \in M$ such that for any α and any $x \in M$,

$$R_\alpha(x)^{\frac{n-k-4}{2}} \left|u_\alpha(x) - u^0(x)\right| \leq C, \quad \text{and} \tag{14}$$

$$\lim_{R\to\infty} \lim_{\alpha\to+\infty} \sup_{x\in M\setminus\Omega_\alpha(R)} R_\alpha(x)^{\frac{n-k-4}{2}} \left|u_\alpha(x) - u^0(x)\right| = 0, \tag{15}$$

where the $(\mu_\alpha^i)_\alpha$ are the inverse of the weights of the bubble B^i, $R_\alpha(x) = \min_{i=1\ldots l} d_g(Gx_\alpha^i, Gx)$ and, for $R > 0$, $\Omega_\alpha(R) = \cup_{i=1}^k B_{Gx_\alpha^i}(R\mu_\alpha^i)$; when there is no symmetry assumption we refer to Struwe [20,21].

Moreover, we have $\nabla f(x_i) = 0$ for any i in the particular case where $u^0 = 0$.

The paper is organized as follow. The first section is devoted to the proof of the H_2^2-decomposition, i.e. the relations (12) and (13) for a Palais–Smale sequence for the functional J_g^α defined by (16), whereas the second one deals with the proof of the pointwise estimates (14) and (15).

2 Proof of the H_2^2-decomposition for Palais–Smale sequences

Let J_g^α be the functional defined on $H_2^2(M)$ by

$$J_g^\alpha(u) = \frac{1}{2}\int_M (\Delta_g u)^2 dv_g + \frac{1}{2}\int_M k_\alpha |\nabla u|_g^2 dv_g + \frac{1}{2}\int_M h_\alpha |u|^2 dv_g - \frac{1}{2^\sharp}\int_M f|u|^{2^\sharp} dv_g, \tag{16}$$

and $(u_\alpha) \subset H_{2,G}^2(M)$ be a Palais–Smale (P–S) sequence for J_g^α, i.e. the sequence $(J_g^\alpha(u_\alpha))$ is bounded and $DJ_g^\alpha(u_\alpha) \to 0$ strongly in $H_2^2(M)'$.

It follows from Hebey–Robert ([11], Step 1) that, up to a subsequence, the sequence (u_α) weakly converges in $H_2^2(M)$ and also a.e. to some $u^0 \in H_{2,G}^2(M)$ which is a weak solution of (E_∞). Let $v_\alpha = u_\alpha - u^0$. Since $v_\alpha \to 0$ strongly in $H_1^2(M)$, we can prove as in Hebey–Robert ([11], Step 2) that (v_α) is a (P-S) sequence for the functional J_g defined on $H_2^2(M)$ by

$$J_g(u) = \frac{1}{2}\int_M (\Delta_g u)^2 dv_g - \frac{1}{2^\sharp}\int_M f|u|^{2^\sharp} dv_g. \tag{17}$$

Moreover

$$J_g(v_\alpha) = J_g^\alpha(u_\alpha) - J_g^\infty(u^0) + o(1),$$

where J_g^∞ is the functional defined on $H_2^2(M)$ by

$$J_g^\infty(u) = \frac{1}{2}\int_M (\Delta_g u)^2 dv_g + \frac{1}{2}\int_M k_\infty |\nabla u|_g^2 dv_g + \frac{1}{2}\int_M h_\infty |u|^2 dv_g - \frac{1}{2^\sharp}\int_M f|u|^{2^\sharp} dv_g. \tag{18}$$

According to Hebey [6], there exists $C > 0$ such that for any $u \in H_2^2(M)$,

$$\left(\int_M |u|^{\frac{2n}{n-4}} dv_g\right)^{\frac{n-4}{n}} \le K_0(n)\int_M (\Delta_g u)^2 dv_g + C\|u\|^2_{H_1^2(M)},$$

where $K_0(n)$ is defined in (9). The constant $K_0(n)$ is optimal. Its value is explicitely known and depends only on n. As for Sobolev spaces of first order, one can improve the order of integrability when we have invariance under isometries. More precisely, the space $H_{2,G}^2(M)$ is continuously embedded in $L^{2^\sharp}(M)$ and there exist constants $\tilde{K}, C > 0$ such that for any $u \in H_{2,G}^2(M)$,

$$\left(\int_M |u|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}} \le \tilde{K}\int_M (\Delta_g u)^2 dv_g + C\|u\|^2_{H_1^2}. \tag{19}$$

This result can be proved as in Hebey–Vaugon [12] (see [19]). We define $\tilde{K}_0$ to be the smallest possible constant $\tilde{K}$ in (19). In other words, for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for any $u \in H_{2,G}^2(M)$,

$$\left(\int_M |u|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}} \leq \left(\tilde{K}_0 + \epsilon\right) \int_M (\Delta_g u)^2 dv_g + C_\epsilon \|u\|^2_{H_1^2}, \tag{20}$$

and $\tilde{K}_0$ is the least constant such that such an inequality holds for any ϵ and u. The value of $\tilde{K}_0$ is studied in Saintier [19]. We can now adapt the argument in Hebey–Robert ([11], Step 3) to prove that if (w_α) is a (P–S) sequence for J_g such that

$$w_\alpha \to 0 \text{ weakly in } H_2^2 \text{ and } \lim\ J_g(w_\alpha) < \|f\|_\infty^{-\frac{n-k-4}{4}} \beta^\sharp,$$

where $\beta^\sharp = \frac{2}{n-k} \tilde{K}_0^{-\frac{n-k}{4}}$, then

$$w_\alpha \to 0 \text{ strongly in} H_2^2.$$

Using this remark and the minoration (10) of the energy of a generalized bubble, we can prove the theorem by induction by repeated use of the following lemma:

Lemma *Let (v_α) be a (P–S) sequence for J_g converging to 0 in H_2^2 weakly but not strongly. Then there exists a generalized bubble $B = (B_\alpha)$ such that $w_\alpha := v_\alpha - B_\alpha$ is a (P-S) sequence for J_g weakly converging to 0 in H_2^2. Moreover*

$$J_g(w_\alpha) = J_g(v_\alpha) - E(B) + o(1).$$

The remainder of this section is devoted to the proof of this lemma. According to the density of the set of smooth G-invariant functions on M in $H^2_{2,G}(M)$ (see [12]), we can assume that the v_α's are smooth. Independently, since the v_α's don't converge strongly to 0, the definition of a (P–S) sequence implies that there exists $\beta > 0$ such that

$$\int_M (\Delta_g v_\alpha)^2 dv_g = \frac{n-k}{2}\beta + o(1) \tag{21}$$

and

$$\int_M f|v_\alpha|^{2^\sharp} dv_g = \frac{n-k}{2}\beta + o(1)$$

with $\beta \geq \|f\|_\infty^{-\frac{n-k-4}{4}} \beta^\sharp > 0$. Since M is compact we deduce the existence of a point $x_0 \in M$ such that for any $\delta > 0$ small,

$$\limsup_{\alpha\to+\infty} \int_{B_{Gx_0}(\delta)} f|v_\alpha|^{2^\sharp} dv_g > 0. \tag{22}$$

Such an orbit is called orbit of concentration. We first give some basic properties of such orbits:

Step 1.1 (1) There are a finite number of orbits of concentration. If Gx_0 is one of them, then $dim\ Gx_0 = k$ and $f(x_0) > 0$. In the particular case where $u^0 = 0$ and $DJ_g^\alpha(u_\alpha) = 0$, we have also $\nabla f(x_0) = 0$. Moreover Gx_0 is an orbit of concentration if and only if for any $\delta > 0$,

$$\limsup_{\alpha\to+\infty} \int_{B_{Gx_0}(\delta)} (\Delta_g v_\alpha)^2 dv_g > 0. \tag{23}$$

(2) Let Gx_0 be an orbit of concentration for (v_α). According to 1) and in view of assumption (H), there exist $\delta_0 > 0$ and a subgroup G' of $Isom_g(M)$ such that we can consider the Riemannian quotient $(n-k)$-manifold $(N := B_{Gx_0}(\delta_0)/G', \bar{g})$. Then $\bar{x}_0$ is a point of concentration for $(\bar{v}_\alpha)$ in the sense that for any $\delta > 0$ small,

$$\limsup_{\alpha \to +\infty} \int_{B^{\tilde{g}}_{\bar{x}_0}(\delta)} (\Delta_{\tilde{g}} \bar{v}_\alpha)^2 dv_{\tilde{g}} > 0 \tag{24}$$

where $\tilde{g}$ is defined by (2), and $\bar{v}_\alpha(\bar{x}) = v_\alpha(x)$.

Proof We first prove (1). Assume that Gx_0 is an orbit of concentration of dimension $k' > k$. Then there exists $\delta > 0$ such that $dim\ Gx \geq k' > k$ for any $x \in B_{Gx_0}(\delta)$ (see [8, lemma 2]). Since $2^\sharp = \frac{2(n-k)}{n-k-4} < \frac{2(n-k')}{n-k'-4}$, it thus follows from Hebey–Vaugon [12] that the injection $\overset{\circ}{H}{}^2_{2,G}\left(B_{Gx_0}(\delta')\right) \hookrightarrow L^{2^\sharp}\left(B_{Gx_0}(\delta')\right)$ is compact for all $\delta' \in (0, \delta)$. In fact, the results proved in Hebey–Vaugon [12] only concern Sobolev spaces of first order but can easily be extended to the second order (see also [19]). Since $v_\alpha \to 0$ weakly in $H_2^2(M)$, we get a contradiction with (22). Hence Gx_0 is of minimal dimension k.

Since the sequence (v_α) is bounded in $H^2_{2,G}(M)$, there exist two positive G-invariant measures μ and ν such that $|v_\alpha|^{2^\sharp} dv_g \rightharpoonup \nu$ and $(\Delta_g v_\alpha)^2 dv_g \rightharpoonup \mu$ weakly in the sense of measures. Let $\epsilon > 0$ and $C_\epsilon > 0$ be such that (20) holds. We thus have for any G-invariant function $\phi \in C^2(M)$ that

$$\left(\int_M |\phi v_\alpha|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}} \leq (\tilde{K}_0 + \epsilon) \int_M (\Delta_g(\phi v_\alpha))^2 dv_g + C_\epsilon \|\phi v_\alpha\|^2_{H_1^2}.$$

Since $v_\alpha \to 0$ strongly in $H_1^2(M)$, we get by passing to the limit $\alpha \to +\infty$ and then $\epsilon \to 0$ in this inequality that

$$\left(\int_M |\phi|^{2^\sharp} d\nu\right)^{\frac{2}{2^\sharp}} \leq \tilde{K}_0 \int_M \phi^2 d\mu$$

for any G-invariant function $\phi \in C^2(M)$. By density, this inequality also holds for any G-invariant function $\phi \in C(M)$. Lemma 1.1 in Lions [15] then gives the existence of $I \subset \mathbb{N}$, a sequence of points $(x_i)_{i \in I} \subset M$ and two sequences of positive reals $(\mu_i)_{i \in I}$ and $(\nu_i)_{i \in I}$ such that

$$|v_\alpha|^{2^\sharp} dv_g \rightharpoonup \nu = \sum_{i \in I} \nu_i \delta_{Gx_i},$$
$$(\Delta_g v_\alpha)^2 dv_g \rightharpoonup \mu \geq \sum_{i \in I} \mu_i \delta_{Gx_i}, \quad \text{and}$$
$$\nu_i^{\frac{2}{2^\sharp}} \leq \tilde{K}_0 \mu_i \ \forall\, i \in I. \tag{25}$$

where δ_{Gx_i} is defined by $\delta_{Gx_i}(\phi) = \int_G \phi(\sigma x_i) dm(\sigma)$ for $\phi \in C(M)$, m being the Haar measure of G such that $m(G) = 1$ [in particular, if ϕ is G-invariant, then $\delta_{Gx_i}(\phi) = \phi(x_i)$]. Let $\phi \in C(M)$. We can write that

$$\begin{aligned}
o(1) &= DJ_g(v_\alpha).(v_\alpha\phi)\\
&= \int_M \Delta_g v_\alpha \Delta_g(v_\alpha\phi)dv_g - \int_M f|v_\alpha|^{2^\sharp}\phi dv_g\\
&= \int_M \phi(\Delta_g v_\alpha)^2 dv_g + \int_M v_\alpha \Delta_g v_\alpha \Delta_g \phi dv_g\\
&\quad -2\int_M \Delta_g v_\alpha (\nabla v_\alpha, \nabla\phi)_g dv_g - \int_M f|v_\alpha|^{2^\sharp}\phi dv_g,
\end{aligned}$$

Using Hölder inequality and the strong convergence $v_\alpha \to 0$ in $H_1^2(M)$, we get by passing to the limit in this relation that $\int_M \phi d\mu = \int_M f\phi d\nu$ for any $\phi \in C(M)$. Hence $\mu = f\nu$. In particular

$$\mu_i \le f(x_i)\nu_i \quad \text{for any } i \in I.$$

Hence $f(x_i) > 0$ for any $i \in I$ and, using (25),

$$\mu_i \ge (\tilde{K}_0)^{-(n-k)/4}(\max_M f)^{-(n-k-4)/4}$$

for any $i \in I$. We thus get with (21) that

$$\begin{aligned}
\frac{n-k}{2}\beta &= \int_M (\Delta_g v_\alpha)^2 dv_g + o(1) = \mu(M) \ge \sum \mu_i\\
&\ge (card\ I)(\tilde{K}_0)^{-(n-k)/4}(\max_M f)^{-(n-k-4)/4}
\end{aligned}$$

which implies that I is finite, i.e. (v_α) has a finite number of orbit of concentration, namely the Gx_i, $i \in I$. Eventually,

$$\mu = f\nu = \sum_{i\in I} \nu_i f(x_i)\delta_{Gx_i} \tag{26}$$

which implies the equivalent definition (23) of an orbit of concentration.

Assume now that $u^0 = 0$ and $DJ_g^\alpha(u_\alpha) = 0$ for all α, and consider an orbit of concentration Gx_i. We are going to prove that $\nabla f(x_i) = 0$. Let G' be the group given by (H) at the point x_i. Let ϕ be a smooth G-invariant function with compact support in some neighbourhood $B_{Gx_i}(\delta)$ of Gx_i not intersecting other concentration orbit, satisfying $\nabla\phi(x_i) = \nabla f(x_i)$ and $\nabla^2\phi(x_i) = 0$. Then the function $(\nabla u_\alpha, \nabla\phi)_g$ is smooth and we can write that

$$\begin{aligned}
&\frac{1}{2^\sharp}\nu_i|\nabla f|_g^2(x_i) + o(1)\\
&= \frac{1}{2^\sharp}\int_M (\nabla f, \nabla\phi)_g |u_\alpha|^{2^\sharp} dv_g\\
&= \frac{1}{2^\sharp}\int_M (\nabla(f|u_\alpha|^{2^\sharp}), \nabla\phi)_g dv_g - \int_M f|u_\alpha|^{2^\sharp-2}u_\alpha(\nabla u_\alpha, \nabla\phi)_g dv_g\\
&= \frac{1}{2^\sharp}\int_M f(\Delta_g\phi)|u_\alpha|^{2^\sharp} dv_g - \int_M \Delta_g u_\alpha \Delta_g(\nabla u_\alpha, \nabla\phi)_g dv_g\\
&\quad - \int_M k_\alpha(\nabla u_\alpha, \nabla(\nabla u_\alpha, \nabla\phi)_g)_g dv_g - \int_M h_\alpha u_\alpha(\nabla u_\alpha, \nabla\phi)_g dv_g.
\end{aligned}$$

Since $\Delta_g \phi(x_i) = 0$, the first integral tends to 0. The same is also true for the last one by Hölder inequality. We can write the third integral as

$$\begin{aligned}
&\int_M k_\alpha (\nabla u_\alpha, \nabla (\nabla u_\alpha, \nabla \phi)_g)_g dv_g \\
&\quad = \int_M (\nabla u_\alpha, \nabla (k_\alpha (\nabla u_\alpha, \nabla \phi)_g))_g dv_g + O(\|\nabla u_\alpha\|_2^2) \\
&\quad = \int_M k_\alpha (\nabla u_\alpha, \nabla \phi)_g \Delta_g u_\alpha dv_g + o(1)
\end{aligned}$$

with, by Hölder inequality,

$$\left| \int_M k_\alpha (\nabla u_\alpha, \nabla \phi)_g \Delta_g u_\alpha dv_g \right| \leq C \|\nabla u_\alpha\|_2 \|\Delta_g u_\alpha\|_2 = o(1) O(1) = o(1).$$

Hence

$$\begin{aligned}
\frac{1}{2^\sharp} \nu_i |\nabla f|_g^2 (x_i) &= - \int_M \Delta_g u_\alpha \Delta_g (\nabla u_\alpha, \nabla \phi)_g dv_g + o(1) \\
&= - \int_N \Delta_{\bar{g}} \bar{u}_\alpha \Delta_{\bar{g}} (\nabla \bar{u}_\alpha, \nabla \bar{\phi})_{\bar{g}} \bar{v} dv_{\bar{g}} + o(1)
\end{aligned}$$

where $N = B_{Gx_i}(\delta)/G'$, $u_\alpha = \bar{u}_\alpha \circ \Pi$, $\phi = \bar{\phi} \circ \Pi$, $\Pi : B_{Gx_i}(\delta) \to N$ being the canonical surjection. Following Robert [16], we write, using the Cartan expansion of $\bar{g}$ in the exponential chart, that

$$\begin{aligned}
\Delta_{\bar{g}} (\nabla \bar{u}_\alpha, \nabla \bar{\phi})_{\bar{g}} = (\nabla (\Delta_{\bar{g}} \bar{u}_\alpha), \nabla \bar{\phi})_{\bar{g}} + O(|\nabla \bar{u}_\alpha|_{\bar{g}}) + O(|x| |\nabla^2 \bar{u}_\alpha|_{\bar{g}}) \\
+ O(|\nabla^2 \bar{u}_\alpha|_{\bar{g}} |\nabla^2 \bar{\phi}|_{\bar{g}}).
\end{aligned}$$

By Hölder inequality (25) and since the sequence $(\bar{u}_\alpha)$ is bounded in $H_2^2(N)$ and converges strongly to 0 in H_1^2, we have:

$$\int_N |\Delta_{\bar{g}} \bar{u}_\alpha| |\nabla \bar{u}_\alpha|_{\bar{g}} \bar{v} dv_{\bar{g}} \leq C \|\Delta_{\bar{g}} \bar{u}_\alpha\|_2 \|\nabla \bar{u}_\alpha\|_2 = O(1) o(1) = o(1),$$

$$\int_N |\Delta_{\bar{g}} \bar{u}_\alpha| |x| |\nabla^2 \bar{u}_\alpha|_{\bar{g}} |\nabla \bar{\phi}|_{\bar{g}} \bar{v} dv_{\bar{g}} \leq C \delta \|\nabla^2 \bar{u}_\alpha\|_2 \|\Delta_{\bar{g}} \bar{u}_\alpha\|_2 = \delta O(1) \text{ and}$$

$$\begin{aligned}
\int_N |\Delta_{\bar{g}} \bar{u}_\alpha| |\nabla^2 \bar{u}_\alpha|_{\bar{g}} |\nabla^2 \bar{\phi}|_{\bar{g}} \bar{v} dv_{\bar{g}} &\leq C \|\nabla^2 \bar{u}_\alpha\|_2 \left(\int_M |\nabla^2 \phi|_g^2 (\Delta_g u_\alpha)^2 dv_g \right)^{\frac{1}{2}} \\
&\leq C \left(\sqrt{|\nabla^2 \phi|_g (x_i)} + o(1) \right) = o(1),
\end{aligned}$$

where $o(1) \to 0$ and $O(1)$ are independent of δ. Hence

$$\begin{aligned}\frac{1}{2^\sharp}\nu_i|\nabla f|_g^2(x_i) &= -\int_N \Delta_{\bar g}\bar u_\alpha(\nabla(\Delta_{\bar g}\bar u_\alpha), \nabla\bar\phi)_{\bar g}\bar v dv_{\bar g} + o(1) + \delta O(1)\\ &= -\int_M \Delta_g u_\alpha(\nabla(\Delta_g u_\alpha), \nabla\phi)_g dv_g + o(1) + \delta O(1)\\ &= -\frac{1}{2}\int_M (\nabla(\Delta_g u_\alpha)^2, \nabla\phi)_g dv_g + o(1) + \delta O(1)\\ &= -\frac{1}{2}\int_M (\Delta_g u_\alpha)^2\Delta_g\phi dv_g + o(1) + \delta O(1)\\ &= o(1) + \delta O(1),\end{aligned}$$

Letting $\alpha \to +\infty$ and then $\delta \to 0$ gives $\nabla f(x_i) = 0$.

We now prove (2). The metric $\tilde g$ being defined by (2), we have $dv_{\bar g} = \bar v^{-\frac{n-k}{n-k-4}} dv_{\tilde g}$ and (see [9]),

$$\Delta_{\bar g}\bar v_\alpha = \bar v^{\frac{2}{n-k-4}}\Delta_{\tilde g}\bar v_\alpha + \frac{n-k-2}{n-k-4}\bar v^{-\frac{n-k-6}{n-k-4}}(\nabla\bar v_\alpha, \nabla\bar v)_{\tilde g}. \tag{27}$$

Then for $\delta > 0$ small,

$$\begin{aligned}\int_{B_{Gx_0}(\delta)} (\Delta_g v_\alpha)^2 dv_g &= \int_{B^{\bar g}_{\bar x_0}(\delta)} (\Delta_{\bar g}\bar v_\alpha)^2\bar v dv_{\bar g}\\ &= \int_{B^{\bar g}_{\bar x_0}(\delta)} (\Delta_{\tilde g}\bar v_\alpha)^2 dv_{\tilde g} + I_1 + I_2\end{aligned}$$

where I_1 and I_2 satisfy estimates of the form

$$\begin{aligned}|I_1| &\le C\int_{B^{\bar g}_{\bar x_0}(\delta)} |\nabla\bar v_\alpha|^2_{\tilde g}\bar v dv_{\tilde g} \le C\int_{B^{\bar g}_{\bar x_0}(\delta)} |\nabla\bar v_\alpha|^2_{\bar g}\bar v dv_{\bar g}\\ &= C\int_{B_{Gx_0}(\delta)} |\nabla v_\alpha|^2 dv_g,\end{aligned}$$

and

$$\begin{aligned}|I_2| &\le C\int_{B^{\bar g}_{\bar x_0}(\delta)} |\Delta_{\tilde g}\bar v_\alpha|\cdot|\nabla\bar v_\alpha|_{\tilde g}\, dv_{\tilde g}\\ &\le C\|\Delta_{\tilde g}\bar v_\alpha\|_{L^2(B^{\bar g}_{\bar x_0}(\delta))}\|\nabla\bar v_\alpha\|_{L^2(B^{\bar g}_{\bar x_0}(\delta))}\\ &\le C\|\Delta_{\tilde g}\bar v_\alpha\|_{L^2(B^{\bar g}_{\bar x_0}(\delta))}\|\nabla v_\alpha\|_{L^2(B_{Gx_0}(\delta))}.\end{aligned}$$

Since $v_\alpha \to 0$ strongly in $H_1^2(M)$, we deduce that

$$\int_{B_{Gx_0}(\delta)} (\Delta_g v_\alpha)^2 dv_g = \int_{B^{\bar{g}}_{\bar{x}_0}(\delta)} (\Delta_{\tilde{g}} \bar{v}_\alpha)^2 dv_{\tilde{g}} + o(1).$$

Letting

$$m = \inf_{B^{\bar{g}}_{\bar{x}_0}(\delta)} \bar{v}^{1/(n-k-4)},$$

we get that $B^{\bar{g}}_{\bar{x}_0}(\delta) \subset B^{\tilde{g}}_{\bar{x}_0}(\delta/m)$, and thus

$$\int_{B_{Gx_0}(\delta)} (\Delta_g v_\alpha)^2 dv_g \le \int_{B^{\tilde{g}}_{\bar{x}_0}(\delta/m)} (\Delta_{\tilde{g}} \bar{v}_\alpha)^2 dv_{\tilde{g}} + o(1)$$

which, together with (23), proves (24). □

The next step shows that the notion of (P–S) sequences passes to the quotient.

Step 1.2 Let Gx_0 be an orbit such that there exist $\delta_0 > 0$ and a subgroup $G' \subset Isom_g(M)$ satisfying (H1) and (H2). Then $(\bar{v}_\alpha)$ is a (P–S) sequence for the functional $\bar{J}_{\tilde{g}}$ defined on $\overset{\circ}{H}{}_2^2(N)$ by

$$\bar{J}_{\tilde{g}}(\bar{u}) = \frac{1}{2}\int_N (\Delta_{\tilde{g}}\bar{u})^2 dv_{\tilde{g}} - \frac{1}{2^\sharp}\int_N \bar{f}|\bar{u}|^{2^\sharp} \bar{v}^{-\frac{4}{n-k-4}} dv_{\tilde{g}}$$

where $N = B_{Gx_0}(\delta_0)/G'$, $\bar{f} \circ \Pi = f$ and $\Pi : B_{Gx_0}(\delta_0) \to N$ is the canonical surjection.

Proof Let $\bar{\phi} \in C_c^\infty(N)$ and $\phi \in C_c^\infty\left(B_{Gx_0}(\delta_0)\right)$ such that $\bar{\phi} \circ \Pi = \phi$. Then

$$\begin{aligned} o(1)\|\phi\|_{H_2^2} &= DJ_g(v_\alpha)\phi \\ &= \int_{B_{Gx_0}(\delta_0)} \Delta_g v_\alpha \Delta_g \phi dv_g - \int_{B_{Gx_0}(\delta_0)} f|v_\alpha|^{2^\sharp-2} v_\alpha \phi dv_g \\ &= \int_{B^{\bar{g}}_{\bar{x}_0}(\delta)} (\Delta_{\tilde{g}}\bar{v}_\alpha)(\Delta_{\tilde{g}}\bar{\phi})\bar{v} dv_{\tilde{g}} - \int_{B^{\bar{g}}_{\bar{x}_0}(\delta)} \bar{f}|\bar{v}_\alpha|^{2^\sharp-2}\bar{v}_\alpha\bar{\phi}\bar{v} dv_{\tilde{g}}. \end{aligned} \tag{28}$$

Using the metric $\tilde{g}$ defined by (2), we have

$$\int_{B^{\bar{g}}_{\bar{x}_0}(\delta)} \bar{f}|\bar{v}_\alpha|^{2^\sharp-2}\bar{v}_\alpha\bar{\phi}\bar{v} dv_{\tilde{g}} = \int_{B^{\bar{g}}_{\bar{x}_0}(\delta)} \bar{f}|\bar{v}_\alpha|^{2^\sharp-2}\bar{v}_\alpha\bar{\phi}\bar{v}^{-\frac{4}{n-k-4}} dv_{\tilde{g}}.$$

In view of (27), we see that

$$\int_{B^{\bar{g}}_{\bar{x}_0}(\delta)} (\Delta_{\bar{g}}\bar{v}_\alpha)(\Delta_{\bar{g}}\bar{\phi})\bar{v} dv_{\bar{g}} = \int_{B^{\bar{g}}_{\bar{x}_0}(\delta)} (\Delta_{\tilde{g}}\bar{v}_\alpha)(\Delta_{\tilde{g}}\bar{\phi}) dv_{\tilde{g}} + I_1 + I_2$$

where I_1 and I_2 are terms satisfying, by Hölder inequality, estimates of the form

$$|I_1| \leq C \sqrt{\int_{B_{\bar{x}_0}^{\tilde{g}}(\delta)} |\nabla \bar{v}_\alpha|_{\tilde{g}}^2 dv_{\tilde{g}} \left(\|\Delta_{\tilde{g}} \bar{\phi}\|_2^2 + \|\nabla \bar{\phi}\|_2^2\right)}$$
$$= o(1)\|\bar{\phi}\|_{H_2^2(N)},$$

and

$$\begin{aligned} I_2 &= \frac{n-k-2}{n-k-4} \int_N \bar{v}^{-1} \left(\nabla \bar{\phi}, \nabla \bar{v}\right)_{\tilde{g}} \Delta_{\tilde{g}} \bar{v}_\alpha \, dv_{\tilde{g}} \\ &= \frac{n-k-2}{n-k-4} \int_N \left(\nabla \bar{v}_\alpha, \nabla \left(\bar{v}^{-1} \left(\nabla \bar{\phi}, \nabla \bar{v}\right)_{\tilde{g}}\right)\right)_{\tilde{g}} dv_{\tilde{g}} \\ &= O(1)\|\nabla \bar{v}_\alpha\|_{L^2(B_{\bar{x}_0}^{\tilde{g}}(\delta))} \|\bar{\phi}\|_{H_2^2(N)} \\ &= o(1)\|\bar{\phi}\|_{H_2^2(N)} \end{aligned}$$

Hence (28) becomes

$$D\bar{J}_{\tilde{g}}(\bar{v}_\alpha)\bar{\phi} = o(1)\|\bar{\phi}\|_{H_2^2(N)}.$$

□

As explained above, there exists an orbit of concentration Gx_0. According to Step 1.1, $dim\, Gx_0 = k$. Assumption (H) then gives $\delta_0 > 0$ and a subgroup $G' \subset Isom_g(M)$ satisfying (H1) and (H2) on $B_{Gx_0}(2\delta_0)$. We let $N = B_{Gx_0}(\delta_0)/G'$ and consider, for $t > 0$,

$$Q_\alpha(t) := \sup_{\bar{x} \in N} \int_{B_{\bar{x}}^{\tilde{g}}(t)} (\Delta_{\tilde{g}} \bar{v}_\alpha)^2 dv_{\tilde{g}}.$$

In view of Step 1.1, there exist λ_0 such that, up to a subsequence, for any α

$$Q_\alpha(\delta_0) \geq \int_{B_{\bar{x}_0}^{\tilde{g}}(\delta_0)} (\Delta_{\tilde{g}} \bar{v}_\alpha)^2 dv_{\tilde{g}} \geq \lambda_0.$$

Since Q_α is continuous, we then get for any $\lambda \in (0, \lambda_0)$ the existence of $t_\alpha \in (0, \delta_0)$ and $\bar{x}_\alpha \in N$, $\bar{x}_\alpha \to \bar{x}_0$, such that for any α

$$Q_\alpha(t_\alpha) = \int_{B_{\bar{x}_\alpha}^{\tilde{g}}(t_\alpha)} (\Delta_{\tilde{g}} \bar{v}_\alpha)^2 dv_{\tilde{g}} = \lambda.$$

In view of Step 1.2, $(\bar{v}_\alpha)$ is a (P–S) sequence for $\bar{J}_{\tilde{g}}$ on $\overset{\circ}{H}{}_2^2(N)$. According to Hebey–Robert [11], there exist a sequence $R_\alpha \to +\infty$ and $v \in D_2^2\left(\mathbb{R}^{n-k}\right)$ (where $D_2^2(\mathbb{R}^{n-k})$ is the completion of $C_c^\infty(\mathbb{R}^{n-k})$ for the norm $u \mapsto \|\Delta_\xi u\|_2$) such that

$$\tilde{v}_\alpha \to v \; in \; H_{2,loc}^2\left(\mathbb{R}^{n-k}\right) \tag{29}$$

and $v \not\equiv 0$, where, if $i_{\tilde{g}}(\bar{x}_0)$ denotes the injectivity radius of $(N, \tilde{g})$ at $\bar{x}_0$,

$$\tilde{v}_\alpha(x) = R_\alpha^{-\frac{n-k-4}{2}} \bar{v}_\alpha \left(exp_{\bar{x}_\alpha}(R_\alpha^{-1} x)\right), \ x \in B_0\left(R_\alpha i_{\tilde{g}}(\bar{x}_0)\right). \tag{30}$$

We now prove that

Step 1.3 v is a solution of the Euclidean equation

$$\begin{aligned} \Delta_\xi^2 v &= \bar{f}(\bar{x}_0)\bar{v}(\bar{x}_0)^{-\frac{4}{n-k-4}} |v|^{2^\sharp-2} v \\ &= f(x_0) Vol(Gx_0)^{-\frac{4}{n-k-4}} |v|^{2^\sharp-2} v. \end{aligned} \tag{31}$$

Proof Let $\phi \in C_c^\infty\left(\mathbb{R}^{n-k}\right)$ and $R > 0$ such that $supp\ \phi \subset B_0(R)$. For α large enough, we define $\phi_\alpha \in C_c^\infty(N)$ by

$$\phi_\alpha(\bar{x}) = R_\alpha^{\frac{n-k-4}{2}} \phi\left(R_\alpha exp_{\bar{x}_\alpha}(\bar{x})\right).$$

Then (ϕ_α) is bounded in $\overset{\circ}{H}{}_2^2(N)$. Thus

$$\begin{aligned} o(1) &= D\bar{J}_{\tilde{g}}(\bar{v}_\alpha)\phi_\alpha \\ &= \int_{B_0(R)} \Delta_{\tilde{g}_\alpha} \tilde{v}_\alpha \Delta_{\tilde{g}_\alpha} \phi dv_{\tilde{g}_\alpha} \\ &\quad - \int_{B_0(R)} |\tilde{v}_\alpha|^{2^\sharp-2} \tilde{v}_\alpha \phi \bar{v}\left(exp_{\bar{x}_\alpha}(R_\alpha^{-1}x)\right)^{-\frac{4}{n-k-4}} \bar{f}\left(exp_{\bar{x}_\alpha}(R_\alpha^{-1}x)\right) dv_{\tilde{g}_\alpha} \end{aligned}$$

where $\tilde{g}_\alpha$ is the metric defined in the Euclidean ball $B_0(i_{\tilde{g}} R_\alpha) \subset \mathbb{R}^{n-k}$ by

$$\tilde{g}_\alpha(x) = \left(exp_{\bar{x}_\alpha}^* \tilde{g}\right)(R_\alpha^{-1} x).$$

Since $R_\alpha \to +\infty$, the $\tilde{g}_\alpha$'s converge locally uniformly to the Euclidean metric ξ. Passing to the limit, we then get with (29) that

$$\int_{\mathbb{R}^{n-k}} \Delta_\xi v \Delta_\xi \phi \, dx - \bar{f}(\bar{x}_0)\bar{v}(\bar{x}_0)^{-\frac{4}{n-k-4}} \int_{\mathbb{R}^{n-k}} |v|^{2^\sharp-2} v\phi \, dx = 0$$

for any $\phi \in C_c^\infty\left(\mathbb{R}^{n-k}\right)$, which proves (31). □

For $\delta > 0$ small, we let

$$\bar{B}_\alpha(\bar{x}) = \eta_{\bar{x}_\alpha,\delta}(\bar{x}) R_\alpha^{\frac{n-k-4}{2}} v\left(R_\alpha exp_{\bar{x}_\alpha}^{-1}(\bar{x})\right)$$

and $\bar{w}_\alpha = \bar{v}_\alpha - \bar{B}_\alpha$. Then, according to Hebey–Robert ([11], Step 3),

$$\bar{B}_\alpha \to 0 \text{ weakly in } H_2^2(N), \tag{32}$$

$$D\bar{J}_{\tilde{g}}(\bar{B}_\alpha) \to 0 \text{ and } D\bar{J}_{\tilde{g}}(\bar{w}_\alpha) \to 0 \text{ strongly in } H_2^2(N), \tag{33}$$

$$\bar{J}_{\tilde{g}}(\bar{w}_\alpha) = \bar{J}_{\tilde{g}}(\bar{v}_\alpha) - E(v) + o(1) \tag{34}$$

where

$$E(v) = \frac{1}{2} \int_{\mathbb{R}^{n-k}} (\Delta_\xi v)^2 dx - \frac{\bar{v}(\bar{x}_0)^{-\frac{4}{n-k-4}} f(x_0)}{2^\sharp} \int_{\mathbb{R}^{n-k}} |v|^{2^\sharp} dx.$$

We now prove that these relations still hold when considering $\bar{B}_{\bar{G},\alpha}$ as defined in (5) and $\bar{w}_{\bar{G},\alpha} := \bar{v}_\alpha - \bar{B}_{\bar{G},\alpha}$.

Step 1.4

$$\bar{B}_{\bar{G},\alpha} \to 0 \text{ weakly in } H_2^2(N), \tag{35}$$

$$DJ_{\bar{g}}(\bar{B}_{\bar{G},\alpha}) \to 0 \quad \text{and} \quad DJ_{\bar{g}}(\bar{w}_{\bar{G},\alpha}) \to 0 \quad \text{strongly in } H_2^2(N),$$

$$\bar{J}_{\bar{g}}(\bar{w}_{\bar{G},\alpha}) = \bar{J}_{\bar{g}}(\bar{v}_\alpha) - E(B) + o(1), \tag{36}$$

where $E(B)$ is defined in (8).

Proof If (i) $dim\ \bar{G}\bar{x}_\alpha \geq 1$ up to a subsequence, we claim that

$$\|\bar{B}_\alpha - \bar{B}_{\bar{G},\alpha}\|^2_{H_2^2} = \|\Delta_{\bar{g}}(\bar{B}_\alpha - \bar{B}_{\bar{G},\alpha})\|_2^2 + \|\nabla(\bar{B}_\alpha - \bar{B}_{\bar{G},\alpha})\|_2^2 + \|\bar{B}_\alpha - \bar{B}_{\bar{G},\alpha}\|_2^2 \to 0.$$

In view of (32)–(34), this will prove Step 1.4 in that case. We prove that $\|\bar{B}_\alpha - \bar{B}_{\bar{G},\alpha}\|_2^2 \to 0$. The convergence of the gradient and laplacian term can be proved in the same way. Since by Jensen's theorem

$$\|\bar{B}_\alpha - \bar{B}_{\bar{G},\alpha}\|_2^2 = \int_N \left(\int_{\bar{G}} \left(\bar{B}_\alpha(\bar{x}) - \bar{B}_\alpha(\bar{\sigma}(\bar{x}))\right) d\bar{m}(\bar{\sigma})\right)^2 dv_{\bar{g}}(\bar{x})$$
$$\leq \int_N \int_G \left(\bar{B}_\alpha(\bar{x}) - \bar{B}_\alpha(\bar{\sigma}(\bar{x}))\right)^2 d\bar{m}(\bar{\sigma}) dv_{\bar{g}}(\bar{x}),$$

it suffices to prove that

$$\|\bar{B}_\alpha - \bar{B}_\alpha \circ \bar{\sigma}\|_{H_2^2} \to 0$$

uniformly in $\bar{\sigma} \in \bar{G}$. To do this we write, given a $\bar{\sigma} \in \bar{G}$, that for any $R > 0$

$$\begin{aligned}
&\|\bar{B}_\alpha - \bar{B}_\alpha \circ \bar{\sigma}\|_{H_2^2(N)} \\
&\leq \|\bar{B}_\alpha - \bar{v}_\alpha\|_{H_2^2(B_{\bar{G}\bar{x}_\alpha}(R\mu_\alpha))} + \|\bar{v}_\alpha - \bar{v}_\alpha \circ \bar{\sigma}\|_{H_2^2(B_{\bar{G}\bar{x}_\alpha}(R\mu_\alpha))} \\
&\quad + \|\bar{v}_\alpha \circ \bar{\sigma} - \bar{B}_\alpha \circ \bar{\sigma}\|_{H_2^2(B_{\bar{G}\bar{x}_\alpha}(R\mu_\alpha))} + 2\|\bar{B}_\alpha\|_{H_2^2(N\setminus B_{\bar{G}\bar{x}_\alpha}(R\mu_\alpha))} \\
&\leq 2\|\bar{B}_\alpha - \bar{v}_\alpha\|_{H_2^2(B_{\bar{G}\bar{x}_\alpha}(R\mu_\alpha))} + 2\|\bar{B}_\alpha\|_{H_2^2(N\setminus B_{\bar{G}\bar{x}_\alpha}(R\mu_\alpha))}
\end{aligned}$$

since $\bar{v}_\alpha$ is $\bar{G}$-invariant, where $\mu_\alpha := R_\alpha^{-1}$. Assume for the moment that

$$\sup_{\bar{\sigma}\in\bar{G}} d_{\bar{g}}(\bar{\sigma}\bar{x}_\alpha, \bar{x}_0) \leq C\mu_\alpha \tag{37}$$

for some constant $C > 0$ independent of α and $\bar{\sigma}$. Then $B_{\bar{G}\bar{x}_\alpha}(R\mu_\alpha) \subset B_{\bar{x}_\alpha}(R'\mu_\alpha)$ for some $R' > R$. It follows that

$$\|\bar{B}_\alpha - \bar{B}_\alpha \circ \bar{\sigma}\|_{H_2^2(N)} \leq 2\|\bar{B}_\alpha - \bar{v}_\alpha\|_{H_2^2(B_{\bar{x}_\alpha}(R'\mu_\alpha))} + 2\|\bar{B}_\alpha\|_{H_2^2(N\setminus B_{\bar{x}_\alpha}(R\mu_\alpha))}$$

which proves the claim in view of (29) and the definition of $\bar{B}_\alpha$.

It remains to prove (37). Since $\bar{G}\bar{x}_0 = \bar{x}_0$, we have to prove that

$$d_{\bar{g}}(\bar{x}_\alpha, \bar{x}_0) \leq C\mu_\alpha.$$

We do this using ideas from Faget [8]. For any $\bar\sigma \in \bar G$, $\bar\sigma(\bar x_0) = \bar x_0$ [because $Gx_0 = G'x_0$ according to (H1)], so that $d\bar\sigma(\bar x_0) : T_{\bar x_0}N \to T_{\bar x_0}N$. Moreover $d\bar\sigma(\bar x_0) = exp_{\bar x_0}^{-1} \circ \bar\sigma \circ exp_{\bar x_0} \in Isom_{\bar g(\bar x_0)}(T_{\bar x_0}N)$. In the exponential chart at $\bar x_0$ that we consider, $\bar g(\bar x_0) = \xi$ the Euclidean metric. We let $S' = \{d\bar\sigma(\bar x_0),\ \bar\sigma \in \bar G\} \subset Isom_\xi(T_{\bar x_0}N)$. There hold $exp_{\bar x_0}^{-1}(\bar G\bar x) = S'(exp_{\bar x_0}^{-1}(\bar x))$ for any $\bar x \in N$ sufficiently close to $\bar x_0$, in particular for $\bar x = \bar x_\alpha$. Considering the exponential chart at $\bar x_0$ and identifying $(T_{\bar x_0}N, \xi)$ with $(\mathbb{R}^n, \xi)$ via an orthogonal map, it follows that we only have to prove that

$$d_\xi(\bar x_\alpha, 0) \le C\mu_\alpha.$$

Since $\bar x_0$ is the unique finite orbit under $\bar G$, $d_\xi(\bar x_\alpha, 0) \le diam(S'_I\bar x_\alpha)$, where S'_I denotes the connected component of the identity in S', and $diam$ the diameter (see [8, lemma 9]). If we assume by contradiction that $diam(S'_I\bar x_\alpha) \ge \mu_\alpha N_\alpha^2$ for some sequence $N_\alpha \to +\infty$, then we can find N_α distinct isometric balls centered at points of $S'_I\bar x_\alpha$ and whose radius r_α satisfies

$$2N_\alpha r_\alpha > diam(S'_I\bar x_\alpha)$$

(see [8, lemma 8]). Since these balls are isometric,

$$O(1) = \int_N |\bar v_\alpha|^{2^\sharp}\, dv_{\bar g} \ge N_\alpha \int_{B^{\bar g}_{\bar x_\alpha}(r_\alpha)} |\bar v_\alpha|^{2^\sharp}\, dv_{\bar g},$$

so that

$$\int_{B^{\bar g}_{\bar x_\alpha}(r_\alpha)} |\bar v_\alpha|^{2^\sharp}\, dv_{\bar g} \to 0.$$

On the other hand,

$$\int_{B^{\bar g}_{\bar x_\alpha}(r_\alpha)} |\bar v_\alpha|^{2^\sharp}\, dv_{\bar g} = \int_{B_0(R_\alpha r_\alpha)} |\tilde v_\alpha|^{2^\sharp}\, dv_{(exp^*_{\bar x_\alpha}\bar g)(R_\alpha^{-1}x)},$$

where $\tilde v_\alpha$ is defined by (30). From $2N_\alpha r_\alpha > diam(S'_I\bar x_\alpha) \ge \mu_\alpha N_\alpha^2$, we get that $R_\alpha r_\alpha \to +\infty$. Moreover $(exp^*_{\bar x_\alpha}\bar g)(R_\alpha^{-1}x) \to \xi$ locally uniformly. Hence, for any $R > 0$, we obtain by passing to the limit using (29) that

$$\int_{B_0(R)} |v|^{2^\sharp}\, dx = 0.$$

This contradicts the fact that $v \not\equiv 0$.

If we are in case (ii) of hypothesis (H3), then each orbit $\bar G\bar x_\alpha$ is discrete. Since $\bar G$ acts continuously on N, the orbit of $\bar x_\alpha$ under the action of any connected component of $\bar G$ is a point. Hence the cardinal of $\bar G\bar x_\alpha$ is less or equal to the number of connected components of $\bar G$ which is finite since $\bar G$ is compact. Hence, up to a subsequence, we can write that $\bar G\bar x_\alpha = \{\bar x_\alpha = \bar x_\alpha^1, \dots, \bar x_\alpha^k\}$ for some k independent of α. Notice that $\bar v_\alpha$ has the same asymptotic behaviour along each sequence $(\bar x_\alpha^i)$ since we pass from one to another by an isometry. Applying the method described in lemma 2.2 in Hebey–Robert [11] successively to the sequences $(\bar x_\alpha^1), \dots, (\bar x_\alpha^k)$, and (32)–(34) each time, we get (35) and (36) but with the function

$$\sum_{i=1}^{k} \eta_{\bar{\sigma}_\alpha^i(\bar{x}_\alpha),\delta'}(\bar{x}) R_\alpha^{\frac{n-k-4}{2}} u\left(R_\alpha exp^{-1}_{\bar{\sigma}_\alpha^i(\bar{x}_\alpha)}(\bar{x})\right) = \sum_{i=1}^{k} \bar{B}_\alpha \circ (\bar{\sigma}_\alpha^i)^{-1}$$

in place of $\bar{B}_{\bar{G},\alpha}$, where the $\bar{\sigma}_\alpha^i$'s are such that $\bar{x}_\alpha^i = \bar{\sigma}_\alpha^i(\bar{x}_\alpha)$, $i = 2, \dots, k$, $\bar{\sigma}_\alpha^1 = Id$. Notice that the function defined by this sum is invariant under the action of $\bar{G}/\bar{S}_\alpha$, where $\bar{S}_\alpha$ denotes the stabilizator of $\bar{x}_\alpha$.

To get the full result, it suffices to prove that each term of this sum can be replaced up to $o(1)$ term by a $\bar{S}_\alpha$-invariant function. We will prove this for $\bar{B}_\alpha$, which is the term corresponding to $(\bar{x}_\alpha^1)$.

Let $\bar{B}_{\bar{S}_\alpha,\alpha} := \int_{\bar{S}_\alpha} \bar{B}_\alpha \circ \bar{\sigma}\, d\bar{m}_\alpha(\bar{\sigma})$ be the symmetrized of $\bar{B}_\alpha$ under $\bar{S}_\alpha$, where $\bar{m}_\alpha$ denotes the Haar measure of $\bar{S}_\alpha$. We are going to prove that

$$\|\bar{B}_\alpha - \bar{B}_{\bar{S}_\alpha,\alpha}\|_{H_2^2} \to 0.$$

As above it suffices to prove that for any ϵ there exists α_0 such that for any $\alpha \geq \alpha_0$ and any $\bar{\sigma}_\alpha \in \bar{S}_\alpha$,

$$\|\bar{B}_\alpha - \bar{B}_\alpha \circ \bar{\sigma}_\alpha\|_{H_2^2} \leq \epsilon. \tag{38}$$

Given some $\bar{\sigma}_\alpha \in \bar{S}_\alpha$ and $R > 0$, we write as previously that

$$\begin{aligned}
&\|\bar{B}_\alpha - \bar{B}_\alpha \circ \bar{\sigma}_\alpha\|_{H_2^2(N)} \\
&\quad \leq \|\bar{B}_\alpha - \bar{v}_\alpha\|_{H_2^2(B_{\bar{x}_\alpha}(R\mu_\alpha))} + \|\bar{v}_\alpha - \bar{v}_\alpha \circ \bar{\sigma}_\alpha\|_{H_2^2(B_{\bar{x}_\alpha}(R\mu_\alpha))} \\
&\qquad + \|\bar{v}_\alpha \circ \bar{\sigma}_\alpha - \bar{B}_\alpha \circ \bar{\sigma}_\alpha\|_{H_2^2(B_{\bar{x}_\alpha}(R\mu_\alpha))} + 2\|\bar{B}_\alpha\|_{H_2^2(N \setminus B_{\bar{x}_\alpha}(R\mu_\alpha))} \\
&\quad \leq 2\|\bar{B}_\alpha - \bar{v}_\alpha\|_{H_2^2(B_{\bar{x}_\alpha}(R\mu_\alpha))} + 2\|\bar{B}_\alpha\|_{H_2^2(N \setminus B_{\bar{x}_\alpha}(R\mu_\alpha))},
\end{aligned}$$

since $B_{\bar{x}_\alpha}(R\mu_\alpha)$ is invariant by $\bar{S}_\alpha$ according to the definition of $\bar{S}_\alpha$. This proves (38) and as explained above proves (35) and (36) in case (ii). Notice that

$$\bar{B}_{\bar{G},\alpha} = \sum_{i=1}^{k} \bar{B}_{\bar{S}_\alpha,\alpha} \circ \bar{\sigma}_i^{-1}. \tag{39}$$

□

We now define a bubble (B_α) by the relation

$$B_\alpha = \bar{B}_{\bar{G},\alpha} \circ \Pi$$

[see (6)] and $w_\alpha = v_\alpha - B_\alpha = \bar{w}_{\bar{G},\alpha} \circ \Pi$. We claim that the following holds:

Step 1.5

$$w_\alpha \to 0 \text{ weakly in } H_2^2(M), \tag{40}$$

$$DJ_g(B_\alpha) \to 0 \text{ and } DJ_g(w_\alpha) \to 0,$$

$$J_g(w_\alpha) = J_g(v_\alpha) - E(B) + o(1) \tag{41}$$

Proof We first prove that $B_\alpha \to 0$ weakly in $H_2^2(M)$ [which implies (40) since $v_\alpha \to 0$ weakly in $H_2^2(M)$]. Since $(B_\alpha) \subset H_{2,G'}^2(M)$ is bounded in $H_2^2(M)$, it suffices to prove that

$B_\alpha \to 0$ weakly in $L^2_{G'}(M)$. Let $\psi \in L^2_{G'}(M)$ and $\bar\psi \in L^2(N)$ be such that $\psi = \bar\psi \circ \Pi$ in $B_{Gx_0}(2\delta)$. Then, using (35),

$$\int_M B_\alpha \psi \, dv_g = \int_N \bar B_{\bar G,\alpha} \bar\psi \bar v^{-\frac{4}{n-k-4}} dv_{\tilde g} \to 0.$$

We prove in the same way that $DJ_g(B_\alpha) \to 0$. We now prove that

$$DJ_g(w_\alpha) \to 0. \tag{42}$$

Let $\phi \in H^2_{2,G}(M)$, $\delta \in (0, \delta_0/6)$ and $\eta_0 \equiv \eta_{\bar x_0, 3\delta} \in C_c^\infty(B_{Gx_0}(6\delta))$. For α large enough so that $d_{\tilde g}(\bar x_\alpha, \bar x_0) < \delta$ [in particular $supp\ \bar B_\alpha \subset B_{\bar x_\alpha}(2\delta) \subset B_{\bar x_0}(3\delta)$], straightforward computations yield

$$\begin{aligned} DJ_g(w_\alpha)\phi &= DJ_g(w_\alpha)(\eta_0\phi) + DJ_g(w_\alpha)((1-\eta_0)\phi) \\ &= D\bar J_{\tilde g}(\bar w_{\bar G,\alpha})(\overline{\eta_0\phi}) + DJ_g(v_\alpha)((1-\eta_0)\phi) \\ &= o\left(\|\overline{\eta_0\phi}\|_{H_2^2(N)}\right) + o\left(\|(1-\eta_0)\phi\|_{H_2^2(M)}\right) \\ &= o\left(\|\phi\|_{H_2^2(M)}\right) \end{aligned} \tag{43}$$

Now consider $\phi \in H_2^2(M)$ et $\phi_G \in H^2_{2,G}(M)$ defined by

$$\phi_G(x) = \int_G \phi(\sigma x) dm(\sigma),$$

where m is the Haar mesure of G such that $m(G) = 1$. Then, according to what we just did, we have

$$DJ_g(w_\alpha)\phi_G = o(1)\|\phi_G\|_{H_2^2},$$

with

$$\begin{aligned} DJ_g(w_\alpha)\phi_G &= \int_G \left(\int_M \Delta_g w_\alpha \Delta_g(\phi \circ \sigma)\, dv_g \right) dm(\sigma) \\ &\quad - \int_G \left(\int_M f|w_\alpha|^{2^\sharp-2} w_\alpha (\phi\circ\sigma) dv_g \right) dm(\sigma) \\ &= DJ_g(w_\alpha)\phi \end{aligned}$$

and, using Hölder inequality,

$$\begin{aligned} &\|\phi_G\|^2_{H_2^2} \\ &= \int_M \left(\int_G \Delta_g(\phi\circ\sigma) dm(\sigma) \right)^2 dv_g + \int_M \left| \int_G \nabla(\phi\circ\sigma) dm(\sigma) \right|^2 dv_g \\ &\quad + \int_M \left(\int_G (\phi\circ\sigma) dm(\sigma) \right)^2 dv_g \end{aligned}$$

$$\begin{aligned}
&\leq \int_G \int_M (\Delta_g(\phi \circ \sigma))^2 dv_g dm(\sigma) + \int_G \int_M |\nabla(\phi \circ \sigma)|_g^2 dv_g dm(\sigma) \\
&\quad + \int_G \int_M |\phi \circ \sigma|^2 dv_g dm(\sigma) \\
&= \|\phi\|_{H_2^2}^2.
\end{aligned}$$

Hence

$$DJ_g(w_\alpha)\phi = o(1)\|\phi\|_{H_2^2}$$

for any $\phi \in H_2^2(M)$, which proves (42).

It remains to prove (41). We write that

$$J_g(w_\alpha) = \frac{1}{2} \int_{M \backslash B_{Gx_0}(2\delta)} (\Delta_g v_\alpha)^2 dv_g - \frac{1}{2^\sharp} \int_{M \backslash B_{Gx_0}(2\delta)} f|v_\alpha|^{2^\sharp} dv_g + \bar{J}_{\tilde{g}}(\bar{w}_{\bar{G},\alpha}).$$

We then get using (36) and the arguments of the Proof of Step 1.2 that

$$\begin{aligned}
J_g(w_\alpha) &= \frac{1}{2} \int_{M \backslash B_{Gx_0}(2\delta)} (\Delta_g v_\alpha)^2 dv_g - \frac{1}{2^\sharp} \int_{M \backslash B_{Gx_0}(2\delta)} f|v_\alpha|^{2^\sharp} dv_g + \bar{J}_{\tilde{g}}(\bar{v}_\alpha) \\
&\quad - E(B) + o(1) \\
&= J_g(v_\alpha) - E(B) + o(1)
\end{aligned}$$

which proves (41). Note that $v \not\equiv 0$. □

This ends the proof of the Lemma and thus of the H_2^2-decomposition for a (P–S) sequences (u_α) for J_g^α of arbitrary sign. If we assume that $u_\alpha > 0$ for any α, then $u^0 \geq 0$ a.e. since $u_\alpha \to u^0$ weakly in H_2^2 and thus also almost everywhere (up to a subsequence). Moreover, according to Hebey–Robert [11], the $\bar{B}^i$ are bubbles and hence so are the B^i, $1 \leq i \leq k$.

To conclude this section, let us remark that if $f \geq 0$, $k_\infty > 0$ and the h_α's are real numbers with $0 < h_\infty \leq k_\infty^2/4$, then u^0 is smooth and either $u^0 \equiv 0$ or $u^0 > 0$. Indeed, since u^0 is a solution of (E_∞), we have

$$\left(\Delta_g + \frac{k_\infty}{2}\right)^2 u^0 = bu^0, \quad b = fu^{2^\sharp - 2} + \frac{k_\infty^2}{4} - h_\infty.$$

Since $b \in L^{n/4}(M)$, lemma 2.1 in [5] gives that $u^0 \in L^s(M)$ for all $s \geq 1$. Hence, according to the standard regularity theory, $u^0 \in H_4^s(M)$ for all $s \geq 1$. In particular $u^0 \in C^4(M)$. From the maximum principle and noting that

$$\left(\Delta_g + \frac{k_\infty}{2}\right)^2 u^0 \geq 0,$$

we then get that either $u^0 \equiv 0$ or $u^0 > 0$. In both cases, we deduce that $u^0 \in C^\infty(M)$.

3 Proof of the C^0-estimates (14) and (15)

We assume that $f \geq 0$, $k_\infty > 0$ and the h_α's are real numbers with $0 < h_\infty \leq k_\infty^2/4$. Let (u_α) be a bounded sequence of positive solutions of (E_α). We prove in this section the pointwise estimates of the Theorem following Hebey–Robert [11] and Robert [16].

We first prove (14). According to the remark concluding the previous section, we know that $u^0 \in C(M)$, where u^0 is the weak limit in H_2^2 of the u_α's. It thus suffices to prove that there exists $C > 0$ such that for every α and every $x \in M$,

$$R_\alpha(x)^{\frac{n-k-4}{2}} u_\alpha(x) \leq C. \tag{44}$$

Actually, we are going to prove the following stronger result: there exists $C > 0$ such that

$$v_\alpha(x) := R'_\alpha(x)^{\frac{n-k-4}{2}} u_\alpha(x) \leq C \tag{45}$$

for all $x \in M$ and all $\alpha > 0$, where

$$R'_\alpha(x) = \min_{i=1,\dots,l} d_g(G'_i x, G'_i x^i_\alpha)$$

and for all $i \in \{1, \dots, l\}$, the group G'_i is given by hypothesis (H) at the orbit of concentration Gx^i_∞, where $\lim_{\alpha\to+\infty} x^i_\alpha = x^i_\infty$.

We assume by contradiction that there exists $y_\alpha \in M$ such that

$$v_\alpha(y_\alpha) = \max_{x\in M} v_\alpha(x) \to +\infty \tag{46}$$

when $\alpha \to +\infty$ and we let $\mu_\alpha := u_\alpha(y_\alpha)^{-2/(n-k-4)} \to 0$ when $\alpha \to +\infty$. We let $\lim_{\alpha\to+\infty} y_\alpha = y_0$, up to extraction.

We claim that the orbit Gy_0 has minimal dimension k. Indeed, we argue by contradiction and assume that $\dim Gy_0 > k$. As in Step 1.1, we then get that there exists $\delta > 0$ such that $\lim_{\alpha\to+\infty} u_\alpha = u^0$ in $L^{2^\sharp}(B_{Gy_0}(\delta))$. It then follows from (E_α) and standard regularity theory that $\lim_{\alpha\to+\infty} u_\alpha = u^0$ in $C^0(B_{Gy_0}(\delta'))$ for all $\delta' < \delta$. A contradiction with the assumption (46). This proves the claim.

We then let G' be the group given by hypothesis (H) at the point y_0. We let $I_0 = \left\{i \in \{1, \dots, l\} / x^i_\infty \in Gy_0\right\}$ (note that I_0 may be empty). Then, for all $i \in I_0$, we have that $G' = G'_i$. We consider the quotient manifold $N := B_{G'y_0}(\delta)/G'$, where $\delta > 0$ is small and given by (H). Here again, we consider the function $\bar{u}_\alpha(\bar{x}) = u_\alpha(x)$ for $\bar{x} \in N$. We fix $R_0 \in (0, i_{\bar{g}}(\bar{y}_0))$ and we consider the function w_α defined on the Euclidean ball $B_0(R_0\mu_\alpha^{-1})$ by

$$w_\alpha(x) := \mu_\alpha^{\frac{n-k-4}{2}} \bar{u}_\alpha(\exp_{\bar{y}_\alpha}(\mu_\alpha x)).$$

In this expression, the exponential map is taken with respect to the metric $\bar{g}$. For $\rho > 0$ and $x \in B_0(\rho) \subset \mathbb{R}^{n-k}$, we let $z_\alpha \in M$ be such that $G'z_\alpha = \bar{z}_\alpha = \exp_{\bar{y}_\alpha}(\mu_\alpha x)$. Given $i \in I_0$, we get that

$$\begin{aligned} d_g(G'z_\alpha, G'x^i_\alpha) &\geq d_g(G'x^i_\alpha, G'y_\alpha) - d_g(G'y_\alpha, G'z_\alpha) \\ &\geq R'_\alpha(y_\alpha) - d_{\bar{g}}(\bar{y}_\alpha, \bar{z}_\alpha) \\ &\geq R'_\alpha(y_\alpha) - \mu_\alpha|x| \\ &\geq \left(1 - \frac{\rho\mu_\alpha}{R'_\alpha(y_\alpha)}\right) R'_\alpha(y_\alpha). \end{aligned}$$

By definition of y_α and μ_α, we have that $\mu_\alpha R'_\alpha(y_\alpha)^{-1} \to 0$ when $\alpha \to +\infty$, and hence the right-hand-side of the above equation is positive. In case $i \notin I_0$, we get that

$$\lim_{\alpha\to+\infty} d_g(\exp_{\bar{y}_\alpha}(\mu_\alpha x), G'_i x^i_\alpha) = d_g(G' y_0, G'_i x^i_\infty)$$
$$= d_g(G y_0, G x^i_\infty) > 0 \text{ in } C^0_{loc}(\mathbb{R}^{n-k}).$$

Since $R'_\alpha(y_\alpha) \to 0$ when $\alpha \to +\infty$, we then get that

$$R'_\alpha(\exp_{\bar{y}_\alpha}(\mu_\alpha x)) \geq \frac{1}{2}\left(1 - \frac{\rho\mu_\alpha}{R'_\alpha(y_\alpha)}\right) R'_\alpha(y_\alpha) > 0$$

for all $x \in B_0(\rho)$ and all $\alpha > 0$. We can then write for $x \in B_0(\rho)$ that

$$\begin{aligned} w_\alpha(x) &= \frac{\mu_\alpha^{\frac{n-k-4}{2}} v_\alpha(z_\alpha)}{R'_\alpha(\exp_{\bar{y}_\alpha}(\mu_\alpha x))^{\frac{n-k-4}{2}}} \\ &\leq 2^{(n-k-4)/2}\left(1 - \frac{\rho\mu_\alpha}{R'_\alpha(y_\alpha)}\right)^{-\frac{n-k-4}{2}} \frac{u_\alpha(y_\alpha)^{-1} v_\alpha(y_\alpha)}{R'_\alpha(y_\alpha)^{\frac{n-k-4}{2}}} \\ &\leq 2^{(n-k-4)/2}\left(1 - \frac{\rho\mu_\alpha}{R'_\alpha(y_\alpha)}\right)^{-\frac{n-k-4}{2}} \end{aligned}$$

uniformly for $x \in B_0(\rho) \subset \mathbb{R}^{n-k}$ when $\alpha \to +\infty$. Thus the sequence (w_α) is uniformly bounded on every compact subset of $\mathbb{R}^{n-k}$. Let $\bar{g}_\alpha$ be the Riemannian metric on $\mathbb{R}^{n-k}$ defined by

$$\bar{g}_\alpha(x) = \exp^*_{\bar{y}_\alpha} \bar{g}(\mu_\alpha x).$$

Equation (E_α) becomes

$$\Delta_{\bar{g}_\alpha}(\tilde{v}_\alpha \Delta_{\bar{g}_\alpha} w_\alpha) - \mu_\alpha^2 k_\alpha div_{\bar{g}_\alpha}(\tilde{v}_\alpha \nabla w_\alpha) + \mu_\alpha^4 \tilde{h}_\alpha \tilde{v}_\alpha w_\alpha = \tilde{f}_\alpha \tilde{v}_\alpha w_\alpha^{2^\sharp - 1}$$

where $\tilde{h}_\alpha(x) = \bar{h}_\alpha(\exp_{\bar{y}_\alpha}(\mu_\alpha x))$, $\tilde{f}_\alpha(x) = \bar{f}(\exp_{\bar{y}_\alpha}(\mu_\alpha x))$, and $\tilde{v}_\alpha(x) = \bar{v}(\exp_{\bar{y}_\alpha}(\mu_\alpha x))$. Since $\mu_\alpha \to 0$ when $\alpha \to +\infty$, the metric $\bar{g}_\alpha$ converges to the Euclidean metric ξ in $C^2_{loc}(\mathbb{R}^{n-k})$ when $\alpha \to +\infty$. It then follows that, up to extraction, there exists $w \in C^4(\mathbb{R}^{n-k})$ such that

$$\lim_{\alpha\to+\infty} w_\alpha = w \text{ in } C^4_{loc}(\mathbb{R}^{n-k}).$$

Since $w_\alpha(0) = 1$, we get that $w(0) = 1$ and then $w \not\equiv 0$. We let $R > 0$. Since

$$\int_{B_0(R)} w_\alpha^{2^\sharp} dv_{\bar{g}_\alpha} = \int_{B_{\bar{y}_\alpha}(R\mu_\alpha)} \bar{u}_\alpha^{2^\sharp} dv_{\bar{g}} = \int_{B_{G' y_\alpha}(R\mu_\alpha)} \mathrm{Vol}(G'x)^{-1} u_\alpha^{2^\sharp}(x)\, dv_g(x),$$

we get that

$$\lim_{\alpha\to+\infty} \int_{B_{G' y_\alpha}(R\mu_\alpha)} u_\alpha^{2^\sharp} dv_g = \mathrm{Vol}(G y_0) \int_{B_0(R)} w^{2^\sharp} dv_\xi > 0.$$

With the H_2^2 decomposition of Theorem, we then get that

$$\begin{aligned} 1 &\le C \int_{B_{G'y_\alpha}(R\mu_\alpha)} \left(u^0 + \sum_{i=1}^{l} B_\alpha^i + S_\alpha\right)^{2^\sharp} dv_g \\ &\le C \sum_{i=1}^{l} \int_{B_{G'y_\alpha}(R\mu_\alpha)} (B_\alpha^i)^{2^\sharp} dv_g + o(1) \\ &\le C \sum_{i\in I_0} \int_{B_{G'y_\alpha}(R\mu_\alpha)} (B_\alpha^i)^{2^\sharp} dv_g + o(1) \\ &\le C \sum_{i\in I_0} \int_{B_{\bar{y}_\alpha}(R\mu_\alpha)} (\bar{B}_\alpha^i)^{2^\sharp} dv_{\bar{g}} + o(1) \end{aligned}$$

where, here again, we have taken the quotient w.r.t. the group G': this is licit since we work at the points x_α^i such that $x_\infty^i = y_0$. We can then prove exactly as in Saintier [12] that the right-hand side of this inequality goes to 0 as $\alpha \to +\infty$. A contradiction, and then (45) holds.

We claim that (44) holds. Indeed, the proof goes by contradiction and we consider a sequence of points (y_α) such that $\lim_{\alpha\to+\infty} R_\alpha(x)^{\frac{n-k-4}{2}} u_\alpha(y_\alpha) = +\infty$. With arguments similar to the ones above, we get that $\lim_{\alpha\to+\infty} y_\alpha = y_0 \in M$ is such that Gy_0 is an orbit of concentration of the u_α's. Hypothesis (H) yields a group G' that satisfies (H1) and (H2). With (H2), we get that $d_g(Gy_\alpha, Gx_\alpha^i) \le d_g(G'y_\alpha, G'x_\alpha^i)$ for the i's such that $\lim_{\alpha\to+\infty} x_\alpha^i \in Gy_0$. Studying separately the remaining i's, we get that $R_\alpha(y_\alpha) \le cR'_\alpha(y_\alpha)$ and we apply (45) to get a contradiction with our initial assumption. This proves that (44) holds.

The proof of (15) goes the same way: if (15) is not satisfies, then we construct a sequence (y_α) which traducts it. We blow-up u_α at y_α and we get a contradiction as above.

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