NoDEA
Nonlinear differ. equ. appl. 15 (2008) 227—245
1021–9722/08/0200227–19
DOI 10.1007/s00030-007-7006-8

Nonlinear Differential Equations
and Applications NoDEA

Blow-up theory for symmetric critical equations involving the p-Laplacian

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Abstract. We describe in this paper the asymptotic behaviour in Sobolev spaces of sequences of solutions of critical equations involving the p-Laplacian (see equations (E_α) below) on a compact Riemannian manifold (M, g) which are invariant by a subgroup of the group of isometries of (M, g). We also prove pointwise estimates.

2000 *Mathematics Subject Classification*: 35B40, 35J20, 35B33
Key words: p-Laplacian, blow-up, invariance under isometries

Let (M, g) be a smooth compact Riemannian n-manifold, G a closed subgroup of the group of isometries $\mathrm{Isom}_g(M)$ of (M, g) and $k = \min_{x\in M} \ dim\ Gx$, where Gx denotes the orbit of a point $x \in M$ under G. We say that a function $\psi : M \to \mathbb{R}$ is G-invariant if $\psi(gx) = \psi(x)$ for any $x \in M$ and $g \in G$. We consider equations like

$$\Delta_{p,g}u + h_\alpha u^{p-1} = fu^{p^*-1} \qquad (E_\alpha)$$

where $1 < p < n - k$, $\Delta_{p,g}u = -div_g \left(|\nabla u|_g^{p-2}\nabla u\right)$ is the p-Laplacian of u, $p^* = \frac{(n-k)p}{n-k-p}$ is the critical exponent for the injection from the Sobolev space $H^p_{1,G}(M)$ of G-invariant functions in $L^p(M)$ whose gradient is also in $L^p(M)$, into the Lebesgue spaces $L^q_G(M)$ of G-invariant functions in $L^q(M)$ (cf Hebey-Vaugon [6]), f is a C^1 G-invariant function, and (h_α) is a sequence of continuous G-invariant functions converging uniformly to some continous G-invariant function h_∞. The solutions we consider are in H^p_1; therefore, a solution to (E_α) has to be taken in the distribution sense. We assume that the operator $\Delta_{p,g} + h_\infty$ is coercive in the sense that there exists $\lambda > 0$ such that for all $u \in H_{1,G}^p(M)$,

$$\int_M \left(|\nabla u|_g^p + h_\infty |u|^p\right) dv_g \geq \lambda \|u\|_{H_1^p}^p. \tag{1}$$

In fact, we can easily prove that $\Delta_{p,g} + h_\infty$ is coercive if and only if there exists $\lambda > 0$ such that for all $u \in H_{1,G}^p(M)$,

$$\int_M \left(|\nabla u|_g^p + h_\infty |u|^p\right) dv_g \geq \lambda \|u\|_p^p.$$

A necessary condition for (E_α) to admit a positive solution u is $\max_M f > 0$. Indeed, multiplying (E_α) by u, integrating by parts and using the coercivity assumption (1) yields

$$\int_M f u^{p^*} dv_g \geq \lambda \|u_\alpha\|_{H_1^p}^p + o(1).$$

We then deduce that f must be positive somewhere, and then $\max_M f > 0$. From now on, we assume that $\max_M f > 0$. We also consider the limit equation obtained by letting formally $\alpha \to +\infty$ in (E_α), namely

$$\Delta_{p,g} u + h_\infty u^{p-1} = f u^{p^*-1}. \tag{E_∞}$$

For each α, let u_α be a G-invariant weak positive solution of (E_α) and assume that the sequence (u_α) is bounded in H_1^p. The purpose of this note is to describe the asymptotic behavior of the u_α's. In the case where the group G is reduced to the identity, it is known (see Saintier [9], Hebey-Robert [5], Struwe [10]) that u_α can be written as the sum of a weak solution of the limit equation (E_∞) plus a finite sum of "bubbles" plus a sequence of functions converging strongly to 0 in H_1^p. A bubble is a sequence of functions obtained by rescaling positive solution of the Euclidean critical equation $\Delta_{p,\xi} u = u^{q-1}$ in $\mathbb{R}^n$, $q = np/(n-p)$, where ξ is the Euclidean metric on $\mathbb{R}^n$. We prove here (cf the theorem below) that this decomposition still holds in the context of G-invariant functions under some assumptions on the orbits of G (assumption (H) below) and with an extended notion of bubble.

We now recall some known facts and fix some notations. We refer to Bredon [1] for more details (see also Hebey-Vaugon [6] and Faget [2]). Let G' be a closed subgroup of $\mathrm{Isom}_g(M)$. Then G' is a Lie group. For each $x \in M$, we let $\bar{x} = \Pi(x)$, where $\Pi : M \to M/G'$ is the canonical surjection, and denote by $G'x = \{gx,\ g \in G'\}$ (resp. $S_x = \{g \in G',\ gx = x\}$) the orbit (resp. the stabilizator) of x under the action of G'. Then $G'x$ is a compact submanifold of M naturally isomorphic to the quotient group G'/S_x. An orbit $G'x$ is said principal if its stabilizator is minimal up to conjugacy i.e. for all $y \in M$, S_y contains a subgroup conjugate to S_x. In particular, the principal orbits are of maximal dimension (but the converse is false). If we denote by Ω the union of all the principal orbits, then Ω is a dense open subset of M and Ω/G' is a smooth connected manifold which can be equiped with a Riemaniann metric $\bar{g}$ in such a way that the canonical surjection from Ω to Ω/G' is a Riemannian submersion. We then consider the metric $\tilde{g}$ belonging to the conformal class of $\bar{g}$ defined by

$$\tilde{g} = \bar{v}^{\frac{2}{n-k-p}} \bar{g} \tag{2}$$

where $\bar{v}(\bar{x}) = Vol(\Pi^{-1}(\bar{x})) = Vol(G'x)$ denotes the volume of $G'x$ computed with respect to the induced metric. We will denote by $B^{\bar{g}}_{\bar{x}}(r)$ and $B^{\tilde{g}}_{\bar{x}}(r)$ the geodesic balls centered at $\bar{x}$ of radius r for the metric $\bar{g}$ and $\tilde{g}$ respectively. Given a Riemannian manifold N, we denote by $H^p_1(N)$ the usual Sobolev space of functions $u \in L^p(N)$ such that $\nabla u \in L^p(N)$ with the norm $\|u\|^p_{H^p_1} = \|u\|^p_p + \|\nabla u\|^p_p$, and by $\overset{\circ}{H}{}^p_1(N)$ the closure of $C^\infty_c(N)$ for the norm $\|.\|_{H^p_1}$. If G' is a subgroup of isometries of N, we let $L^p_{G'}(N)$, $H^p_{1,G'}(N)$ and $\overset{\circ}{H}{}^p_{1,G'}(N)$ be the space of G'-invariant functions in $L^p(N)$, $H^p_1(N)$ and $\overset{\circ}{H}{}^p_1(N)$ respectively:

$$\begin{aligned} L^p_{G'}(N) &= \{u \in L^p(N) \ s.t. \ \forall \ g \in G', \ u(gx) = u(x) \ a.e. \ in \ N\}, \\ H^p_{1,G'}(N) &= \{u \in H^p_1(N) \ s.t. \ \forall \ g \in G', \ u(gx) = u(x) \ a.e. \ in \ N\}, \\ \overset{\circ}{H}{}^p_{1,G'}(N) &= \left\{u \in \overset{\circ}{H}{}^p_1(N) \ s.t. \ \forall \ g \in G', \ u(gx) = u(x) \ a.e. \ in \ N\right\}, \end{aligned}$$

We assume that $k = \min_{x \in M} \ dim \ Gx \geq 1$ and make the following assumption on the G-orbits of dimension k:
(H) *for each G - orbit Gx_0 of minimal dimension k, there exist $\delta > 0$ and a closed subgroup G' of $Isom_g(M)$ such that*

$$G'x_0 = Gx_0 \tag{H1}$$

and, for all $x \in B_{Gx_0}(\delta) := \{y \in M, \ d_g(y, Gx_0) < \delta\}$,

$$G'x \ \textit{is principal and} \ G'x \subset Gx. \tag{H2}$$

We refer to Faget [2] for examples of manifolds and groups satisfying (H). In particular, $dim \ G'x = dim \ Gx_0 = k$ for all $x \in B_{Gx_0}(\delta)$ and we can consider the Riemannian quotient $(n-k)$-manifold $N := B_{Gx_0}(\delta)/G'$. We fix a smooth cut-off function $\eta \in C^\infty_c(\mathbb{R}^{n-k})$ with support in $B_0(2)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $B_0(1)$. Given $\bar{x}_1 \in N$ and $\delta' \in (0, i_{\tilde{g}}(\bar{x}_1)/2)$, we let

$$\eta_{\bar{x}_1,\delta'}(\bar{x}) = \eta\left(\frac{d_{\tilde{g}}(\bar{x}_1, \bar{x})}{\delta'}\right)$$

for $\bar{x} \in N$. Here, $i_{\tilde{g}}(\bar{x}_1)$ denotes the injectivity radius of N at $\bar{x}_1$.

We now define a bubble in this context. Let (x_α) be a sequence of points in M converging to some $x_0 \in M$ such that Gx_0 is of dimension k. Then assumption

(H) provides us with a subgroup G' of $Isom_g(M)$ and a $\delta > 0$ such that (H1) and (H2) hold. Let $2\delta' > 0$ be inferior to the injectivity radius of the quotient $(n-k)$-manifold $N := B_{Gx_0}(\delta)/G'$. Consider also a sequence $(R_\alpha) \subset [0, +\infty)$ such that $R_\alpha \to +\infty$. Given a positive solution $u \in H_1^p(\mathbb{R}^{n-k})$ of the Euclidean equation

$$\Delta_{p,\xi} u = f(x_0) Vol(Gx_0)^{\frac{-p}{n-k-p}} u^{p^*-1},$$

where ξ is the Euclidean metric, we define a bubble $(\bar{B}_\alpha)$ of centers $(\bar{x}_\alpha)$ and weights (R_α) in the usual way by

$$\bar{B}_\alpha(\bar{x}) = \eta_{\bar{x}_\alpha, \delta'}(\bar{x}) R_\alpha^{\frac{n-k-p}{p}} u\left(R_\alpha exp_{\bar{x}_\alpha}^{-1}(\bar{x})\right), \ \bar{x} \in N. \tag{3}$$

where exp is the exponential map of N for the metric $\tilde{g}$. We then define a bubble $B = (B_\alpha)$ of centers (x_α) and weights (R_α) as the G'-invariant function satisfying

$$B_\alpha = \bar{B}_\alpha \circ \Pi$$

where $\Pi : B_{Gx_0}(\delta) \to N$ is the canonical surjection. A generalized bubble is defined in the same way by considering a nontrivial, not necessarily positive, solution $u \in H_1^p(\mathbb{R}^{n-k})$ of the Euclidean equation

$$\Delta_{p,\xi} u = f(x_0) Vol(Gx_0)^{\frac{-p}{n-k-p}} |u|^{p^*-2} u. \tag{4}$$

This definition clearly extends the usual definition of a bubble to the case of G-invariant functions. We also define the energy $E(B)$ of the (generalized) bubble B by

$$E(B) = \frac{1}{p} \int_{\mathbb{R}^{n-k}} |\nabla u|_\xi^p dx - \frac{f(x_0) Vol(Gx_0)^{-\frac{p}{n-k-p}}}{p^*} \int_{\mathbb{R}^{n-k}} |u|^{p^*} dx. \tag{5}$$

We can prove as in Saintier ([9] step 1.5) that

$$E(B) \geq f(x_0)^{-\frac{n-k-p}{p}} Vol(Gx_0) \frac{1}{n-k} K(n-k, p)^{k-n} \tag{6}$$

where $K(n-k,p)$ denotes the best Sobolev constant for the injection of $H_1^p(\mathbb{R}^{n-k})$ into $L^{p^*}(\mathbb{R}^{n-k})$, namely

$$\frac{1}{K(n-k,p)} = \inf_{u \in C_c^\infty(\mathbb{R}^{n-k}) \setminus \{0\}} \frac{\int_{\mathbb{R}^{n-k}} |\nabla u|_\xi^p \, dx}{\left(\int_{\mathbb{R}^{n-k}} |u|^{p^*} \, dx\right)^{p/p^*}} > 0.$$

If we denote by A the minimum volume of G-orbit of dimension k, we then have the minoration

$$E(B) \geq \left(\max_M f\right)^{-\frac{n-k-p}{p}} A \frac{1}{n-k} K(n-k,p)^{k-n} \tag{7}$$

which holds for any generalized bubble.

Our result is then the following:

Theorem *Let (M,g) be a Riemaniann manifold, G a closed subgroup of $Isom_g(M)$ satisfying (H) and (u_α) be a sequence of positive G-invariant solutions of (E_α) bounded in $H_1^p(M)$. There exist $u^0 \in H_{1,G}^p(M)$ such that either $u^0 \equiv 0$ or u^0 is a positive solution of (E_∞), and there exist l bubbles $B^i = \left(B_\alpha^i\right)_\alpha$, $i = 1\dots l$, such that, up to a subsequence,*

$$u_\alpha = u^0 + \sum_{i=1}^{l} B_\alpha^i + S_\alpha \tag{8}$$

where the sequence $(S_\alpha) \subset H_1^p(M)$ converges strongly to 0 in H_1^p, and

$$I_g^\alpha(u_\alpha) = I_g^\infty(u^0) + \sum_{i=1}^{k} E(B^i) + o(1) \tag{9}$$

where I_g^α and I_g^∞ are the functional defined on $H_1^p(M)$ by (12) and (13) respectively, and the energy $E(B^i)$ of the bubble B^i is defined by (5).
Moreover, there exists a constant $C > 0$ independent of α and $x \in M$ such that for any α and any $x \in M$,

$$R_\alpha(x)^{\frac{n-k-p}{p}} \left|u_\alpha(x) - u^0(x)\right| \le C, \text{ and} \tag{10}$$

$$\lim_{R\to\infty} \lim_{\alpha\to+\infty} \sup_{x\in M\setminus\Omega_\alpha(R)} R_\alpha(x)^{\frac{n-k-p}{p}} \left|u_\alpha(x) - u^0(x)\right| = 0 \tag{11}$$

where the $(x_\alpha^i)_\alpha$ and $(\mu_\alpha^i)_\alpha$ are the centers and the inverse of the weights of the bubble B^i, $R_\alpha(x) = \min_{i=1\dots l} d_g(Gx_\alpha^i, Gx)$ and, for $R > 0$, $\Omega_\alpha(R) = \cup_{i=1}^k B_{Gx_\alpha^i}(R\mu_\alpha^i)$.
In the particular case where $p \le 2$, $u^0 = 0$ and u_α is a solution of (E_α), we can prove that $\nabla f(x^i) = 0$ for any i, where $x^i = \lim_\alpha x_\alpha^i$.

The paper is organized as follow. The first section is devoted to the proof of the H_1^p-decomposition, i.e. the relations (8) and (9) for a Palais-Smale sequence for the functional I_g^α defined by (12), whereas the second one deals with the proof of the pointwise estimates (10) and (11).

1 Proof of the H_1^p-decomposition for Palais-Smale sequences

Let I_g^α be the functional defined on $H_1^p(M)$ by

$$I_g^\alpha(u) = \frac{1}{p}\int_M |\nabla u|_g^p dv_g + \frac{1}{p}\int_M h_\alpha |u|^p dv_g - \frac{1}{p^*}\int_M f|u|^{p^*} dv_g, \tag{12}$$

and $(u_\alpha) \in H^p_{1,G}(M)$ be a Palais-Smale (P-S) sequence for I^α_g i.e. the sequence $\big(I^\alpha_g(u_\alpha)\big)$ is bounded and $DI^\alpha_g(u_\alpha) \to 0$ strongly in $H^p_1(M)'$. We are going to prove that the relations (8) and (9) hold for (u_α) with generalized bubbles B^i. We will then prove that if the u_α are positive then the B^i are bubbles.

It follows from Saintier [9] that the sequence (u_α) weakly converges, up to a subsequence, to a solution $u^0 \in H^p_1(M)$ of the limit equation (E_∞). Since we can also assume that the convergence holds almost everywhere, we have $u^0 \in H^p_{1,G}(M)$. Let $v_\alpha = u_\alpha - u^0 \in H^p_{1,G}(M)$. Then (cf Saintier [9]) (v_α) weakly converges to 0 in $H^p_1(M)$ and is a (P-S) sequence for the functional I_g defined on $H^p_1(M)$ by

$$I_g(u) = \frac{1}{p}\int_M |\nabla u|^p_g dv_g - \frac{1}{p^*}\int_M f|u|^{p^*} dv_g.$$

Moreover

$$I_g(v_\alpha) = I^\alpha_g(u_\alpha) - I^\infty_g(u^0) + o(1)$$

where I^∞_g is the functional defined on $H^p_1(M)$ by

$$I^\infty_g(u) = \frac{1}{p}\int_M |\nabla u|^p_g dv_g + \frac{1}{p}\int_M h_\infty |u|^p dv_g - \frac{1}{p^*}\int_M f|u|^{p^*} dv_g. \tag{13}$$

According to Faget [2], for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for any $u \in H^p_{1,G}(M)$,

$$\left(\int_M |u|^{p^*} dv_g\right)^{\frac{p}{p^*}} \le \left(\frac{K(n-k,p)^p}{A^{\frac{p}{n-k}}} + \epsilon\right)\int_M |\nabla u|^p_g dv_g + C_\epsilon \int_M |u|^p dv_g, \tag{14}$$

where A denotes the minimal volume of k-dimensional G-orbits. We can then adapt Saintier ([9] step 1.4) to prove that if (w_α) is a (P-S) sequence for I_g such that

$$w_\alpha \to 0 \text{ weakly in } H^p_1(M) \text{ and } \lim_\alpha I_g(w_\alpha) < \|f\|_\infty^{-\frac{n-k-p}{p}} A\beta^*,$$

where $\beta^* = \frac{1}{(n-k)K(n-k,p)^{n-k}}$, then

$$w_\alpha \to 0 \text{ strongly in } H^p_1.$$

Using this remark and the minoration (6) of the energy of a bubble, we can prove the theorem by induction by repeated use of the following lemma:

Lemma. *Let (v_α) be a (P-S) sequence for I_g converging to* 0 *in H^p_1 weakly but not strongly. Then there exists a generalized bubble $B = (B_\alpha)$ such that $w_\alpha := v_\alpha - B_\alpha$ is a (P-S) sequence for I_g weakly converging to* 0 *in H^p_1. Moreover*

$$I_g(w_\alpha) = I_g(v_\alpha) - E(B) + o(1).$$

The remainder of this section is devoted to the proof of this Lemma. The set of smooth G-invariant functions on M being dense in $H^p_{1,G}(M)$ (see Hebey-Vaugon [6]), we can assume that the v_α's are smooth. Independently, since the v_α's don't converge strongly to 0, the definition of a (P-S) sequence implies that

$$\int_M |\nabla v_\alpha|_g^p dv_g = (n-k)\beta + o(1) \tag{15}$$

and

$$\int_M f|v_\alpha|^{p^*} dv_g = (n-k)\beta + o(1)$$

for some $\beta \geq \|f\|_\infty^{-\frac{n-k-p}{p}} A\beta^* > 0$. The compactness of M then gives the existence of a point $x_0 \in M$ such that for any $\delta > 0$ small enough,

$$\limsup_{\alpha\to+\infty} \int_{B_{Gx_0}(\delta)} f|v_\alpha|^{p^*} dv_g > 0. \tag{16}$$

The orbit Gx_0 is called *orbit of concentration*. We give some preliminary properties of such an orbit:

Step 1.1 1) There are a finite number of concentration orbits. If Gx_0 is one of them, then $dim\ Gx_0 = k$ and $f(x_0) > 0$. In the particular case where $p \leq 2$, $u^0 = 0$ and u_α is a solution of (E_α) for any α, we also have $\nabla f(x_0) = 0$. Moreover Gx_0 is an orbit of concentration if and only if for any $\delta > 0$,

$$\limsup_{\alpha\to+\infty} \int_{B_{Gx_0}(\delta)} |\nabla v_\alpha|_g^p dv_g > 0. \tag{17}$$

2) Let Gx_0 be an orbit of concentration for (v_α). According to 1) and in view of assumption (H), there exist $\delta_0 > 0$ and a subgroup G' of $Isom_g(M)$ such that we can consider the Riemannian quotient $(n-k)$-manifold $(N := B_{Gx_0}(\delta_0)/G', \bar{g})$. Then $\bar{x}_0$ is a point of concentration for $(\bar{v}_\alpha)$ in the sense that for any $\delta > 0$ small,

$$\limsup_{\alpha\to+\infty} \int_{B^{\tilde{g}}_{\bar{x}_0}(\delta)} |\nabla \bar{v}_\alpha|_{\tilde{g}}^p dv_{\tilde{g}} > 0$$

where $\tilde{g}$ is defined by (2) and $\bar{v}_\alpha(\bar{x}) = v_\alpha(x)$.

Proof. We first prove 1). Assume that Gx_0 is an orbit of concentration of dimension $k' > k$. Then there exists $\delta > 0$ such that $dim\ Gx \geq k' > k$ for any $x \in B_{Gx_0}(\delta)$ (see Faget [3] lemma 2). It thus follows from Hebey-Vaugon (corollary 2 of [6]) and the inequality $\frac{(n-k')p}{n-k'-p} > \frac{(n-k)p}{n-k-p} = p^*$ that the injection $\overset{\circ}{H}{}^p_{1,G}(B_{Gx_0}(\delta')) \hookrightarrow L^{p^*}(B_{Gx_0}(\delta'))$ is compact for all $\delta' \in (0,\delta)$. Since $v_\alpha \to 0$ weakly in $H^p_1(M)$, we get a contradiction with (16). Hence Gx_0 is of minimal dimension k.

Since (v_α) is bounded in $H^p_{1,G}(M)$, there exist two finite positive G-invariant measures μ and ν such that $|v_\alpha|^{p^*} dv_g \rightharpoonup \nu$ and $|\nabla v_\alpha|_g^p dv_g \rightharpoonup \mu$ weakly in the sense of measure. Let $\epsilon > 0$. According to Faget [2], there exists $C_\epsilon > 0$ such that for any α and any G-invariant function $\phi \in C(M)$,

$$\left(\int_M |\phi v_\alpha|^{p^*} dv_g\right)^{\frac{1}{p^*}} \leq \left(\frac{K(n-k,p)}{A^{\frac{1}{n-k}}} + \epsilon\right)\left(\int_M |\nabla(\phi v_\alpha)|_g^p dv_g\right)^{\frac{1}{p}} + C_\epsilon \left(\int_M |\phi v_\alpha|^p dv_g\right)^{\frac{1}{p}}. \tag{18}$$

Passing to the limit in α and then in ϵ in this inequality we get

$$\left(\int_M |\phi|^{p^*} d\nu\right)^{\frac{1}{p^*}} \leq \left(\frac{K(n-k,p)}{A^{\frac{1}{n-k}}}\right)\left(\int_M |\phi|^p d\mu\right)^{\frac{1}{p}}$$

for any G-invariant function $\phi \in C(M)$. Lemma 1.1 in Lions [7] then gives the existence of $I \subset \mathbb{N}$, a sequence of points $(x_i)_{i\in I} \subset M$ and two sequences of positive reals $(\mu_i)_{i\in I}$ and $(\nu_i)_{i\in I}$ such that

$$\begin{aligned} |v_\alpha|^{p^*} dv_g &\rightharpoonup \nu = \sum_{i\in I} \nu_i \delta_{Gx_i}, \\ |\nabla v_\alpha|_g^p dv_g &\rightharpoonup \mu \geq \sum_{i\in I} \mu_i \delta_{Gx_i}, \text{ and} \\ \nu_i^{\frac{p}{p^*}} &\leq \frac{K(n-k,p)^p}{A^{\frac{p}{n-k}}} \mu_i \ \forall\, i \in I. \end{aligned} \tag{19}$$

where δ_{Gx_i} is defined by $\delta_{Gx_i}(\phi) = \int_G \phi(\sigma x_i) dm(\sigma)$ for $\phi \in C(M)$, m being the Haar measure of G such that $m(G) = 1$ (in particular, if ϕ is G-invariant, then $\delta_{Gx_i}(\phi) = \phi(x_i)$). Let $\phi \in C(M)$. Then

$$\begin{aligned} o(1) &= DI_g(v_\alpha).(v_\alpha \phi) \\ &= \int_M |\nabla v_\alpha|_g^p \phi dv_g + \int_M v_\alpha |\nabla v_\alpha|^{p-2} (\nabla v_\alpha, \nabla\phi)_g dv_g - \int_M f\phi |v_\alpha|^{p^*} dv_g. \end{aligned}$$

By Hölder inequality, the second integral tends to 0. We thus get by passing to the limit in the above expression that

$$\int_M \phi d\mu = \int_M \phi f d\nu$$

for any $\phi \in C(M)$. Hence $\mu = f\nu$. In particular $\mu(Gx_i) = \int_{Gx_i} f d\nu$ for any $i \in I$, and thus $\mu_i \leq f(x_i)\nu_i$ for any $i \in I$. This implies that $f(x_i) > 0$ for any $i \in I$. Using (19), we obtain

$$\nu_i \geq \frac{AK(n-k,p)^{k-n}}{f(x_i)^{\frac{n-k}{p}}} \geq \frac{AK(n-k,p)^{k-n}}{(\max f)^{\frac{n-k}{p}}},$$

and

$$\mu_i \geq \frac{AK(n-k,p)^{k-n}}{f(x_i)^{\frac{n-k-p}{p}}} \geq \frac{AK(n-k,p)^{k-n}}{(\max f)^{\frac{n-k-p}{p}}}$$

for any $i \in I$. We now write using (15) that

$$(n-k)\beta = \int_M |\nabla v_\alpha|_g^p dv_g + o(1) = \sum_{i\in I} \mu_i \ge card(I) \frac{AK(n-k,p)^{k-n}}{(\max f)^{\frac{n-k-p}{p}}}$$

which implies that I is finite i.e. (v_α) has a finite number of orbits of concentration, namely the Gx_i, $i \in I$. Eventually,

$$\mu = f\nu = \sum_{i\in I} \nu_i f(x_i)\delta_{Gx_i} \tag{20}$$

which implies the equivalent definition (17) of an orbit of concentration.

We assume that $p \le 2$, $u^0 \equiv 0$ and $DI_g^\alpha(u_\alpha) = 0$. Note that it follows from (19) and (20) that

$$\lim_{\alpha\to+\infty} \int_M \varphi |u_\alpha|^{p^*} dv_g = \sum_{i\in I} \nu_i \varphi(x_i), \tag{21}$$

$$\lim_{\alpha\to+\infty} \int_M \varphi |\nabla u_\alpha|_g^p dv_g = \sum_{i\in I} f(x_i)\nu_i\varphi(x_i) \tag{22}$$

for all $\varphi \in C^0(M)$ $G-$invariant.

We fix $i \in I$ and Gx_i an orbit of concentration. We consider the group G' given by the hypothesis (H) taken at x_i (note that an orbit of concentration has minimal dimension and therefore we can appply (H)) and, given $\epsilon > 0$, let $\eta \equiv \eta_{\bar{x}_i,\epsilon}$.

We assume that $\nabla f(x_i) \neq 0$. We consider a smooth $G'-$invariant function ϕ with compact support in $B_{Gx_i}(\delta)$ such that $\nabla\phi(x_i) = \nabla f(x_i)$ and $\nabla^2\phi(x_i) = 0$. We let $\sigma := |\nabla\phi|_g^{p-2}\nabla\phi$. Since $p \le 2$, it follows from [12] that $u_\alpha \in H_2^p(M)$. In particular, the function $(\sigma, \nabla u_\alpha)_g$ belongs to $H_1^p(M)$. We let $\epsilon > 0$ such that $\nabla\phi(x) \neq 0$ for all $x \in B_{Gx_i}(\epsilon/2)$ and we let η defined above. With (21), we get that

$$\frac{1}{p^*}\int_M (\sigma, \nabla f)_g |u_\alpha|^{p^*} dv_g = \frac{\nu_i}{p^*} |\nabla f(x_i)|_g^p(x_i) + o(1)$$

Independently, we have that

$$\begin{aligned}
\frac{1}{p^*}\int_M (\sigma, \nabla f)_g |u_\alpha|^{p^*} dv_g &= \frac{1}{p^*}\int_M \eta(\sigma, \nabla f)_g |u_\alpha|^{p^*} dv_g + o(1)\\
&= \frac{-1}{p^*}\int_M \mathrm{div}\ \left(\eta |u_\alpha|^{p^*} |\nabla\phi|_g^{p-2}\nabla\phi\right) f\, dv_g + o(1)\\
&= \frac{1}{p^*}\int_M \eta f |u_\alpha|^{p^*} \Delta_p\phi\, dv_g - \frac{1}{p^*}\int_M |\nabla\phi|^{p-2}(\nabla\phi, \nabla\eta)_g |u_\alpha|^{p^*} f\, dv_g\\
&\quad - \int_M \eta f |u_\alpha|^{p^*-2} u_\alpha (\sigma, \nabla u_\alpha)_g\, dv_g + o(1)\\
&= \frac{\nu_i}{p^*}\eta(x_i) f(x_i)\Delta_p\phi(x_i) - \int_M |\nabla u_\alpha|_g^{p-2}(\nabla u_\alpha, \nabla(\eta(\sigma, \nabla u_\alpha)_g))_g\, dv_g\\
&\quad - \int_M h_\alpha |u_\alpha|^{p-2} u_\alpha \eta(\sigma, \nabla u_\alpha)_g\, dv_g + o(1)
\end{aligned}$$

Thanks to Hölder's inequality, the last integral goes to 0, and straightforward computations yield that $\Delta_p\phi(x_i)=0$. We then get that

$$\frac{\nu_i}{p^*}|\nabla f(x_i)|_g^p(x_i) = -\int_M |\nabla u_\alpha|_g^{p-2}(\nabla u_\alpha, \nabla(\eta(\sigma,\nabla u_\alpha)_g))_g\, dv_g + o(1).$$

Passing to the quotient manifolds $N := B_{G'x_i}(\delta)/G'$ and using the G'−invariance, we get that

$$\frac{\nu_i}{p^*}|\nabla f(x_i)|_g^p(x_i) = -\int_N |\nabla \bar u_\alpha|_{\bar g}^{p-2}(\nabla \bar u_\alpha, \nabla(\bar\eta(\bar\sigma,\nabla \bar u_\alpha)_{\bar g}))_{\bar g}\bar v\, dv_{\bar g} + o(1),$$

where we have that $\bar\eta\circ\Pi=\eta$, $\bar u_\alpha\circ\Pi=u_\alpha$ and $\bar\sigma\circ\Pi=\sigma$. We have that

$$\begin{aligned}|\nabla \bar u_\alpha|_{\bar g}^{p-2}(\nabla \bar u_\alpha, \nabla(\bar\eta(\bar\sigma,\nabla \bar u_\alpha)_{\bar g}))_{\bar g} &= \frac{1}{p}\bar\eta(\bar\sigma,\nabla|\nabla\bar u_\alpha|_{\bar g}^p)_g \\ &\quad +|\nabla\bar u_\alpha|_{\bar g}^{p-2}\nabla^i\bar u_\alpha\nabla^j\bar u_\alpha\nabla_j(\bar\eta\bar\sigma_j).\end{aligned}$$

Using Cartan's expansion of $\bar g$ in the exponential chart at $\bar x_i$ and noting that $\nabla\sigma(x_i)=0$ (here, one uses that $\nabla^2\phi(x_i)=0$), we get that

$$|\nabla \bar u_\alpha|_{\bar g}^{p-2}(\nabla \bar u_\alpha, \nabla(\bar\eta(\bar\sigma,\nabla \bar u_\alpha)_{\bar g}))_{\bar g} = \frac{1}{p}\bar\eta(\bar\sigma,\nabla|\nabla\bar u_\alpha|_{\bar g}^p)_{\bar g} + O\left(d_{\bar g}(\bar x_i,\bar x)|\nabla\bar u_\alpha|_{\bar g}^p\right).$$

With (22), we get that

$$\int_N d_{\bar g}(\bar x_i,\bar x)|\nabla\bar u_\alpha|_{\bar g}^p = o(1)$$

when $\alpha\to+\infty$. We then get that

$$\begin{aligned}\frac{\nu_i}{p^*}|\nabla f(x_i)|_g^p(x_i) &= -\frac{1}{p}\int_N \bar\eta(\bar\sigma,\nabla|\nabla\bar u_\alpha|_{\bar g}^p)_{\bar g}\bar v\, dv_{\bar g} + o(1) \\ &= -\frac{1}{p}\int_M \eta(\sigma,\nabla|\nabla u_\alpha|_g^p)_g\, dv_g + o(1) \\ &= -\frac{1}{p}\int_M (\eta\Delta_p\phi - |\nabla\phi|_g^{p-2}(\nabla\eta,\nabla\phi)_g)|\nabla u_\alpha|_g^p\, dv_g + o(1) \\ &= -\frac{1}{p}\Delta_p\phi(x_i)f(x_i)\nu_i + o(1) = o(1)\end{aligned}$$

since $\nabla^2\phi(x_i)=0$. We then get that $\nabla f(x_i)=0$, a contradiction with our initial hypothesis. Then $\nabla f(x_i)=0$ for all $i\in I$. This proof requires the use of $(\sigma,\nabla u_\alpha)_g$ as a test-function and thus cannot work if we deal with (P-S) sequence since then we need the u_α to be bounded in H_2^p (which is irrevelant here because it implies the strong convergence in H_1^p of the u_α to u^0). This explains the restriction imposed to get $\nabla f(x_i)=0$.

We now prove 2). According to 1), $dim\ Gx_0 = k$. Assumption (H) then gives $\delta_0 > 0$ and a subgroup G' of $Isom_g(M)$ such that (H1) and (H2) are satisfied. We have for $\delta \in (0, \delta_0)$

$$\int_{B_{Gx_0}(\delta)} |\nabla v_\alpha|_g^p dv_g = \int_{B^{\bar{g}}_{\bar{x}_0}(\delta)} |\nabla \bar{v}_\alpha|_{\bar{g}}^p \bar{v} dv_{\bar{g}} = \int_{B^{\bar{g}}_{\bar{x}_0}(\delta)} |\nabla \bar{v}_\alpha|_{\tilde{g}}^p dv_{\tilde{g}},$$

where $\bar{v}_\alpha \circ \Pi = v_\alpha$, $\Pi : B_{Gx_0}(\delta) \to N$ being the canonical surjection. Let

$$m = \inf_{\bar{x} \in B^{\bar{g}}_{\bar{x}_0}(\delta)} \bar{v}(\bar{x})^{\frac{1}{n-k-p}} \ and\ M = \sup_{\bar{x} \in B^{\bar{g}}_{\bar{x}_0}(\delta)} \bar{v}(\bar{x})^{\frac{1}{n-k-p}}.$$

Then for any $\bar{x} \in B^{\bar{g}}_{\bar{x}_0}(\delta)$,

$$\frac{1}{M} d_{\bar{g}}(\bar{x}, \bar{x}_0) \le d_{\tilde{g}}(\bar{x}, \bar{x}_0) \le \frac{1}{m} d_{\bar{g}}(\bar{x}, \bar{x}_0).$$

Hence

$$\int_{B_{Gx_0}(\delta)} |\nabla v_\alpha|^p dv_g \le \int_{B^{\tilde{g}}_{\bar{x}_0}(\frac{\delta}{m})} |\nabla \bar{v}_\alpha|_{\tilde{g}}^p dv_{\tilde{g}}$$

which proves the claim. □

Step 1.2 Let Gx_0 be an orbit such that there exist $\delta_0 > 0$ and a subgroup $G' \subset Isom_g(M)$ satisfying (H1) and (H2). Then $(\bar{v}_\alpha)$ is a (P-S) sequence for the functional $\bar{I}_{\tilde{g}}$ defined on $\overset{\circ}{H}{}^p_1(N)$ by

$$\bar{I}_{\tilde{g}}(\bar{u}) = \frac{1}{p} \int_N |\nabla \bar{u}|_{\tilde{g}}^p dv_{\tilde{g}} - \frac{1}{p^*} \int_N \bar{f} |\bar{u}|^{p^*} \bar{v}^{-\frac{p}{n-k-p}} dv_{\tilde{g}}$$

where $N = B_{Gx_0}(\delta_0)/G'$, $\bar{f} \circ \Pi = f$ and $\Pi : B_{Gx_0}(\delta_0) \to N$ is the canonical surjection.

Proof. Let $\bar{\phi} \in C_c^\infty(N)$ and $\phi \in C_c^\infty(B_{Gx_0}(\delta_0))$ such that $\bar{\phi} \circ \Pi = \phi$. Then

$$\begin{aligned}
o(1)\|\bar{\phi}\|_{H_1^p(N)} &= o(1)\|\phi\|_{H_1^p(M)} = DI_g(v_\alpha)\phi \\
&= \int_{B_{Gx_0}(\delta_0)} |\nabla v_\alpha|_g^{p-2} (\nabla v_\alpha, \nabla \phi)_g dv_g - \int_{B_{Gx_0}(\delta_0)} f |v_\alpha|^{p^*-2} v_\alpha \phi dv_g \\
&= \int_{B^{\bar{g}}_{\bar{x}_0}(\delta_0)} |\nabla \bar{v}_\alpha|_{\bar{g}}^{p-2} (\nabla \bar{v}_\alpha, \nabla \bar{\phi})_{\bar{g}} \bar{v} dv_{\bar{g}} - \int_{B^{\bar{g}}_{\bar{x}_0}(\delta_0)} \bar{f} |\bar{v}_\alpha|^{p^*-2} \bar{v}_\alpha \bar{\phi} \bar{v} dv_{\bar{g}} \\
&= \int_{B^{\tilde{g}}_{\bar{x}_0}(\delta_0)} |\nabla \bar{v}_\alpha|_{\tilde{g}}^{p-2} (\nabla \bar{v}_\alpha, \nabla \bar{\phi})_{\tilde{g}} dv_{\tilde{g}} \\
&\quad - \int_{B^{\tilde{g}}_{\bar{x}_0}(\delta_0)} \bar{f} |\bar{v}_\alpha|^{p^*-2} \bar{v}_\alpha \bar{\phi} \bar{v}^{-\frac{p}{n-k-p}} dv_{\tilde{g}} \\
&= D\bar{I}_{\tilde{g}}(\bar{v}_\alpha).\bar{\phi}
\end{aligned}$$

□

As explained above, there exists an orbit of concentration Gx_0. According to Step 1.1, $dim\ Gx_0 = k$. Assumption (H) then gives $\delta_0 > 0$ and a subgroup $G' \subset Isom_g(M)$ satisfying (H1) and (H2) on $B_{Gx_0}(2\delta_0)$. We let $N = B_{Gx_0}(\delta_0)/G'$ and consider, for $t > 0$,

$$\mu_\alpha(t) = \max_{\bar{x}\in N} \int_{B^{\tilde{g}}_{\bar{x}}(t)} |\nabla \bar{v}_\alpha|^p_{\tilde{g}} dv_{\tilde{g}}.$$

In view of Step 1.1, there exist λ_0 such that, up to a subsequence, for any α

$$\mu_\alpha(\delta_0) \geq \int_{B^{\tilde{g}}_{\bar{x}_0}(\delta_0)} |\nabla \bar{v}_\alpha|^p_{\tilde{g}} dv_{\tilde{g}} \geq \lambda_0.$$

Since μ_α is continuous, we then get for any $\lambda \in (0, \lambda_0)$ the existence of $t_\alpha \in (0, \delta_0)$ and $\bar{x}_\alpha \in N$, $\bar{x}_\alpha \to \bar{x}_0$, such that for any α

$$\mu_\alpha(t_\alpha) = \int_{B^{\tilde{g}}_{\bar{x}_\alpha}(t_\alpha)} |\nabla \bar{v}_\alpha|^p_{\tilde{g}} dv_{\tilde{g}} = \lambda.$$

In view of to Step 1.2, $(\bar{v}_\alpha)$ is a (P-S) sequence for $\bar{I}_{\tilde{g}}$ on $\overset{\circ}{H}{}^p_1(N)$. According to Saintier [9], there exist a sequence $R_\alpha \to +\infty$ and $v \in D^p_1\left(\mathbb{R}^{n-k}\right)$, (where $D^p_1(\mathbb{R}^{n-k})$ is the completion of $C^\infty_c(\mathbb{R}^{n-k})$ for the norm $u \mapsto \|\nabla u\|_p$) such that

$$\tilde{v}_\alpha \to v \ in\ H^p_{1,loc}\left(\mathbb{R}^{n-k}\right) \tag{23}$$

and $v \not\equiv 0$, where, if $i_{\tilde{g}}(\bar{x}_0)$ denotes the injectivity radius of $(N, \tilde{g})$ at $\bar{x}_0$,

$$\tilde{v}_\alpha(x) = R_\alpha^{-\frac{n-k-p}{p}} \bar{v}_\alpha\left(exp_{\bar{x}_\alpha}(R_\alpha^{-1}x)\right),\ x \in B_0\left(R_\alpha i_{\tilde{g}}(\bar{x}_0)\right).$$

Actually, the analysis in [9] is performed with a constant function in front of $|\bar{u}|^{p^*}$ in the functional $\bar{I}_{\tilde{g}}(\bar{u})$. In our context here, $\bar{f}\bar{v}^{-\frac{p}{n-k-p}}$ is not constant: however, the analysis for the proof of the result above works the same.

We now prove that

Step 1.3 v is a solution of the Euclidean equation

$$\begin{aligned}\Delta_{p,\xi} v &= \bar{f}(\bar{x}_0)\bar{v}(\bar{x}_0)^{-\frac{p}{n-k-p}} |v|^{p^*-2} v \\ &= f(x_0) Vol(Gx_0)^{-\frac{p}{n-k-p}} |v|^{p^*-2} v\end{aligned}$$

Proof. Let $\phi \in C^\infty_c\left(\mathbb{R}^{n-k}\right)$ and $R > 0$ such that $supp\ \phi \subset B_0(R)$. For α large enough, we define $\phi_\alpha \in C^\infty_c(N)$ by

$$\phi_\alpha(\bar{x}) = R_\alpha^{\frac{n-k-p}{p}} \phi\left(R_\alpha exp_{\bar{x}_\alpha}(\bar{x})\right).$$

Then (ϕ_α) is bounded in $\overset{\circ}{H}{}_1^p(N)$. Thus

$$
\begin{aligned}
o(1) &= D\bar{I}_{\tilde{g}}(\bar{v}_\alpha)\phi_\alpha \\
&= \int_{B_0(R)} |\nabla \tilde{v}_\alpha|_{\tilde{g}_\alpha}^{p-2} (\nabla \tilde{v}_\alpha, \nabla \phi)_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} \\
&\quad - \int_{B_0(R)} |\tilde{v}_\alpha|^{p^*-2} \tilde{v}_\alpha \phi \bar{v}\left(exp_{\bar{x}_\alpha}(R_\alpha^{-1}x)\right)^{-\frac{p}{n-k-p}} \bar{f}\left(exp_{\bar{x}_\alpha}(R_\alpha^{-1}x)\right) dv_{\tilde{g}_\alpha}
\end{aligned}
$$

where $\tilde{g}_\alpha$ is the metric defined in the Euclidean ball $B_0(i_{\tilde{g}}R_\alpha) \subset \mathbb{R}^{n-k}$ by

$$\tilde{g}_\alpha(x) = \left(exp_{\bar{x}_\alpha}^* \tilde{g}\right)(R_\alpha^{-1}x).$$

Since $R_\alpha \to +\infty$, the $\tilde{g}_\alpha$ converge locally uniformly to the Euclidean metric ξ. Passing to the limit, we then get using (23) that

$$\int_{\mathbb{R}^{n-k}} |\nabla v|_\xi^{p-2} (\nabla v, \nabla \phi)_\xi dx - \bar{f}(\bar{x}_0)\bar{v}(\bar{x}_0)^{-\frac{p}{n-k-p}} \int_{\mathbb{R}^{n-k}} |v|^{p^*-2} v \phi dx = 0$$

which proves Step 1.3. □

For $\delta > 0$ small, we let

$$\bar{B}_\alpha(\bar{x}) = \eta_{\bar{x}_\alpha,\delta}(\bar{x}) R_\alpha^{\frac{n-k-p}{p}} v\left(R_\alpha exp_{\bar{x}_\alpha}^{-1}(\bar{x})\right)$$

and $\bar{w}_\alpha = \bar{v}_\alpha - \bar{B}_\alpha$. Then, according to Saintier ([9] Step 2.4),

$$\bar{B}_\alpha \to 0 \text{ weakly in } \overset{\circ}{H}{}_1^p(N), \tag{24}$$

$$D\bar{I}_{\tilde{g}}(\bar{B}_\alpha) \to 0 \text{ and } D\bar{I}_{\tilde{g}}(\bar{w}_\alpha) \to 0 \text{ strongly in } \overset{\circ}{H}{}_1^p(N)', \tag{25}$$

$$\bar{I}_{\tilde{g}}(\bar{w}_\alpha) = \bar{I}_{\tilde{g}}(\bar{v}_\alpha) - E(v) + o(1) \tag{26}$$

where

$$E(v) = \frac{1}{p}\int_{\mathbb{R}^{n-k}} |\nabla v|_\xi^p dx - \frac{\bar{v}(\bar{x}_0)^{-\frac{p}{n-k-p}} f(x_0)}{p^*} \int_{\mathbb{R}^{n-k}} |v|^{p^*} dx.$$

We now define a bubble (B_α) by the relation

$$B_\alpha = \bar{B}_\alpha \circ \Pi$$

and $w_\alpha = v_\alpha - B_\alpha$. We now claim that the following holds:

Step 1.4

$$w_\alpha \to 0 \text{ weakly in } H_1^p(M), \tag{27}$$

$$DI_g(B_\alpha) \to 0 \text{ and } DI_g(w_\alpha) \to 0, \tag{28}$$

$$I_g(w_\alpha) = I_g(v_\alpha) - E(v) + o(1). \tag{29}$$

Proof. We first prove that $B_\alpha \to 0$ weakly in H_1^p (which implies (27) since $v_\alpha \to 0$ weakly in H_1^p). Since $(B_\alpha) \subset H_{1,G'}^p(M)$ is bounded in H_1^p, it suffices to prove that $B_\alpha \to 0$ weakly in $L_{G'}^p(M)$. Let $\psi \in L_{G'}^q(M)$, $q = \frac{p}{p-1}$, and $\bar\psi \in L^q(N, \tilde g)$ be such that $\psi = \bar\psi \circ \Pi$ in $B_{Gx_0}(2\delta)$. Then, using (24),

$$\int_M B_\alpha \psi dv_g = \int_N \bar B_\alpha \bar\psi \bar v^{-\frac{p}{n-k-p}} dv_{\tilde g} \to 0.$$

We prove in the same way that $DI_g(B_\alpha) \to 0$. We now prove that

$$DI_g(w_\alpha) \to 0.$$

Let $\phi \in H_{1,G}^p(M)$, $\delta \in (0, \delta_0/6)$ and $\eta_0 \equiv \eta_{\bar x_0, 3\delta} \in C_c^\infty(B_{Gx_0}(6\delta))$. For α large enough so that $d_{\bar g}(\bar x_\alpha, \bar x_0) < \delta$ (in particular *supp* $\bar B_\alpha \subset B_{\bar x_\alpha}(2\delta) \subset B_{\bar x_0}(3\delta)$), straightforward computations yield

$$\begin{aligned} DI_g(w_\alpha)\phi &= DI_g(w_\alpha)(\eta_0\phi) + DI_g(w_\alpha)((1-\eta_0)\phi) \\ &= D\bar I_{\tilde g}(\bar w_\alpha)(\overline{\eta_0\phi}) + DI_g(v_\alpha)((1-\eta_0)\phi) \\ &= o\left(\|\overline{\eta_0\phi}\|_{H_1^p(N)}\right) + o\left(\|(1-\eta_0)\phi\|_{H_1^p(M)}\right) \\ &= o\left(\|\phi\|_{H_1^p(M)}\right) \end{aligned}$$

Now consider $\phi \in H_1^p(M)$ et $\phi_G \in H_{1,G}^p(M)$ defined by

$$\phi_G(x) = \int_G \phi(\sigma x) dm(\sigma)$$

where m is the Haar measure of G such that $m(G) = 1$. Then, according to what we just did,

$$DI_g(w_\alpha)\phi_G = o(1)\|\phi_G\|_{H_1^p}$$

with

$$\begin{aligned} DI_g(w_\alpha)\phi_G &= \int_G \left(\int_M |\nabla w_\alpha|^{p-2} (\nabla w_\alpha, \nabla(\phi \circ \sigma))_g dv_g \right) dm(\sigma) \\ &\quad - \int_G \left(\int_M f |w_\alpha|^{p^*-2} w_\alpha (\phi \circ \sigma) dv_g \right) dm(\sigma) \\ &= m(G) DI_g(w_\alpha)\phi \end{aligned}$$

and, using Hölder inequality,

$$\begin{aligned}\|\phi_G\|_{H_1^p}^p &= \int_M \left|\int_G \nabla(\phi\circ\sigma)dm(\sigma)\right|^p dv_g + \int_M \left|\int_G (\phi\circ\sigma)dm(\sigma)\right|^p dv_g \\ &\le m(G)^{p-1}\int_M \left(\int_G |\nabla(\phi\circ\sigma)|^p dm(\sigma)\right) dv_g \\ &\quad + m(G)^{p-1}\int_M \left(\int_G |\phi\circ\sigma|^p dm(\sigma)\right) dv_g \\ &\le \|\phi\|_{H_1^p}^p .\end{aligned}$$

Hence

$$DI_g(w_\alpha)\phi = o(1)\|\phi\|_{H_1^p}.$$

It remains to prove (29). We write that

$$I_g(w_\alpha) = \frac{1}{p}\int_{M\setminus B_{Gx_0}(2\delta)} |\nabla v_\alpha|_g^p dv_g - \frac{1}{p^*}\int_{M\setminus B_{Gx_0}(2\delta)} f|v_\alpha|^{p^*} dv_g + \bar{I}_{\tilde{g}}(\bar{w}_\alpha).$$

We then get using (26) that

$$\begin{aligned}I_g(w_\alpha) &= \frac{1}{p}\int_{M\setminus B_{Gx_0}(2\delta)} |\nabla v_\alpha|_g^p dv_g - \frac{1}{p^*}\int_{M\setminus B_{Gx_0}(2\delta)} f|v_\alpha|^{p^*} dv_g + \bar{I}_{\tilde{g}}(\bar{v}_\alpha) \\ &\quad -E(v) + o(1) \\ &= I_g(v_\alpha) - E(v) + o(1)\end{aligned}$$

which proves (29). Note that $v \not\equiv 0$. □

This ends the proof of the Lemma and thus of the H_1^p-decomposition for a Palais-Smale sequences (u_α) for I_g^α of arbitrary sign. If we assume that $u_\alpha > 0$ for any α, then $u^0 \ge 0$ a.e. since $u_\alpha \to u^0$ weakly in H_1^p and thus also almost everywhere (up to a subsequence). Since $u^0 \in H_1^p(M)$ is a weak solution to (E_∞), it follows from Tolksdorf [12] that $u^0 \in C^{1,\theta}(M)$ for some $\theta \in (0,1)$. We then deduce from Vazquez' maximum principle [13] that $u_0 \equiv 0$ or $u^0 > 0$ everywhere. Moreover, according to Saintier [9], the $\bar{B}^i$ are bubbles and hence so are the B^i, $1 \le i \le k$.

2 Proof of the C^0-estimates (10) and (11)

Let (u_α) be a bounded sequence of positive solutions of (E_α). We prove in this section the pointwise estimates of Theorem 0.1. We first prove (10). By standard regularity results (see Tolksdorff [12]), we know that $u^0 \in C^{1,\theta}(M)$ for some $\theta \in (0,1)$, where u^0 is the weak limit in H_1^p of the u_α's. It thus suffices to prove that there exists $C > 0$ such that for every α and every $x \in M$,

$$R_\alpha(x)^{\frac{n-k-p}{p}} u_\alpha(x) \le C. \tag{30}$$

Actually, we are going to prove the following stronger result: there exists $C > 0$ such that

$$v_\alpha(x) := R'_\alpha(x)^{\frac{n-k-p}{p}} u_\alpha(x) \le C \tag{31}$$

for all $x \in M$ and all $\alpha > 0$, where

$$R'_\alpha(x) = \min_{i=1,\dots,l} d_g(G'_i x, G'_i x^i_\alpha)$$

and for all $i \in \{1, ..., l\}$, the group G'_i is given by hypothesis (H) at the orbit of concentration Gx^i_∞, where $\lim_{\alpha\to+\infty} x^i_\alpha = x^i_\infty$.

We assume by contradiction that there exists $y_\alpha \in M$ such that

$$v_\alpha(y_\alpha) = \max_{x\in M} v_\alpha(x) \to +\infty \tag{32}$$

when $\alpha \to +\infty$ and we let $\mu_\alpha := u_\alpha(y_\alpha)^{-p/(n-k-p)} \to 0$ when $\alpha \to +\infty$. We let $\lim_{\alpha\to+\infty} y_\alpha = y_0$, up to extraction.

We claim that the orbit Gy_0 has minimal dimension k. Indeed, we argue by contradiction and assume that dim $Gy_0 > k$. As in Step 1.1, we then get that there exist $q_0 > p^*$ and $\delta > 0$ such that $\lim_{\alpha\to+\infty} u_\alpha = u^0$ in $L^{q_0}(B_{Gy_0}(\delta))$. It then follows from (E_α) and standard regularity theory that $\lim_{\alpha\to+\infty} u_\alpha = u^0$ in $C^0(B_{Gy_0}(\delta'))$ for all $\delta' < \delta$. A contradiction with the assumption (32). This proves the claim.

We then let G' be the group given by hypothesis (H) at the point y_0. We let $I_0 = \left\{i \in \{1, ..., l\}/x^i_\infty \in Gy_0\right\}$ (note that I_0 may be empty). Then, for all $i \in I_0$, we have that $G' = G'_i$. We consider the quotient manifold $N := B_{G'y_0}(\delta)/G'$, where $\delta > 0$ is small and given by (H). Here again, we consider the function $\bar{u}_\alpha(\bar{x}) = u_\alpha(x)$ for $\bar{x} \in N$. We fix $R_0 \in (0, i_{\bar{g}}(\bar{y}_0))$ and we consider the function w_α defined on the Euclidean ball $B_0(R_0\mu_\alpha^{-1})$ by

$$w_\alpha(x) := \mu_\alpha^{\frac{n-k-p}{p}} \bar{u}_\alpha(\exp_{\bar{y}_\alpha}(\mu_\alpha x)).$$

In this expression, the exponential map is taken wrt the metric $\bar{g}$. For $\rho > 0$ and $x \in B_0(\rho) \subset \mathbb{R}^{n-k}$, we let $z_\alpha \in M$ be such that $G'z_\alpha = \bar{z}_\alpha = \exp_{\bar{y}_\alpha}(\mu_\alpha x)$. Given $i \in I_0$, we get that

$$\begin{aligned} d_g(G'z_\alpha, G'x^i_\alpha) &\ge d_g(G'x^i_\alpha, G'y_\alpha) - d_g(G'y_\alpha, G'z_\alpha) \\ &\ge R'_\alpha(y_\alpha) - d_{\bar{g}}(\bar{y}_\alpha, \bar{z}_\alpha) \\ &\ge R'_\alpha(y_\alpha) - \mu_\alpha|x| \\ &\ge \left(1 - \frac{\rho\mu_\alpha}{R'_\alpha(y_\alpha)}\right) R'_\alpha(y_\alpha). \end{aligned}$$

By definition of y_α and μ_α, we have that $\mu_\alpha R'_\alpha(y_\alpha)^{-1} \to 0$ when $\alpha \to +\infty$, and hence the right-hand-side of the above equation is positive. In case $i \notin I_0$, we get that

$$\lim_{\alpha\to+\infty} d_g(\exp_{\bar{y}_\alpha}(\mu_\alpha x), G'_i x^i_\alpha) = d_g(G'y_0, G'_i x^i_\infty)$$
$$= d_g(Gy_0, Gx^i_\infty) > 0 \text{ in } C^0_{loc}(\mathbb{R}^{n-k}).$$

Since $R'_\alpha(y_\alpha) \to 0$ when $\alpha \to +\infty$, we then get that

$$R'_\alpha(\exp_{\bar{y}_\alpha}(\mu_\alpha x)) \geq \frac{1}{2}\left(1 - \frac{\rho\mu_\alpha}{R'_\alpha(y_\alpha)}\right) R'_\alpha(y_\alpha) > 0$$

for all $x \in B_0(\rho)$ and all $\alpha > 0$. We can then write for $x \in B_0(\rho)$ that

$$\begin{aligned} w_\alpha(x) &= \frac{\mu_\alpha^{\frac{n-k-p}{p}} v_\alpha(z_\alpha)}{R'_\alpha(\exp_{\bar{y}_\alpha}(\mu_\alpha x))^{\frac{n-k-p}{p}}} \\ &\leq 2^{(n-k-p)/p}\left(1 - \frac{\rho\mu_\alpha}{R'_\alpha(y_\alpha)}\right)^{-\frac{n-k-p}{p}} \frac{u_\alpha(y_\alpha)^{-1} v_\alpha(y_\alpha)}{R'_\alpha(y_\alpha)^{\frac{n-k-p}{p}}} \\ &\leq 2^{(n-k-p)/p}\left(1 - \frac{\rho\mu_\alpha}{R'_\alpha(y_\alpha)}\right)^{-\frac{n-k-p}{p}} \end{aligned}$$

uniformly for $x \in B_0(\rho) \subset \mathbb{R}^{n-k}$ when $\alpha \to +\infty$. Thus the sequence (w_α) is uniformly bounded on every compact subset of $\mathbb{R}^{n-k}$. Let $\bar{g}_\alpha$ be the Riemannian metric on $\mathbb{R}^{n-k}$ defined by

$$\bar{g}_\alpha(x) = \exp^*_{\bar{y}_\alpha} \bar{g}(\mu_\alpha x).$$

Equation (E_α) becomes

$$-div_{\bar{g}_\alpha}(\tilde{v}_\alpha |\nabla w_\alpha|^{p-2}_{\bar{g}_\alpha} \nabla w_\alpha) + \mu_\alpha^p \tilde{h}_\alpha \tilde{v}_\alpha w_\alpha^{p-1} = \tilde{f}_\alpha \tilde{v}_\alpha w_\alpha^{p^*-1}$$

where $\tilde{h}_\alpha(x) = \bar{h}_\alpha(\exp_{\bar{y}_\alpha}(\mu_\alpha x))$, $\tilde{f}_\alpha(x) = \bar{f}(\exp_{\bar{y}_\alpha}(\mu_\alpha x))$, $\tilde{v}_\alpha(x) = \bar{v}(\exp_{\bar{y}_\alpha}(\mu_\alpha x))$. Since $\mu_\alpha \to 0$ when $\alpha \to +\infty$, the metric $\bar{g}_\alpha$ converges to the Euclidean metric ξ in $C^2_{loc}(\mathbb{R}^{n-k})$ when $\alpha \to +\infty$. It then follows from Tolksdorff [13] that, up to extraction, there exists $w \in C^{1,\theta}(\mathbb{R}^{n-k})$ such that

$$\lim_{\alpha\to+\infty} w_\alpha = w \text{ in } C^{1,\theta}_{loc}(\mathbb{R}^{n-k}).$$

Since $w_\alpha(0) = 1$, we get that $w(0) = 1$ and then $w \not\equiv 0$. We let $R > 0$. Since

$$\int_{B_0(R)} w_\alpha^{p^*} dv_{\bar{g}_\alpha} = \int_{B_{\bar{y}_\alpha}(R\mu_\alpha)} \bar{u}_\alpha^{p^*} dv_{\bar{g}} = \int_{B_{G'y_\alpha}(R\mu_\alpha)} \mathrm{Vol}(G'x)^{-1} u_\alpha^{p^*}(x)\, dv_g(x),$$

we get that

$$\lim_{\alpha\to+\infty} \int_{B_{G'y_\alpha}(R\mu_\alpha)} u_\alpha^{p^*} dv_g = \mathrm{Vol}(Gy_0) \int_{B_0(R)} w^{p^*} dv_\xi > 0.$$

With the H_1^p decomposition of Theorem 0.1, we then get that

$$\begin{aligned} 1 &\leq C \int_{B_{G' y_\alpha}(R\mu_\alpha)} \left(u^0 + \sum_{i=1}^{l} B_\alpha^i + S_\alpha \right)^{p^*} dv_g \\ &\leq C \sum_{i=1}^{l} \int_{B_{G' y_\alpha}(R\mu_\alpha)} (B_\alpha^i)^{p^*} dv_g + o(1) \\ &\leq C \sum_{i\in I_0} \int_{B_{G' y_\alpha}(R\mu_\alpha)} (B_\alpha^i)^{p^*} dv_g + o(1) \\ &\leq C \sum_{i\in I_0} \int_{B_{\bar{y}_\alpha}(R\mu_\alpha)} (\bar{B}_\alpha^i)^{p^*} dv_{\bar{g}} + o(1) \end{aligned}$$

where, here again, we have taken the quotient wrt the group G': this is licit since we work at the points x_α^i such that $x_\infty^i = y_0$. We can then prove exactly as in Saintier [12] that the right-hand side of this inequality goes to 0 as $\alpha \to +\infty$. A contradiction, and then (31) holds.

We claim that (30) holds. Indeed, the proof goes by contradiction and we consider a sequence of points (y_α) such that $\lim_{\alpha\to+\infty} R_\alpha(x)^{\frac{n-k-p}{p}} u_\alpha(y_\alpha) = +\infty$. With arguments similar to the ones above, we get that $\lim_{\alpha\to+\infty} y_\alpha = y_0 \in M$ is such that Gy_0 is an orbit of concentration of the u_α's. Hypothesis (H) yields a group G' that satisfies (H1) and (H2). With (H2), we get that $d_g(Gy_\alpha, Gx_\alpha^i) \leq d_g(G'y_\alpha, G'x_\alpha^i)$ for the i's such that $\lim_{\alpha\to+\infty} x_\alpha^i \in Gy_0$. Studying separately the remaining i's, we get that $R_\alpha(y_\alpha) \leq cR'_\alpha(y_\alpha)$ and we apply (31) to get a contradiction with our initial assumption. This proves that (30) holds.

The proof of (11) goes the same way: if (11) is not satisfies, then we construct a sequence (y_α) which traducts it. We blow-up u_α at y_α and we get a contradiction as above.

Acknowledgements

The author would like to express his gratitude to Prof. Frédéric Robert for his careful reading and helpful comments.

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Received 5 February 2007; accepted 25 June 2007;
published Online First 10 January 2008