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Asymptotics of best Sobolev constants on thin manifolds

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ABSTRACT

We study the asymptotic behaviour of best Sobolev constants on a compact manifold with boundary that we contract in k directions or to a point. We find in the limit best Sobolev constants for weighted Sobolev spaces of the limit manifold.

The study of thin structures is of great importance in physics (e.g. the study of thin pipes or plates), but complicated from a mathematical point of view due to their two-scale nature. One way to overcome this difficulty is trying to reduce the dimension of the problem by looking at the limit problem on the structure we get as the thickness goes to zero. Independently, Sobolev inequalities and the associated best Sobolev constants are known to be relevant for the study of nonlinear equations. We are thus led to the study of the asymptotic behaviour of these best Sobolev constants on thin domains as the thickness goes to zero. This problem has already been considered in the subcritical case in [3] and [10] for open subsets of $\mathbb{R}^n$, and in [11] for thin pipes of $\mathbb{R}^3$. In this paper, we extend these results to the Riemannian setting in both the subcritical and critical case.

We now describe precisely our problem. Let $(N, \bar{g})$ be a Riemannian manifold of dimension n, and M an embedded compact manifold of N without boundary of codimension $k \in [1, n-1]$ which does not intersect the boundary of N if any. We will deal with the case $k = n$, i.e. when M is a point, later. We equipped M with the induced Riemannian metric $g = i^*\bar{g}$, where $i : M \hookrightarrow N$ is the canonical injection. Given a point $y \in N$ and a tangent vector $Y \in T_yN$, we denote by $t \to \gamma_{y,Y}(t)$ the geodesic (for $\bar{g}$) starting from y with velocity Y, i.e. $\gamma_{y,Y}(0) = y$ and $\dot{\gamma}_{y,Y}(0) = Y$. We assume that there exists an orthonormal family $\{\nu_1, \dots, \nu_k\}$ of smooth vector-fields on N such that $\nu_i(x) \perp T_xM$ for every $x \in M$ and $i = 1, \dots, k$. In the case $k = 1$ with N orientable, this amounts to assume that M

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is orientable. Given $x \in M$ and $t > 0$, we denote by $S_x(t) \subset T_xM^\perp$ the sphere of radius t and center x given by

$$S_x(t) = \left\{ \nu = \sum_{i=1}^{k} t_i \nu_i(x),\ t_1^2 + \cdots + t_k^2 = t^2 \right\}.$$

Given a continuous function $r : \bigcup_{x\in M} S_x(1) \to (0, +\infty)$, we then define, for a sufficiently small $\varepsilon_0 > 0$, the hypersurfaces M_t, $0 < t \leqslant \varepsilon_0$, by

$$M_t = \left\{ \gamma_{x,\nu(x)}\big(r\big(x, \nu(x)\big)t\big),\ x \in M,\ \nu(x) \in S_x(1) \right\}.$$

Then the M_t does not intersect ∂N for $\varepsilon_0 > 0$ small enough. In the case $k = 1$ and $r \equiv 1$, $S_x(1) = \{\nu_1(x), -\nu_1(x)\}$, and M_t is composed of two copies of M. We also consider the open subset M^ε, $\varepsilon \in (0, \varepsilon_0)$, of N defined by

$$M^\varepsilon = \bigcup_{0\leqslant t<\varepsilon} M_t = \left\{ \gamma_{x,\nu(x)}\big(r\big(x, \nu(x)\big)t\big),\ x \in M,\ \nu(x) \in S_x(1),\ 0 \leqslant t < \varepsilon \right\}.$$

Then M^ε is an open subset of N with boundary $\partial M^\varepsilon = M_\varepsilon$.

We now deal with the case $k = n$, i.e. the case where M is a point that we denote by $0 \in N$. Let Ω be a smooth connected open subset of N containing 0 included in some geodesic ball $B_0(\delta)$ with δ less than the injectivity radius of $(N, \bar{g})$ at 0. We contract Ω at 0 by considering the open subsets $\Omega_\varepsilon := \exp_0(\varepsilon\tilde{\Omega})$, $\varepsilon > 0$, where $\exp_0$ denotes the exponential map at 0, and $\tilde{\Omega} = \exp_0^{-1}(\Omega) \subset T_0N \approx \mathbb{R}^n$.

Given $p \in (1, n)$, we denote by $H_1^p(M^\varepsilon)$ the Sobolev space of the functions in $L^p(M^\varepsilon)$ such that their gradient is also in $L^p(M^\varepsilon)$. It is well known that $H_1^p(M^\varepsilon) \hookrightarrow L^q(\partial M^\varepsilon)$ continuously for any $q \in [1, p_*]$, where $p_* := p(n-1)/(n-p)$. Moreover this embedding is compact when $q < p_*$. We let $S^\varepsilon(p, q)$ be the best constant for this embedding, namely

$$S^\varepsilon(p,q) = \inf_{u\in H_1^p(M^\varepsilon),\, u\not\equiv 0 \text{ on } \partial M^\varepsilon} \frac{\int_{M^\varepsilon}(|\nabla u|_{\bar{g}}^p + |u|^p)\, dv_{\bar{g}}}{(\int_{\partial M^\varepsilon} |u|^q\, d\sigma_{\bar{g}})^{p/q}} > 0,$$

when $k \leqslant n-1$, and where $d\sigma_{\bar{g}}$ denotes the volume element induced by $\bar{g}$ on ∂M^ε. In the case $k = n$, we define $S^\varepsilon(p, q)$ in the same way but with Ω_ε (resp. $\partial\Omega_\varepsilon$) in place of M^ε (resp. ∂M^ε).

The aim of this paper is to describe the asymptotic behaviour of $S^\varepsilon(p, q)$ as $\varepsilon \to 0$. This problem was solved in [3] (resp. [4]) in the case of an open subset of $\mathbb{R}^n$ with $q < p_*$ and $k \leqslant n-1$ (resp. $k = n$).

Before stating our result, we let, when $k \leqslant n-1$, $K(p, q)$ be the best constant for the embedding of $H_1^p(M)$ into $L^q(M)$ in the sense that

$$K(p,q) = \inf_{u\in H_1^p(M),\, u\not\equiv 0} \frac{\int_M (|\nabla u|_g^p + |u|^p)|B_x(r(x))|_\xi\, dv_g(x)}{(\int_M |u|^q |S_x(r(x))|_\xi\, dv_g(x))^{p/q}} > 0, \tag{1}$$

where $|S_x(r(x))|_\xi$ (resp. $|B_x(r(x))|_\xi$) denotes the volume for the Euclidean metric ξ of the sphere-like subset $S_x(r(x)) \subset T_xM^\perp$ (resp. the ball-like subset $B_x(r(x)) \subset T_xM^\perp$) given in polar coordinate by $r = r(x, \theta)$ (resp. $r \leqslant r(x, \theta)$), $\theta \in S_x(1)$.

The result is the following:

Theorem 0.1. *For $1 < p < n$ and $1 \leqslant q \leqslant p_*$, with $p < n-k$ if $q = p_*$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{k(q-p)+p}{q}} S^\varepsilon(p,q) = K(p,q), \tag{2}$$

when $k \leqslant n-1$, and for $1 < p < n$ and $1 \leqslant q \leqslant p^*$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{k(q-p)+p}{q}} S^{\varepsilon}(p,q) = \frac{|\tilde{\Omega}|_{\xi}}{|\partial \tilde{\Omega}|_{\xi}^{p/q}}$$

when $k = n$. Moreover, the extremals for $S^{\varepsilon}(p,q)$, suitably normalized and rescaled, converge to an extremal for $K(p,q)$ as $\varepsilon \to 0$ when $k \leqslant n-1$, and to a constant when $k = n$.

For example, if r is constant and $k \leqslant n-1$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{k(q-p)+p}{q}} S^{\varepsilon}(p,q) = \frac{r^{k(1-p/q)+p/q} \omega_{k-1}^{1-p/q}}{k} \inf_{u \in H_1^p(M),\, u \not\equiv 0} \frac{\int_M (|\nabla u|_g^p + |u|^p)\, dv_g}{(\int_M |u|^q \, dv_g)^{p/q}}$$
$$= \frac{r^{k(1-p/q)+p/q} \omega_{k-1}^{1-p/q}}{k} \operatorname{Vol}_g(M)^{1-p/q} \quad \text{if } q \leqslant p,$$

where ω_{k-1} is the volume of the standard sphere of $\mathbb{R}^k$. The second equality follows by taking the constant function equal to 1 as a test-function to get the $\leqslant$ inequality, and by applying Hölder's inequality to $\int_M |u|^q \, dv_g$ to get the converse one.

As an application, consider the problem of finding a conformal metric to $\bar{g}$ with zero scalar curvature in the interior of M^{ε} and constant mean curvature on ∂M^{ε}. To prove the existence of such a metric, it suffices to show that

$$\lambda_{\varepsilon} := \inf_{u \in H_1^2(M^{\varepsilon}),\, u \not\equiv 0 \text{ on } \partial M^{\varepsilon}} \frac{\int_{M^{\varepsilon}} (|\nabla u|_{\bar{g}}^2 + h|u|^2)\, dv_{\bar{g}} + \int_{\partial M^{\varepsilon}} k u^2 \, d\sigma_{\bar{g}}}{(\int_{\partial M^{\varepsilon}} |u|^{2_*} \, d\sigma_{\bar{g}})^{2/2_*}} < \tilde{K}(n,2), \tag{3}$$

for some suitable smooth functions h and k, and where $\tilde{K}(n,2)$ is defined by (24) (see [8]). Various works have been devoted to this problem. Existence of solutions are usually proved under geometric conditions on ∂M^{ε}. In contrast, the first part of the proof of Theorem 0.1 can easily be adapted to λ_{ε} to show that $\limsup_{\varepsilon \to 0} \varepsilon^{-(k-1)/(n-1)} \lambda_{\varepsilon} = 0$. Hence the problem of finding such a metric on a sufficiently thin manifold always has a solution.

We can also describe in a similar way the asymptotic behaviour of the best Sobolev constant $S_{\varepsilon}(p,q)$ corresponding to the embedding of $H_1^p(M^{\varepsilon})$ into $L^q(M^{\varepsilon})$, namely

$$S_{\varepsilon}(p,q) = \inf_{u \in H_1^p(M^{\varepsilon}) \setminus \{0\}} \frac{\int_{M^{\varepsilon}} (|\nabla u|_{\bar{g}}^p + |u|^p)\, dv_{\bar{g}}}{(\int_{M^{\varepsilon}} |u|^q \, dv_{\bar{g}})^{p/q}} > 0$$

when $k \leqslant n-1$. In the case $k = n$, we define $S_{\varepsilon}(p,q)$ in the same way but with Ω_{ε} in place of M^{ε}. The result is

Theorem 0.2. *For $1 < p < n$ and $1 \leqslant q \leqslant p^*$, with $p < n-k$ if $q = p^*$, we have*

$$\lim_{\varepsilon \to 0} \varepsilon^{k(\frac{p}{q}-1)} S_{\varepsilon}(p,q) = \inf_{u \in H_1^p(M),\, u \not\equiv 0} \frac{\int_M (|\nabla u|_g^p + |u|^p) |B_x(r(x))|_{\xi} \, dv_g(x)}{(\int_M |u|^q |B_x(r(x))|_{\xi} \, dv_g(x))^{p/q}}, \tag{4}$$

when $k \leqslant n-1$, and for $1 < p < n$ and $1 \leqslant q \leqslant p^*$,

$$\lim_{\varepsilon \to 0} \varepsilon^{n(p/q-1)} S_{\varepsilon}(p,q) = |\tilde{\Omega}|_{\xi}^{1-p/q}$$

when $k=n$. Moreover, the extremals for $S_\varepsilon(p,q)$, suitably normalized and rescaled, converge to an extremal for the minimization problem of the right-hand side of (4) as $\varepsilon\to 0$ when $k\leqslant n-1$, and to a constant when $k=n$.

This result generalizes both [10] and [11] to the Riemannian setting.

1. Proof of Theorem 0.1

We first prove the theorem in the case $k\leqslant n-1$. Let $(x_1,\dots,x_{n-k})$ be a coordinate system of M at a point $0\in M$. The existence of the ν_i's allows us to consider global polar coordinates $(x_1,\dots,x_{n-k},\theta_1,\dots,\theta_{k-1},t)$ around 0 in N, where the θ_i's form a coordinate system of each $S_x(1)$, and t is the parameter of a geodesic $t\to\gamma_{x,r(x,\theta)\theta}(t)$, $\theta\in S_x(1)$ (which has constant speed $r(x,\theta)^2$), so that the M_t's are the level-sets of t. Since $S_x(t)\subset T_xM^\perp$ for any $x\in M$ and $t>0$, and a geodesic $t\to\gamma_{x,r(x,\theta)\theta}(t)$ intersects perpendicularly the M_t's, the metric $\bar g$ in these coordinates takes the form

$$\bar g(x,t,\theta)=g_{ij}(x,t,\theta)\,dx^i\,dx^j+r(x,\theta)^2\,dt^2+t^2 f_{ij}(x,\theta)\,d\theta^i\,d\theta^j,$$

for some smooth functions g_{ij}, f_{ij}, where $\sigma_t(x):=t^2 f_{ij}(x,\theta)\,d\theta^i d\theta^j$ is the metric on the geodesic sphere $\bar S_x(t):=\exp_x(S_x(t))$, exp being the exponential mapping for $\bar g$, and $g_t:=\bar g(.,t,.)=g_{t,.}+\sigma_t$ is the metric induced by $\bar g$ on M_t, with $g_{t,\theta}(x)=g_{ij}(x,t,\theta)\,dx^i\,dx^j$. Note that $g_0=g$ the metric induced on M. We also let $g^\perp_{t,\theta}(x)=r(x,\theta)^2dt^2+\sigma_t(x)$.

Let $\varepsilon\in(0,\varepsilon_0)$ and $R_\varepsilon:M^{\varepsilon_0}\to M^\varepsilon$ be defined by

$$R_\varepsilon\big(\gamma_{x,r(x,\nu(x))\nu(x)}(t)\big)=\gamma_{x,r(x,\nu(x))\nu(x)}(\varepsilon t/\varepsilon_0),\quad \nu(x)\in S_x(1),\ 0\leqslant t<\varepsilon_0,$$

i.e. $R_\varepsilon(x,t,\theta)=(x,\varepsilon t/\varepsilon_0,\theta)$ in coordinates. We then have for $(x,t,\theta)\in M^{\varepsilon_0}$ that

$$\begin{aligned}\big(R^*_\varepsilon\bar g\big)(x,t,\theta)&=g_{ij}(x,\varepsilon t/\varepsilon_0,\theta)\,dx^i\,dx^j+(\varepsilon/\varepsilon_0)^2 r(x,\theta)^2\,dt^2+(\varepsilon/\varepsilon_0)^2t^2 f(x,\theta)\,d\theta^2\\&=g_{\varepsilon t/\varepsilon_0,\theta}(x,\theta)+(\varepsilon/\varepsilon_0)^2 g^\perp_{t,\theta}(x).\end{aligned}\tag{5}$$

In particular,

$$dv_{(R^*_\varepsilon\bar g)(x,t,\theta)}=(\varepsilon/\varepsilon_0)^k\,dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\,dv_{g^\perp_{t,\theta}}.$$

Given $\bar u,\bar\phi\in H^p_1(M^\varepsilon)$, we let u and ϕ be the functions defined on M^{ε_0} by

$$u=\bar u\circ R_\varepsilon,\qquad \phi=\bar\phi\circ R_\varepsilon.$$

We then have

$$\int\limits_{M^\varepsilon}|\bar u|^{p-2}\bar u\bar\phi\,dv_{\bar g}=\int\limits_{M^{\varepsilon_0}}|u|^{p-2}u\phi\,dv_{R^*_\varepsilon\bar g}=(\varepsilon/\varepsilon_0)^k\int\limits_{M^{\varepsilon_0}}|u|^{p-2}u\phi\,dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\,dv_{g^\perp_{t,\theta}},\tag{6}$$

$$\begin{aligned}\int\limits_{M^\varepsilon}|\nabla\bar u|^{p-2}_{\bar g}(\nabla\bar u,\nabla\bar\phi)_{\bar g}\,dv_{\bar g}&=(\varepsilon/\varepsilon_0)^k\int\limits_{M^{\varepsilon_0}}|\nabla u|^{p-2}_{R^*_\varepsilon\bar g}(\nabla u,\nabla\phi)_{R^*_\varepsilon\bar g}\,dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\,dv_{g^\perp_{t,\theta}}\\&=(\varepsilon/\varepsilon_0)^k\int\limits_{M^{\varepsilon_0}}\big\{(\varepsilon/\varepsilon_0)^{-2}|\nabla_{t,\theta}u|^2_{g^\perp_{t,\theta}}+|\nabla_x u|^2_{g_{\varepsilon t/\varepsilon_0,\theta}}\big\}^{\frac{p-2}{2}}\\&\quad\times\big\{(\varepsilon/\varepsilon_0)^{-2}(\nabla_{t,\theta}u,\nabla_{t,\theta}\phi)_{g^\perp_{t,\theta}}+(\nabla_x u,\nabla_x\phi)_{g_{\varepsilon t/\varepsilon_0,\theta}}\big\}\,dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\,dv_{g^\perp_{t,\theta}},\end{aligned}\tag{7}$$

and eventually,

$$\int_{\partial M^\varepsilon} |\bar u|^{q-2}\bar u\bar\phi\, d\sigma_{\bar g} = \int_{M_\varepsilon} |\bar u|^{q-2}\bar u\bar\phi\, dv_{g_\varepsilon} = (\varepsilon/\varepsilon_0)^{k-1}\int_{M_{\varepsilon_0}} |u|^{q-2}u\phi\, d\sigma_{\varepsilon_0}\, dv_{g_{\varepsilon,\theta}}, \tag{8}$$

since $(R_\varepsilon^*\bar g)(x,\varepsilon_0,\theta) = \bar g(x,\varepsilon,\theta) = g_{\varepsilon,\theta} + (\varepsilon/\varepsilon_0)^2\sigma_{\varepsilon_0}$ on M_{ε_0}. Taking $\phi = u$ in (6), (7) and (8), it follows that

$$\begin{aligned}
&(\varepsilon/\varepsilon_0)^{-\frac{k(q-p)+p}{q}} S^\varepsilon(p,q)\\
&\quad= \inf_{u\in H_1^p(M^{\varepsilon_0}),\, u\not\equiv 0 \text{ on } M_{\varepsilon_0}} \frac{\int_{M^{\varepsilon_0}} (\{(\varepsilon/\varepsilon_0)^{-2}|\nabla_{t,\theta}u|^2_{g^\perp_{t,\theta}} + |\nabla_x u|^2_{g_{\varepsilon t/\varepsilon_0,\theta}}\}^{\frac{p}{2}} + |u|^p)\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\, dv_{g^\perp_{t,\theta}}}{(\int_{M_{\varepsilon_0}} |u|^q\, d\sigma_{\varepsilon_0}\, dv_{g_{\varepsilon,\theta}})^{p/q}}\\
&\quad\leqslant \inf_{u\in \tilde H_1^p(M^{\varepsilon_0}),\, u\not\equiv 0 \text{ on } M_{\varepsilon_0}} \frac{\int_{M^{\varepsilon_0}} (|\nabla_x u|^p_{g_{\varepsilon t/\varepsilon_0,\theta}} + |u|^p)\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\, dv_{g^\perp_{t,\theta}}}{(\int_{M_{\varepsilon_0}} |u|^q\, d\sigma_{\varepsilon_0}\, dv_{g_{\varepsilon,\theta}})^{p/q}},
\end{aligned} \tag{9}$$

where $\tilde H_1^p(M^{\varepsilon_0})$ denotes the subspace of $H_1^p(M^{\varepsilon_0})$ of (t,θ)-independent functions. We identify $\tilde H_1^p(M^{\varepsilon_0})$ with $H_1^p(M)$. Since $\bar g$ is continuous, we get

$$\limsup_{\varepsilon\to 0}(\varepsilon/\varepsilon_0)^{-\frac{k(q-p)+p}{q}} S^\varepsilon(p,q) \leqslant \inf_{u\in \tilde H_1^p(M^{\varepsilon_0}),\, u\not\equiv 0 \text{ on } M_{\varepsilon_0}} \frac{\int_{M^{\varepsilon_0}} (|\nabla_x u|^p_g + |u|^p)\, dv_g\, dv_{g^\perp_{t,\theta}}}{(\int_{M_{\varepsilon_0}} |u|^q\, dv_g\, d\sigma_{\varepsilon_0})^{p/q}}. \tag{10}$$

Independently, for a (t,θ)-independent function v,

$$\begin{aligned}
\int_{M^{\varepsilon_0}} v\, dv_g\, dv_{g^\perp_{t,\theta}} &= \int_M v(x)\Big(\int_{\exp_x(B_x(\varepsilon_0 r(x)))} dv_{g^\perp_{t,\theta}}\Big) dv_g(x)\\
&= \int_M v(x)\Big(\int_{B_x(\varepsilon_0 r(x))} dv_{\exp_x^*\bar g}\Big) dv_g(x),
\end{aligned}$$

where $B_x(\varepsilon_0 r(x)) \subset T_xM^\perp$ is defined in polar coordinate by $r = \varepsilon_0 r(x,\theta)$, $\theta \in S_x(1)$, and $\exp_x$ denotes the exponential map at x (for the metric $\bar g$) restricted to $T_xM^\perp$. Since $dv_{(\exp_x^*\bar g)(y)} \to dv_\xi$ as $y \to 0$ in $T_xM^\perp$, where ξ denotes the Euclidean metric, we have as $\varepsilon_0 \to 0$ that

$$\int_{B_x(\varepsilon_0 r(x))} dv_{\exp_x^*\bar g} \sim \varepsilon_0^k \big|B_x\big(r(x)\big)\big|_\xi,$$

so that

$$\int_{M^{\varepsilon_0}} v\, dv_g\, dv_{g^\perp_{t,\theta}} \sim \varepsilon_0^k \int_M v(x)\big|B_x\big(r(x)\big)\big|_\xi\, dv_g(x) \tag{11}$$

for any (t,θ)-independent function v. In particular, given $u \in \tilde H_1^p(M^{\varepsilon_0})$, applying (11) to $v = |\nabla_x u|_g^p + |u|^p$ gives

$$\int_{M^{\varepsilon_0}} \big(|\nabla_x u|^p_g + |u|^p\big)\, dv_g\, dv_{g^\perp_{t,\theta}} \sim \varepsilon_0^k \int_M \big(|\nabla u|^p_g + |u|^p\big)\big|B_x\big(r(x)\big)\big|_\xi\, dv_g \tag{12}$$

as $\varepsilon_0 \to 0$. In the same way we also have that

$$\int_{M_{\varepsilon_0}} |u|^q \, dv_g \, d\sigma_{\varepsilon_0} \sim \varepsilon_0^{k-1} \int_M |u|^q \left|S_x\big(r(x)\big)\right|_\xi \, dv_g(x) \tag{13}$$

as $\varepsilon_0 \to 0$. Plugging (12) and (13) into (10) yields

$$\limsup_{\varepsilon \to 0} \varepsilon^{-\frac{k(q-p)+p}{q}} S^\varepsilon(p,q) \leqslant K(p,q). \tag{14}$$

We now prove the converse inequality by analyzing the behaviour as $\varepsilon \to 0$ of the normalized extremal of $S^\varepsilon(p,q)$. Let $\varepsilon \in (0, \varepsilon_0)$. We first assume that $q < p_*$. Then a standard variational argument gives the existence of a nonnegative function $\bar{u}_\varepsilon \in H_1^p(M^\varepsilon)$ normalized by $\int_{\partial M^\varepsilon} \bar{u}_\varepsilon^q \, d\sigma_{\bar{g}} = \varepsilon^{k-1}$ which realizes the infimum in the definition of $S^\varepsilon(p,q)$. Then

$$\int_{M^\varepsilon} \big(|\nabla \bar{u}_\varepsilon|_{\bar{g}}^{p-2} (\nabla \bar{u}_\varepsilon, \nabla \bar{\phi}_\varepsilon)_{\bar{g}} + \bar{u}_\varepsilon^{p-1} \bar{\phi}_\varepsilon\big) \, dv_{\bar{g}} = \varepsilon^{(k-1)\frac{p-q}{q}} S^\varepsilon(p,q) \int_{\partial M^\varepsilon} \bar{u}_\varepsilon^{q-1} \bar{\phi}_\varepsilon \, d\sigma_{\bar{g}} \tag{15}$$

for every $\bar{\phi}_\varepsilon \in H_1^p(M^\varepsilon)$. In particular, with $\bar{\phi}_\varepsilon = \bar{u}_\varepsilon$, we get

$$\varepsilon^{\frac{(k-1)p}{q}} S^\varepsilon(p,q) = \int_{M^\varepsilon} \big(|\nabla \bar{u}_\varepsilon|_{\bar{g}}^p + \bar{u}_\varepsilon^p\big) \, dv_{\bar{g}}. \tag{16}$$

Let u_ε be the function defined on M^{ε_0} by $u_\varepsilon = \bar{u}_\varepsilon \circ R_\varepsilon$. In view of (6)–(7), (16) can be rewritten as

$$\begin{aligned} &\int_{M^{\varepsilon_0}} \Big(\big\{(\varepsilon/\varepsilon_0)^{-2} |\nabla_{t,\theta} u_\varepsilon|^2_{g^\perp_{t,\theta}} + |\nabla_x u_\varepsilon|^2_{g_{\varepsilon t/\varepsilon_0,\theta}}\big\}^{\frac{p}{2}} + u_\varepsilon^p\Big) \, dv_{g_{\varepsilon t/\varepsilon_0,\theta}} \, dv_{g^\perp_{t,\theta}} \\ &\quad = \varepsilon_0^k \varepsilon^{-\frac{k(q-p)+p}{q}} S^\varepsilon(p,q) = O(1), \end{aligned} \tag{17}$$

where the last equality follows from (14). Since $\bar{M}^{\varepsilon_0} = \bigcup_{0 \leqslant t \leqslant \varepsilon_0} M_t$ is compact and the g_{ij}'s are continuous, there exists a positive constant $C > 0$ such that

$$C^{-1} \, dx^i \, dx^j \geqslant g_{t,\theta}(x) = g_{ij}(x,t,\theta) \, dx^i \, dx^j \geqslant C \, dx^i \, dx^j$$

in the sense of bilinear forms. Thus

$$C'^{-1} g_{t,\theta}(x) \geqslant g_{\varepsilon t/\varepsilon_0,\theta}(x) \geqslant C' g_{t,\theta}(x) \tag{18}$$

in the sense of bilinear forms for any $(x,\theta,t) \in \bar{M}^{\varepsilon_0}$. Hence

$$\int_{M^{\varepsilon_0}} \Big(\big\{(\varepsilon/\varepsilon_0)^{-2} |\nabla_{t,\theta} u_\varepsilon|^2_{g^\perp_{t,\theta}} + |\nabla_x u_\varepsilon|^2_{g_{t,\theta}}\big\}^{\frac{p}{2}} + u_\varepsilon^p\Big) \, dv_{\bar{g}} = O(1). \tag{19}$$

It follows that (u_ε) is bounded in $H_1^p(M^{\varepsilon_0})$. We then deduce the existence of a function $u \in H_1^p(M^{\varepsilon_0})$ such that, up to a subsequence, $u_\varepsilon \to u$ weakly in $H_1^p(M^{\varepsilon_0})$, strongly in $L^p(M^{\varepsilon_0})$ and in $L^q(M_{\varepsilon_0})$, and a.e. In particular, $u \geqslant 0$ a.e.

Since $u_\varepsilon \to u$ strongly in $L^p(M^{\varepsilon_0})$ and in $L^q(M_{\varepsilon_0})$, we also have that $\nabla u_\varepsilon \to \nabla u$ weakly in $L^p(M^{\varepsilon_0})$. As a consequence, $\nabla_x u_\varepsilon \to \nabla_x u$ and $\nabla_{t,\theta} u_\varepsilon \to \nabla_{t,\theta} u$ weakly in L^p. In particular, in view of (19),

$$\int_{M^{\varepsilon_0}} |\nabla_{t,\theta} u|^p_{g^\perp_{t,\theta}}\, dv_{\bar g} \leqslant \liminf_{\varepsilon\to 0} \int_{M^{\varepsilon_0}} |\nabla_{t,\theta} u_\varepsilon|^p_{g^\perp_{t,\theta}}\, dv_{\bar g} = 0. \tag{20}$$

It follows that u does not depend on (t,θ), i.e. $u = u(x)$. Independently, since $g_{ij}(x, \varepsilon t/\varepsilon_0, \theta) \to g_{ij}(x)$ as $\varepsilon \to 0$ uniformly in (x,t,θ), and $u_\varepsilon \to u$ in $L^p(M^{\varepsilon_0})$, we have that

$$\lim_{\varepsilon\to 0} \int_{M^{\varepsilon_0}} u_\varepsilon^p\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\, dv_{g^\perp_{t,\theta}} = \int_{M^{\varepsilon_0}} u^p\, dv_g\, dv_{g^\perp_{t,\theta}}.$$

Moreover, since $\nabla_x u_\varepsilon \to \nabla_x u$ weakly in L^p,

$$\liminf_{\varepsilon\to 0} \int_{M^{\varepsilon_0}} |\nabla_x u_\varepsilon|^p_{g_{\varepsilon t/\varepsilon_0,\theta}}\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\, dv_{g^\perp_{t,\theta}} = \liminf_{\varepsilon\to 0} \int_{M^{\varepsilon_0}} |\nabla_x u_\varepsilon|^p_g\, dv_g\, dv_{g^\perp_{t,\theta}} \geqslant \int_{M^{\varepsilon_0}} |\nabla u|^p_g\, dv_g\, dv_{g^\perp_{t,\theta}}.$$

Passing to the limit in (17), we obtain

$$\int_{M^{\varepsilon_0}} \left(|\nabla u|^p_g + u^p\right) dv_g\, dv_{g^\perp_{t,\theta}} \leqslant \liminf_{\varepsilon\to 0} \varepsilon_0^k \varepsilon^{-\frac{k(q-p)+p}{q}} S^\varepsilon(p,q).$$

Since u is (t,θ)-independent, we eventually get in view of (11) that

$$\int_M \left(|\nabla u|^p_g + u^p\right) \left|B_x\big(r(x)\big)\right|_\xi dv_g \leqslant \liminf_{\varepsilon\to 0} \varepsilon^{-\frac{k(q-p)+p}{q}} S^\varepsilon(p,q). \tag{21}$$

Recalling the normalization of $\bar u_\varepsilon$, (8) with $\bar u = \bar\phi = \bar u_\varepsilon$ gives

$$\varepsilon_0^{1-k} \int_{M_{\varepsilon_0}} u_\varepsilon^q\, d\sigma_{\varepsilon_0}\, dv_{g_{\varepsilon,\theta}} = 1. \tag{22}$$

Passing to the limit $\varepsilon \to 0$ and then using the (t,θ)-invariance of u as previously, we obtain

$$1 = \varepsilon_0^{1-k} \int_{M_{\varepsilon_0}} u^q\, dv_g\, d\sigma_{\varepsilon_0} \sim \int_M u^q \left|S_x\big(r(x)\big)\right|_\xi dv_g$$

as $\varepsilon_0 \to 0$. Inserting this into (21) yields

$$K(p,q) \leqslant \frac{\int_M (|\nabla u|^p_g + u^p)|\bar B_x(r(x))|_\xi\, dv_g}{(\int_M u^q |S_x(r(x))|_\xi\, dv_g)^{p/q}} \leqslant \liminf_{\varepsilon\to 0}\ \varepsilon^{-\frac{k(q-p)+p}{q}} S^\varepsilon(p,q).$$

This together with (14) proves (2) in the subcritical case $q < p_*$.

We now assume that $q = p_*$. The only difficulty in proving (2) in the critical case comparing to the subcritical one lies in the existence of the u_ε's and in their strong convergence to u in $L^{p_*}(\partial M^{\varepsilon_0})$. All the other steps of the proof are identical.

According to [12] (see also [8]), $S^\varepsilon(p,p_*)$ is attained as soon as

$$S^\varepsilon(p,p_*) < \tilde K(n,p), \tag{23}$$

where $\tilde K(n,p)$ denotes the best constant for the embedding of $D_1^p(\mathbb{R}_n^+)$ into $L^{p_*}(\partial\mathbb{R}_n^+)$, namely

$$\tilde K(n,p) = \inf_{u\in L^{p_*}(\partial\mathbb{R}_n^+)\setminus\{0\},\ \nabla u\in L^p(\mathbb{R}_n^+)} \frac{\int_{\mathbb{R}_n^+}|\nabla u|^p\,dx}{(\int_{\partial\mathbb{R}_n^+}|u|^{p_*}\,d\sigma)^{p/p_*}} > 0. \tag{24}$$

The value of $\tilde K(n,p)$ is explicitely known (see [8] for $p=2$ and [6] for the general case $p\in(1,n)$). According to (14), which is still valid when $q=p_*$, this condition holds for small ε. This proves the existence of the $\bar u_\varepsilon$. We will now prove the strong convergence of u_ε to u in $H_1^p(M^{\varepsilon_0})$ for ε_0 sufficiently small.

Consider on M^{ε_0} the metric $\hat g(x,t,\theta) = g_{ij}(x)\,dx^i\,dx^j + g^\perp_{t,\theta}(x)$. We have $C^{-1}\hat g \leqslant \bar g \leqslant C\hat g$ in the sense of bilinear forms. Hence (u_ε) is bounded in $H_1^p(M^{\varepsilon_0},\hat g)$ and thus converges to some $\hat u$ weakly in $H_1^p(M^{\varepsilon_0},\hat g)$ and strongly in $L^p(M^{\varepsilon_0},\hat g)$. Since $L^p(M^{\varepsilon_0},\hat g) = L^p(M^{\varepsilon_0},\bar g)$ and $u_\varepsilon \to u$ strongly in $L^p(M^{\varepsilon_0},\bar g)$, we have $u=\hat u$. Independently, according to [2], for any $\eta>0$ there exists $C_\eta>0$ such that

$$\Big(\int_{\partial M^{\varepsilon_0}} |v|^{p_*}\,dv_g\,d\sigma_{\varepsilon_0}\Big)^{p/p_*} \leqslant \big(\tilde K(n,p)^{-1}+\eta\big)\int_{M^{\varepsilon_0}} \big(|\nabla_{t,\theta}v|^2_{g^\perp_{t,\theta}} + |\nabla_x v|_g^2\big)^{p/2}\,dv_g\,dv_{g^\perp_{t,\theta}} + C_\eta\int_{M^{\varepsilon_0}}|v|^p\,dv_g\,dv_{g^\perp_{t,\theta}} \tag{25}$$

for every $v\in H_1^p(M^{\varepsilon_0})$. Using Lions' concentration–compactness principle [9], we then deduce the existence of two measures μ and ν supported in $\partial M^{\varepsilon_0}$, a sequence of points $(p_i)_{i\in I}\in\partial M^{\varepsilon_0}$, and two sequences of positive real numbers $(\mu_i)_{i\in I}$ and $(\nu_i)_{i\in I}$ such that

$$\begin{cases} \big(|\nabla_{t,\theta}u_\varepsilon|^2_{g^\perp_{t,\theta}} + |\nabla_x u_\varepsilon|^2_g\big)^{p/2}\,dv_g\,dv_{g^\perp_{t,\theta}} \to \mu \geqslant |\nabla_x u|_g^p\,dv_g\,dv_{g^\perp_{t,\theta}} + \sum_{i\in I}\mu_i\delta_{p_i}, \\ u_\varepsilon^{p_*}\,dv_g\,d\sigma_{\varepsilon_0|\partial M^{\varepsilon_0}} \to \nu = u^{p_*}\,dv_g\,d\sigma_{\varepsilon_0|\partial M^{\varepsilon_0}} + \sum_{i\in I}\nu_i\delta_{p_i}, \\ \nu_i^{p/p_*} \leqslant \tilde K(n,p)^{-1}\mu_i \quad \forall i\in I, \end{cases} \tag{26}$$

where the convergence holds in the sense of measures. We consider a point $p=p_i$ appearing in this decomposition with coordinates $(x_p,\varepsilon_0,\theta_p)$, and let $\psi_\delta\in C_c^\infty(B_{x_p}(2\delta))$ be such that $0\leqslant\psi_\delta\leqslant 1$, $\psi_\delta\equiv 1$ in $B_{x_p}(\delta)$, and $\|\nabla\psi_\delta\|_\infty = O(1/\delta)$, where $B_{x_p}(2\delta)\subset M$ is the geodesic ball for the metric g centered at x_p of radius 2δ. We extend ψ_δ to M^{ε_0} as a (t,θ)-independent function. We rewrite (15) using (6)–(8) as

$$\int_{M^{\varepsilon_0}} \big(|\nabla u_\varepsilon|^{p-2}_{R_\varepsilon^*\bar g}\big((\varepsilon/\varepsilon_0)^{-2}(\nabla_{t,\theta}u_\varepsilon,\nabla_{t,\theta}\phi_\varepsilon)_{g^\perp_{t,\theta}} + (\nabla_x u_\varepsilon,\nabla_x\phi_\varepsilon)_{g_{\varepsilon t/\varepsilon_0,\theta}}\big) + u_\varepsilon^{p-1}\phi_\varepsilon\big)\,dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\,dv_{g^\perp_{t,\theta}}$$
$$= \varepsilon_0\varepsilon^{-\frac{k(p_*-p)+p}{p_*}} S^\varepsilon(p,p_*)\int_{M_{\varepsilon_0}} u_\varepsilon^{p_*-1}\phi_\varepsilon\,d\sigma_{\varepsilon_0}\,dv_{g_{\varepsilon,\theta}},$$

where $\phi_\varepsilon = \bar\phi_\varepsilon\circ R_\varepsilon$. With $\phi_\varepsilon = u_\varepsilon\psi_\delta$, we get

$$\int_{M^{\varepsilon_0}} \psi_\delta |\nabla u_\varepsilon|^p_{R^*_\varepsilon \bar g}\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\, dv_{g^\perp_{t,\theta}} - \varepsilon_0 \varepsilon^{-\frac{k(p_*-p)+p}{p_*}} S^\varepsilon(p,p_*) \int_{M_{\varepsilon_0}} u_\varepsilon^{p_*} \psi_\delta\, d\sigma_{\varepsilon_0}\, dv_{g_{\varepsilon,\theta}}$$
$$\leqslant \int_{M^{\varepsilon_0}} |\nabla u_\varepsilon|^{p-1}_{R^*_\varepsilon \bar g} u_\varepsilon |\nabla_x \psi_\delta|_{g_{\varepsilon t/\varepsilon_0,\theta}}\, dv_{g^\perp_{t,\theta}}\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}.$$

We estimate the right-hand side of this inequality using Hölder's inequality and (17) which gives that $\int_{M^{\varepsilon_0}} |\nabla u_\varepsilon|^p_{R^*_\varepsilon \bar g}\, dv_{g^\perp_{t,\theta}}\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}} = O(1)$. We obtain

$$\int_{M^{\varepsilon_0}} \psi_\delta \bigl\{|\nabla_{t,\theta} u_\varepsilon|^2_{g^\perp_{t,\theta}} + |\nabla_x u_\varepsilon|^2_{g_{\varepsilon t/\varepsilon_0,\theta}}\bigr\}^{\frac p2}\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\, dv_{g^\perp_{t,\theta}} - \varepsilon_0 \varepsilon^{-\frac{k(p_*-p)+p}{p_*}} S^\varepsilon(p,p_*) \int_{M_{\varepsilon_0}} u_\varepsilon^{p_*}\psi_\delta\, d\sigma_{\varepsilon_0}\, dv_{g_{\varepsilon,\theta}}$$
$$\leqslant O(1/\delta)\Biggl(\int_{\operatorname{supp}\psi_\delta} u_\varepsilon^p\, dv_{g^\perp_{t,\theta}}\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\Biggr)^{\frac1p},$$

where $\operatorname{supp}\psi_\delta$ denotes the support of ψ_δ. Since $g_{\varepsilon t/\varepsilon_0,\theta} \to g$ uniformly in $\bar M^{\varepsilon_0}$ as $\varepsilon \to 0$, and $u_\varepsilon \to u$ in $L^p(M^{\varepsilon_0})$, we can pass to the limit in this equality using (26) to get

$$\int_{M^{\varepsilon_0}} \psi_\delta\, d\mu - A \int_{M^{\varepsilon_0}} \psi_\delta\, d\nu \leqslant O(1/\delta)\Biggl(\int_{\operatorname{supp}\psi_\delta} u^p\, dv_g\, dv_{g^\perp_{t,\theta}}\Biggr)^{\frac1p}$$
$$\leqslant O(1/\delta)\Biggl(\int_{B_{x_p}(2\delta)} u^p\, dv_g\Biggr)^{\frac1p}, \tag{27}$$

where $A = \varepsilon_0 \lim_{\varepsilon\to 0} \varepsilon^{-\frac{k(p_*-p)+p}{p_*}} S^\varepsilon(p,p_*)$, which exists up to a subsequence in view of (14), and the last inequality follows from the fact that u is (t,θ)-independent. Remark that

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{k(p_*-p)+p}{p_*}} S^\varepsilon(p,p_*) > 0$$

since otherwise we would have $u_\varepsilon \to 0$ in $H_1^p(M^{\varepsilon_0})$ according to (17), which contradicts the normalization condition $\int_{M_{\varepsilon_0}} u_\varepsilon^q\, d\sigma_{\varepsilon_0}\, dv_{g_{\varepsilon,\theta}} = \varepsilon_0^{k-1}$ (see (22)). In view of (21), $u \in H_1^p(M)$ and thus $u \in L^{p^*}(M)$ with $p^* = (n-k)p/(n-k-p)$ if $p < n-k$. Hence in that case

$$\int_{B_{x_p}(2\delta)} u^p\, dv_g \leqslant \Biggl(\int_{B_{x_p}(2\delta)} u^{p^*}\, dv_g\Biggr)^{\frac{p}{p^*}} \bigl|B_{x_p}(2\delta)\bigr|^{\frac{p^*-p}{p^*}} = o(1)O\bigl(\delta^p\bigr) = o\bigl(\delta^p\bigr).$$

Letting $\delta \to 0$ in (27) then gives $\mu_i \leqslant A\nu_i$, from which we get using (26) that

$$\mu_i \geqslant A^{-\frac{n-p}{p-1}} \tilde K(n,p)^{\frac{n-1}{p-1}}$$

for any $i \in I$. We now pass to the limit in (17) and obtain

$$\varepsilon_0^{k-1} A \geqslant \mu\bigl(M^{\varepsilon_0}\bigr) \geqslant \sum_{i\in I} \mu_i \geqslant |I| A^{-\frac{n-p}{p-1}} \tilde K(n,p)^{\frac{n-1}{p-1}},$$

i.e. $|I|(A^{-1}\tilde K(n,p))^{\frac{n-1}{p-1}} \leqslant \varepsilon_0^{k-1}$. Since $A = O(\varepsilon_0) < \tilde K(n,p)$ for ε_0 small enough, and $k \geqslant 1$, we must have $I = \emptyset$, i.e. $u_\varepsilon \to u$ strongly in $H_1^p(M^{\varepsilon_0})$. As said above, this ends the proof of Theorem 0.1 in the critical case.

On what concerns the remark (3) about the problem of finding a conformal metric to $\bar g$ with zero scalar curvature in the interior of M^ε and constant mean curvature on ∂M^ε, we note that (6)–(8) gives

$$\lambda_\varepsilon = \inf_{u\in H_1^2(M^{\varepsilon_0}),\, u\not\equiv 0 \text{ on } M_\varepsilon} \frac{(\varepsilon/\varepsilon_0)^k I_\varepsilon + (\varepsilon/\varepsilon_0)^{k-1}\int_{M_{\varepsilon_0}} k(x,\varepsilon)u^2\, d\sigma_{\varepsilon_0}\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}}{((\varepsilon/\varepsilon_0)^{k-1}\int_{M_{\varepsilon_0}} |u|^{2_*}\, d\sigma_{\varepsilon_0}\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}})^{2/2_*}},$$

where

$$I_\varepsilon = \int_{M^{\varepsilon_0}} \big((\varepsilon/\varepsilon_0)^{-2}|\nabla_{t,\theta}u|^2_{g_{t,\theta}^\perp} + |\nabla_x u|^2_{g_{\varepsilon t/\varepsilon_0,\theta}} + h(x,\varepsilon,\theta)|u|^2\big)\, dv_{g_{\varepsilon t/\varepsilon_0,\theta}}\, dv_{g_{t,\theta}^\perp}.$$

Hence

$$\limsup_{\varepsilon\to 0} \varepsilon^{-(k-1)(1-2/2_*)}\lambda_\varepsilon \leqslant \inf_{u\in H_1^2(M),\, u\not\equiv 0} \frac{\int_M k_{|M} u^2 |B_x(r(x))|_\xi\, dv_g}{(\int_M |u|^{2_*}|B_x(r(x))|_\xi\, dv_g)^{2/2_*}} = 0,$$

where the second equality follows by taking $u_\eta(x) = \eta u(\eta^{-1}\exp_y^{-1}(x))$, with $\eta \to 0$, as a test-function to estimate the inf, for some point $y \in M$ and function $u \in C_c^\infty(\mathbb{R}^{n-k})$, $u \not\equiv 0$.

We now prove the theorem when $k = n$. Using the constant function equal to 1 in the definition of $S^\varepsilon(p,q)$, we get

$$S^\varepsilon(p,q) \leqslant \frac{|\Omega_\varepsilon|_{\bar g}}{|\partial\Omega_\varepsilon|_{\bar g}^{p/q}} = \frac{|\tilde\Omega_\varepsilon|_{\tilde g}}{|\partial\tilde\Omega_\varepsilon|_{\tilde g}^{p/q}}, \tag{28}$$

where $\tilde\Omega_\varepsilon = \varepsilon\tilde\Omega$, so that $\Omega_\varepsilon = \exp_0(\tilde\Omega_\varepsilon)$, and $\tilde g = \exp_0^* \bar g$. Letting $R_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$, $R_\varepsilon(x) = x/\varepsilon$, we have

$$|\tilde\Omega_\varepsilon|_{\tilde g} = \int_{\tilde\Omega} dv_{(R_\varepsilon^{-1})^*\tilde g} = \varepsilon^n \int_{\tilde\Omega} dv_{\tilde g(\varepsilon x)} \sim \varepsilon^n |\tilde\Omega|_\xi$$

as $\varepsilon \to 0$, since $\tilde g(0) = \xi$ the Euclidean metric. In the same way,

$$|\partial\tilde\Omega_\varepsilon|_{\tilde g} \sim \varepsilon^{n-1}|\partial\tilde\Omega|_\xi$$

as $\varepsilon \to 0$. We can thus rewrite (28) as

$$\limsup_{\varepsilon\to 0} \varepsilon^{-\frac{(q-p)n+p}{q}} S^\varepsilon(p,q) \leqslant \frac{|\tilde\Omega|_\xi}{|\partial\tilde\Omega|_\xi^{p/q}}. \tag{29}$$

In the subcritical case $q < p_*$, the standard variational method implies that $S^\varepsilon(p,q)$ is attained by some nonnegative function $v_\varepsilon \in H_1^p(\Omega_\varepsilon)$ such that $\int_{\partial\Omega_\varepsilon} v_\varepsilon^q\, d\sigma_{\bar g} = 1$. In the critical case $q = p_*$, this method does not work anymore since the embedding $H_1^p(\Omega_\varepsilon) \hookrightarrow L^{p_*}(\partial\Omega_\varepsilon)$ is not compact. However, according to (29), we have $S^\varepsilon(p,p_*) \to 0$ as $\varepsilon \to 0$. In particular

$$S^\varepsilon(p,p_*) < \tilde K(n,p)$$

for $\varepsilon > 0$ sufficiently small, where $\tilde{K}(n,p)$ is given by (24). It then follows that $S^{\varepsilon}(p,p_*)$ is attained by some nonnegative function $v_\varepsilon \in H_1^p(\Omega_\varepsilon)$ such that $\int_{\partial\Omega_\varepsilon} v_\varepsilon^{p_*}\,d\sigma_{\bar{g}} = 1$ (see for example [5,6]).

We let $\tilde{v}_\varepsilon$ and $\tilde{u}_\varepsilon$ be the functions defined in $\tilde{\Omega}_\varepsilon$ and $\tilde{\Omega}$ respectively by

$$\tilde{v}_\varepsilon(x) = v_\varepsilon\big(\exp_0(x)\big), \quad x \in \tilde{\Omega}_\varepsilon,$$

and

$$\tilde{u}_\varepsilon(x) = \varepsilon^{\frac{n-1}{q}}\,\tilde{v}_\varepsilon(\varepsilon x), \quad x \in \tilde{\Omega}.$$

We then have

$$\int_{\Omega} |\tilde{u}_\varepsilon|^p\,dv_{\tilde{g}} = \varepsilon^{\frac{(n-1)p}{q}} \int_{\tilde{\Omega}_\varepsilon} |\tilde{v}_\varepsilon|^p\,dv_{R_\varepsilon^*\tilde{g}} = \varepsilon^{\frac{(p-q)n-p}{q}} \int_{\tilde{\Omega}_\varepsilon} \big|\tilde{v}_\varepsilon(x)\big|^p\,dv_{\tilde{g}(x/\varepsilon)},$$

and

$$\begin{aligned}\int_{\Omega} |\nabla\tilde{u}_\varepsilon|_{\tilde{g}}^p\,dv_{\tilde{g}} &= \varepsilon^{\frac{(n-1)p}{q}} \int_{\tilde{\Omega}} \big|\nabla\big(\tilde{v}_\varepsilon(\varepsilon x)\big)\big|_{\tilde{g}}^p\,dv_{\tilde{g}} = \varepsilon^{\frac{(n-1)p}{q}} \int_{\tilde{\Omega}} |\nabla\tilde{v}_\varepsilon|_{R_\varepsilon^*\tilde{g}}^p\,dv_{R_\varepsilon^*\tilde{g}} \\ &= \varepsilon^{\frac{(p-q)n+(q-1)p}{q}} \int_{\tilde{\Omega}_\varepsilon} |\nabla\tilde{v}_\varepsilon|_{\tilde{g}(x/\varepsilon)}^p\,dv_{\tilde{g}(x/\varepsilon)}.\end{aligned}$$

Independently, there exists $C > 0$ such that for every $x \in \tilde{\Omega}$

$$C^{-1}\delta_{ij} \leqslant \tilde{g}(x) \leqslant C\delta_{ij} \tag{30}$$

in the sense of bilinear forms. Thus

$$\int_{\tilde{\Omega}_\varepsilon} \big|\tilde{v}_\varepsilon(x)\big|^p\,dv_{\tilde{g}(x/\varepsilon)} \leqslant C \int_{\tilde{\Omega}_\varepsilon} \big|\tilde{v}_\varepsilon(x)\big|^p\,dv_{\tilde{g}},$$

and

$$\int_{\tilde{\Omega}_\varepsilon} |\nabla\tilde{v}_\varepsilon|_{\tilde{g}(x/\varepsilon)}^p\,dv_{\tilde{g}(x/\varepsilon)} \leqslant C \int_{\tilde{\Omega}_\varepsilon} |\nabla\tilde{v}_\varepsilon|_{\tilde{g}}^p\,dv_{\tilde{g}}.$$

Hence

$$\begin{aligned}\varepsilon^{-p}\int_{\tilde{\Omega}} |\nabla\tilde{u}_\varepsilon|_{\tilde{g}}^p\,dv_{\tilde{g}} + \int_{\tilde{\Omega}} \tilde{u}_\varepsilon^p\,dv_{\tilde{g}} &\leqslant C\varepsilon^{\frac{(p-q)n-p}{q}} \int_{\tilde{\Omega}_\varepsilon} \big(|\nabla\tilde{v}_\varepsilon|_{\tilde{g}}^p + \tilde{v}_\varepsilon^p\big)\,dv_{\tilde{g}} \\ &= C\varepsilon^{\frac{(p-q)n-p}{q}} \int_{\Omega_\varepsilon} \big(|\nabla v_\varepsilon|_{\tilde{g}}^p + v_\varepsilon^p\big)\,dv_{\tilde{g}} \\ &= C\varepsilon^{\frac{(p-q)n-p}{q}} S^{\varepsilon}(p,q).\end{aligned} \tag{31}$$

We then deduce with (29) that $(\tilde{u}_\varepsilon)$ is bounded in $H_1^p(\tilde{\Omega})$ and that $\nabla\tilde{u}_\varepsilon \to 0$ in $L^p(\tilde{\Omega})$. There thus exists a nonnegative function $\tilde{u} \in H_1^p(\tilde{\Omega})$ such that $\tilde{u}_\varepsilon \to \tilde{u}$ weakly in $H_1^p(\tilde{\Omega})$, strongly in $L^p(\tilde{\Omega})$ and strongly (resp. weakly) in $L^q(\partial\tilde{\Omega})$ if $q < p_*$ (resp. $q = p_*$), and a.e. We have

$$\int_{\tilde{\Omega}} |\nabla\tilde{u}|^p \, dv_{\tilde{g}} \leqslant \liminf_{\varepsilon\to 0} \int_{\tilde{\Omega}} |\nabla\tilde{u}_\varepsilon|^p \, dv_{\tilde{g}} = 0$$

according to (31) and (29). Hence $\tilde{u}$ is a nonnegative constant, and $\tilde{u}_\varepsilon \to \tilde{u}$ strongly in $H_1^p(\tilde{\Omega})$. As a consequence $\tilde{u}_\varepsilon \to \tilde{u}$ strongly in $L^q(\partial\tilde{\Omega})$ for any $q \leqslant p_*$. To find the value of $\tilde{u}$, we write that

$$1 = \int_{\partial\Omega_\varepsilon} v_\varepsilon^q \, d\sigma_{\tilde{g}} = \int_{\partial\tilde{\Omega}_\varepsilon} \tilde{v}_\varepsilon^q \, d\sigma_{\tilde{g}} = \int_{\partial\tilde{\Omega}} \tilde{u}_\varepsilon^q \, d\sigma_{\tilde{g}(\varepsilon x)} \to \tilde{u}^q |\partial\tilde{\Omega}|_\xi$$

as $\varepsilon \to 0$, i.e. $\tilde{u} = |\partial\tilde{\Omega}|_\xi^{-q}$. Eventually,

$$S^\varepsilon(p,q) \geqslant \int_{\tilde{\Omega}_\varepsilon} \tilde{v}_\varepsilon^p \, dv_{\tilde{g}} = \varepsilon^{-\frac{(n-1)p}{q}} \int_{\tilde{\Omega}} \tilde{u}_\varepsilon^p \, dv_{(R_\varepsilon^{-1})^*\tilde{g}} = \varepsilon^{\frac{(q-p)n+p}{q}} \int_{\tilde{\Omega}} \tilde{u}_\varepsilon^p \, dv_{\tilde{g}(\varepsilon x)},$$

which gives

$$\liminf_{\varepsilon\to 0} \varepsilon^{-\frac{(q-p)n+p}{q}} S^\varepsilon(p,q) \geqslant \tilde{u}^p |\tilde{\Omega}|_\xi = \frac{|\tilde{\Omega}|_\xi}{|\partial\tilde{\Omega}|_\xi^{p/q}}.$$

Together with (29), this proves the result.

2. Proof of Theorem 0.2

The proof of Theorem 0.2 is similar to the one of Theorem 0.1 so that we briefly outline it. We first assume that $k \leqslant n-1$. In view of (6) and (7), we have

$$\begin{aligned}
&(\varepsilon/\varepsilon_0)^{k(p/q-1)} S_\varepsilon(p,q) \\
&\quad = \inf_{u\in H_1^p(M^{\varepsilon_0}),\, u\not\equiv 0} \frac{\int_{M^{\varepsilon_0}} (\{(\varepsilon/\varepsilon_0)^{-2}(\nabla_{t,\theta}u)^2_{g^\perp_{t,\theta}} + |\nabla_x u|^2_{g_{\varepsilon t/\varepsilon_0,\theta}}\}^{\frac{p}{2}} + |u|^p) \, dv_{g_{\varepsilon t/\varepsilon_0,\theta}} \, dv_{g^\perp_{t,\theta}}}{(\int_{M^{\varepsilon_0}} |u|^q \, dv_{g_{\varepsilon t/\varepsilon_0,\theta}} \, dv_{g^\perp_{t,\theta}})^{p/q}} \\
&\quad \leqslant \inf_{u\in \tilde{H}_1^p(M^{\varepsilon_0}),\, u\not\equiv 0} \frac{\int_{M^{\varepsilon_0}} (|\nabla_x u|^p_{g_{\varepsilon t/\varepsilon_0,\theta}} + |u|^p) \, dv_{g_{\varepsilon t/\varepsilon_0,\theta}} \, dv_{g^\perp_{t,\theta}}}{(\int_{M^{\varepsilon_0}} |u|^q \, dv_{g_{\varepsilon t/\varepsilon_0,\theta}} \, dv_{g^\perp_{t,\theta}})^{p/q}}.
\end{aligned}$$

Using (11) we then obtain

$$\limsup_{\varepsilon\to 0} \varepsilon^{k(\frac{p}{q}-1)} S_\varepsilon(p,q) \leqslant \inf_{u\in H_1^p(M),\, u\not\equiv 0} \frac{\int_M (|\nabla u|_g^p + |u|^p)|B_x(r(x))|_\xi \, dv_g(x)}{(\int_M |u|^q |B_x(r(x))|_\xi \, dv_g(x))^{p/q}}.$$

As in the proof of Theorem 0.1, $S_\varepsilon(p,q)$, $q < p^*$, is attained by some nonnegative $\bar{u}_\varepsilon \in H_1^p(M^\varepsilon)$ normalized by $\int_{M^\varepsilon} \bar{u}_\varepsilon^q \, dv_{\tilde{g}} = \varepsilon^k$. We then have

$$\int_{M^\varepsilon} \left(|\nabla\bar{u}_\varepsilon|_{\bar{g}}^{p-2}(\nabla\bar{u}_\varepsilon, \nabla\bar{\phi}_\varepsilon)_{\bar{g}} + \bar{u}_\varepsilon^{p-1}\bar{\phi}_\varepsilon\right) dv_{\bar{g}} = \varepsilon^{k(p/q-1)} S_\varepsilon(p,q) \int_{M^\varepsilon} \bar{u}_\varepsilon^{q-1}\bar{\phi}_\varepsilon \, dv_{\bar{g}}$$

for every $\bar{\phi}_\varepsilon \in H_1^p(M^\varepsilon)$. In particular, with $\bar{\phi}_\varepsilon = \bar{u}_\varepsilon$,

$$\varepsilon^{kp/q} S_\varepsilon(p,q) = \int_{M^\varepsilon} \left(|\nabla\bar{u}_\varepsilon|_{\bar{g}}^p + \bar{u}_\varepsilon^p\right) dv_{\bar{g}}.$$

Let u_ε be the function defined on M^{ε_0} by $u_\varepsilon = \bar{u}_\varepsilon \circ R_\varepsilon$. We then rewrite the previous equality as

$$\begin{aligned} &\int_{M^{\varepsilon_0}} \left(\left\{(\varepsilon/\varepsilon_0)^{-2}|\nabla_{t,\theta}u_\varepsilon|^2_{g^\perp_{t,\theta}} + |\nabla_x u_\varepsilon|^2_{g_{\varepsilon t/\varepsilon_0,\theta}}\right\}^{\frac{p}{2}} + u_\varepsilon^p\right) dv_{g_{\varepsilon t/\varepsilon_0,\theta}} \, dv_{g^\perp_{t,\theta}} \\ &\quad = \varepsilon_0^k \varepsilon^{k(p/q-1)} S_\varepsilon(p,q) = O(1), \end{aligned} \tag{32}$$

from which we deduce that the u_ε's converge to some nonnegative (t,θ)-independent $u \in H_1^p(M^{\varepsilon_0})$ weakly in $H_1^p(M^{\varepsilon_0})$ and strongly in $L^p(M^{\varepsilon_0})$ and $L^q(M^{\varepsilon_0})$ when $q < p^*$. Moreover the normalization of the $\bar{u}_\varepsilon$ gives $\int_{M^{\varepsilon_0}} u^q |B_x(r(x))|_\xi \, dv_g = 1$. Passing to the limit $\varepsilon \to 0$ in (32) and then using (11) eventually yields

$$\liminf_{\varepsilon\to 0} \varepsilon^{k(\frac{p}{q}-1)} S_\varepsilon(p,q) \geqslant \inf_{u\in H_1^p(M),\, u\not\equiv 0} \frac{\int_M (|\nabla u|_g^p + |u|^p)|B_x(r(x))|_\xi \, dv_g(x)}{(\int_M |u|^q |B_x(r(x))|_\xi \, dv_g(x))^{p/q}},$$

which ends the proof of Theorem 0.2 in the subcritical case $q < p^*$.

To deal with the critical case $q = p^*$, we introduce the best constant $K(n,p)$ for the embedding of $D_1^p(\mathbb{R}^n)$ into $L^{p^*}(\mathbb{R}^n)$ namely

$$K(n,p) = \inf_{u\in C_c^\infty(\mathbb{R}^n),\, u\not\equiv 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^p \, dx}{(\int_{\mathbb{R}^n} |u|^{p^*} dx)^{p/p^*}} > 0.$$

Since $S_\varepsilon(p,p^*) = O(\varepsilon^{k(1-p/p^*)}) = o(1) < K(n,p)$, $S_\varepsilon(p,p^*)$ is attained by some nonnegative $\bar{u}_\varepsilon \in H_1^p(M^\varepsilon)$ normalized as previously (see e.g. [1] or [7]). To get the strong convergence of the $\bar{u}_\varepsilon$'s to u in $L^{p^*}(M^{\varepsilon_0})$, we consider the inequality

$$\begin{aligned} &\left(\int_{M^{\varepsilon_0}} |v|^{p^*} dv_g \, dv_{g^\perp_{t,\theta}}\right)^{p/p_*} \\ &\quad \leqslant \left(K(n,p)^{-1} + \eta\right) \int_{M^{\varepsilon_0}} \left(|\nabla_{t,\theta}v|^2_{g^\perp_{t,\theta}} + |\nabla_x v|^2_g\right)^{p/2} dv_g \, dv_{g^\perp_{t,\theta}} + C_\eta \int_{M^{\varepsilon_0}} |v|^p \, dv_g \, dv_{g^\perp_{t,\theta}}, \end{aligned}$$

which holds for every $v \in H_1^p(M^{\varepsilon_0})$ (see [1,7]). We then obtain that

$$\begin{cases} \left(|\nabla_{t,\theta}u_\varepsilon|^2_{g^\perp_{t,\theta}} + |\nabla_x u_\varepsilon|^2_g\right)^{p/2} dv_g \, dv_{g^\perp_{t,\theta}} \rightharpoonup \mu \geqslant |\nabla_x u|_g^p \, dv_g \, dv_{g^\perp_{t,\theta}} + \sum_{i\in I} \mu_i \delta_{p_i}, \\ u_\varepsilon^{p^*} dv_g \, dv_{g^\perp_{t,\theta}} \rightharpoonup \nu = u^{p_*} dv_g \, dv_{g^\perp_{t,\theta}} + \sum_{i\in I} \nu_i \delta_{p_i}, \\ \nu_i^{p/p^*} \leqslant K(n,p)^{-1}\mu_i \quad \forall i \in I. \end{cases}$$

As in the proof of Theorem 0.1, we obtain $\mu_i \leqslant A\nu_i$ for every $i \in I$, where $A = \lim_{\varepsilon\to 0} \varepsilon^{k(p/p^*-1)} S_\varepsilon(p, p^*)$. Hence $\mu_i \geqslant A(A^{-1}K(n,p))^{n/p}$. Passing to the limit in (32), we obtain $\varepsilon_0^k A \geqslant \sum_{i\in I}\mu_i$, and thus $\varepsilon_0^k \geqslant |I|(A^{-1}K(n,p))^{n/p}$ for any $\varepsilon_0 > 0$ with $k \geqslant 1$. It follows that $I = \emptyset$, and thus that $u_\varepsilon \to u$ strongly in $H_1^p(M^{\varepsilon_0})$. We can end the proof as previously.

We now assume that $k = n$. Using the constant function equal to 1, we get

$$S_\varepsilon(p,q) \leqslant |\Omega_\varepsilon|_{\tilde g}^{1-p/q} \sim \varepsilon^{-n(p/q-1)} |\tilde\Omega|_\xi^{1-p/q}$$

as $\varepsilon \to 0$, so that

$$\limsup_{\varepsilon\to 0} \varepsilon^{n(p/q-1)} S_\varepsilon(p,q) \leqslant |\tilde\Omega|_\xi^{1-p/q}. \tag{33}$$

As before, $S_\varepsilon(p,q)$, $q \leqslant p^*$, is attained by some nonnegative $v_\varepsilon \in H_1^p(\Omega_\varepsilon)$ such that $\int_{\Omega_\varepsilon} v_\varepsilon^q \, dv_{\tilde g} = 1$. We then consider $\tilde u_\varepsilon(x) = \varepsilon^{n/q} \tilde v_\varepsilon(\varepsilon x)$, $x \in \tilde\Omega$, where $\tilde v_\varepsilon(x) = v_\varepsilon(\exp_0(x))$, $x \in \tilde\Omega_\varepsilon$. We then have

$$\int_{\tilde\Omega} \tilde u_\varepsilon^p \, dv_{\tilde g} = \varepsilon^{n(p/q-1)} \int_{\tilde\Omega_\varepsilon} \tilde v_\varepsilon^p \, dv_{\tilde g(x/\varepsilon)},$$

and

$$\int_{\tilde\Omega} |\nabla \tilde u_\varepsilon|_{\tilde g}^p \, dv_{\tilde g} = \varepsilon^{n(p/q-1)+p} \int_{\tilde\Omega_\varepsilon} |\nabla \tilde v_\varepsilon|^p \, dv_{\tilde g(x/\varepsilon)},$$

so that, with (30),

$$\int_{\tilde\Omega} \varepsilon^{-p}\big(|\nabla \tilde u_\varepsilon|_{\tilde g}^p + \tilde u_\varepsilon^p\big)\, dv_{\tilde g} \leqslant C\varepsilon^{n(p/q-1)} \int_{\tilde\Omega_\varepsilon} \big(|\nabla \tilde v_\varepsilon|^p + \tilde v_\varepsilon^p\big)\, dv_{\tilde g} \leqslant C\varepsilon^{n(p/q-1)} S_\varepsilon(p,q) \leqslant C.$$

We then deduce as above that the $\tilde u_\varepsilon$'s converge strongly in $H_1^p(\tilde\Omega)$ to some nonnegative constant $\tilde u$. In fact $\tilde u = |\tilde\Omega|_\xi^{-1/q}$ since

$$1 = \int_{\Omega_\varepsilon} v_\varepsilon^q \, dv_{\tilde g} = \int_{\tilde\Omega} \tilde u_\varepsilon^q \, dv_{\tilde g(\varepsilon x)} \to \tilde u^q |\tilde\Omega|_\xi$$

as $\varepsilon \to 0$. From

$$S_\varepsilon(p,q) \geqslant \int_{\Omega_\varepsilon} v_\varepsilon^p \, dv_{\tilde g} = \varepsilon^{n(1-p/q)} \int_{\tilde\Omega} \tilde u_\varepsilon^p \, dv_{\tilde g(\varepsilon x)} \sim \varepsilon^{n(1-p/q)} \tilde u^p |\tilde\Omega|_\xi,$$

we obtain

$$\liminf_{\varepsilon\to 0} \varepsilon^{n(p/q-1)} S_\varepsilon(p,q) \geqslant |\tilde\Omega|_\xi^{1-p/q},$$

which ends the proof of Theorem 0.2.

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References

[1] T. Aubin, Equations différentielles non-linéaires et le problème de Yamabé concernant la courbure scalaire, J. Math. Pures Appl. (9) 55 (3) (1976) 269–296.
[2] R.J. Biezuner, Best constants in Sobolev trace inequalities, Nonlinear Anal. 54 (3) (2003) 575–589.
[3] J. Fernández Bonder, S. Martínez, J.D. Rossi, The behavior of the best Sobolev trace constant and extremals in thin domains, J. Differential Equations 198 (1) (2004) 129–148.
[4] J. Fernández Bonder, J.D. Rossi, Asymptotic behavior of the best Sobolev trace constant in expanding and contracting domains, Commun. Pure Appl. Anal. 1 (3) (2002) 359–378.
[5] J. Fernández Bonder, J.D. Rossi, On the existence of extremals for the Sobolev trace embedding theorem with critical exponent, Bull. London Math. Soc. 37 (1) (2005) 119–125.
[6] J. Fernández Bonder, N. Saintier, Estimates for the Sobolev trace constant with critical exponent and applications, Ann. Mat. Pura Appl. 187 (4) (2008) 683–704.
[7] O. Druet, E. Hebey, The AB program in geometric analysis: Sharp Sobolev inequalities and related problems, Mem. Amer. Math. Soc. 160 (2002).
[8] J.F. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, Ann. of Math. (2) 136 (1) (1992) 1–50.
[9] P.L. Lions, The concentration–compactness principle in the calculus of variations. The limit case, part 2, Rev. Mat. Iberoamericana 1 (2) (1985) 45–121.
[10] S. Marusić, Asymptotic behaviour of sharp Sobolev constants in thin domains, J. Math. Anal. Appl. 272 (2) (2002) 575–581.
[11] S. Marusić, Asymptotic behaviour of Sobolev constants for thin curved rods or pipes, Rend. Istit. Mat. Univ. Trieste 34 (1–2) (2002) 57–65.
[12] N. Saintier, Generalized mean curvature type equations on compact manifolds with boundary, in preparation.