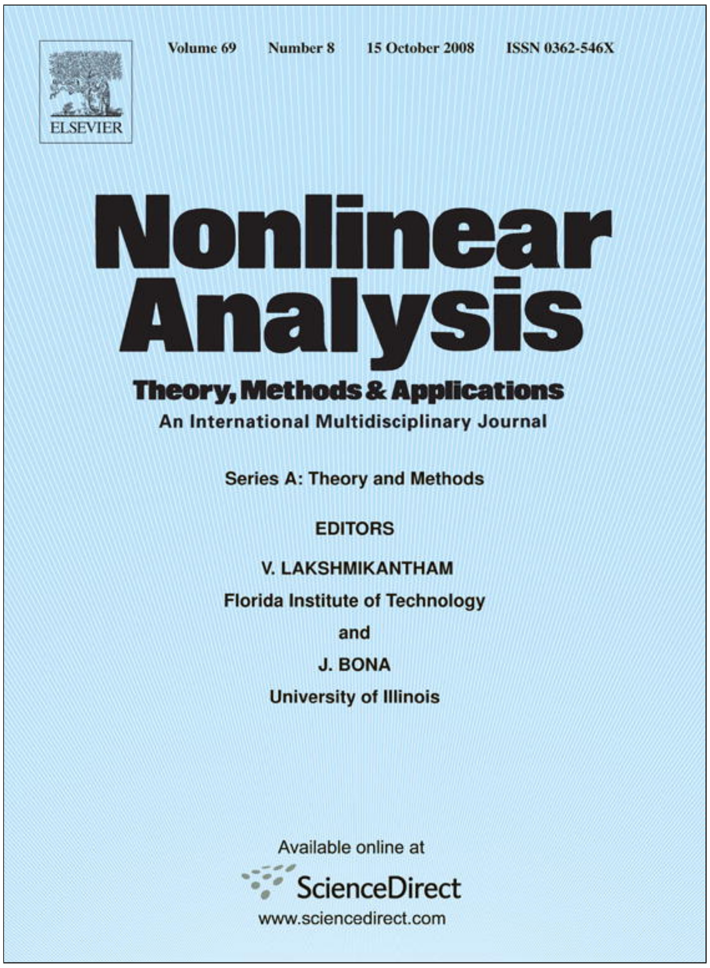
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Estimates of the best Sobolev constant of the embedding of $BV(\Omega)$ into $L^1(\partial\Omega)$ and related shape optimization problems

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Abstract

In this paper we find estimates for the optimal constant in the critical Sobolev trace inequality $\lambda_1(\Omega)\|u\|_{L^1(\partial\Omega)} \le \|u\|_{W^{1,1}(\Omega)}$ that are independent of Ω. These estimates generalize those of [J. Fernandez Bonder, N. Saintier, Estimates for the Sobolev trace constant with critical exponent and applications, Ann. Mat. Pura. Aplicata (in press)] concerning the p-Laplacian to the case $p = 1$.

We apply our results to prove the existence of an extremal for this embedding. We then study an optimal design problem related to λ_1, and eventually compute the shape derivative of the functional $\Omega \to \lambda_1(\Omega)$.

MSC: 35P15 (49Q10; 49Q20)

Keywords: Sobolev trace embedding; Optimal design problems; Critical exponents; Shape analysis; Functions of bounded variations; 1-Laplacian

Let Ω be a bounded smooth domain of $\mathbb{R}^N$. It is well known that the trace embedding from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$ is continuous, where $W^{1,1}(\Omega)$ is the usual Sobolev space of functions $u \in L^1(\Omega)$ such that $\nabla u \in L^1(\Omega)$. The best constant for this embedding is then defined by

$$\lambda_1(\Omega) = \inf_{u\in W^{1,1}(\Omega)\setminus W_0^{1,1}(\Omega)} \frac{\int_\Omega |\nabla u| \mathrm{d}x + \int_\Omega |u| \mathrm{d}x}{\int_{\partial\Omega} |u| \mathrm{d}H^{N-1}}, \tag{1}$$

where $W_0^{1,1}(\Omega)$ denotes the closure for the $W^{1,1}$-norm of the space of smooth functions with compact support in Ω, and H^{N-1} is the $(N-1)$-dimensional Hausdorff measure. The purpose of this paper is to obtain estimates of $\lambda_1(\Omega)$ under geometric assumptions on $\partial\Omega$, and to apply them to some shape optimization problems related to $\lambda_1(\Omega)$.

It turns out to be more convenient when dealing with $\lambda_1(\Omega)$ to rewrite (1) as a minimization problem in the space $BV(\Omega)$ of functions of bounded variation (see [1,8,22]) in the following way:

$$\lambda_1(\Omega) = \inf_{u\in BV(\Omega), u\not\equiv 0 \text{ on } \partial\Omega} \frac{\int_\Omega |\nabla u| + \int_\Omega |u| \mathrm{d}x}{\int_{\partial\Omega} |u| \mathrm{d}H^{N-1}}. \tag{2}$$

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The equivalence between (1) and (2) follows from the fact that given $u \in BV(\Omega)$, there exist $u_n \in C^\infty(\Omega)$ such that $u_n = u$ on $\partial\Omega$ and the u_n's approximate u in the sense that $u_n \to u$ in $L^1(\Omega)$ and $\int_\Omega |\nabla u_n| \mathrm{d}x \to \int_\Omega |\nabla u|$ (see [5, 12]).

We can also express $\lambda_1(\Omega)$ in a more geometric way as an isoperimetric type problem. We recall that a set $A \subset \bar{\Omega}$ is said to be of finite perimeter if its characteristic function χ_A belongs to $BV(\mathbb{R}^n)$. It then follows from the coarea formula that

$$\lambda_1(\Omega) = \inf_{A \subset \bar{\Omega}, \chi_A \in BV(\mathbb{R}^n)} \frac{|\partial A \cap \Omega| + |A|}{|A \cap \partial\Omega|}, \tag{3}$$

where $|\partial A \cap \Omega|$ and $|A \cap \partial\Omega|$ stand for $H^{n-1}(\partial A \cap \Omega)$ and $H^{n-1}(A \cap \partial\Omega)$ respectively. This infimum is always attained by some set of finite perimeter $A \subset \bar{\Omega}$ that we call an eigenset. We refer the reader to [15] for a detailed proof of this result.

We end this presentation of $\lambda_1(\Omega)$ by recalling its value in the case where $\Omega = B_0(R)$ is a ball or an annulus $\Omega = B_0(R) \setminus \bar{B}_0(r)$. As remarked in [2, Remark 1], it follows from [19] that

$$\lambda_1(\Omega) = \begin{cases} \frac{|\Omega|}{|\partial\Omega|} & \text{if } \frac{|\Omega|}{|\partial\Omega|} \le 1 \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

Moreover, if $|\Omega|/|\partial\Omega| \le 1$, then $u = |\partial\Omega|^{-1}\chi_\Omega$ is a minimizer, and the only normalized one if $|\Omega|/|\partial\Omega| = 1$, whereas if $|\Omega|/|\partial\Omega| \ge 1$, there is no extremal for $\lambda_1(\Omega)$.

We first consider the problem of the existence of an extremal for $\lambda_1(\Omega)$. Since the immersion $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$ is not compact, the existence of minimizers for $\lambda_1(\Omega)$ does not follow by standard methods. Indeed this problem has already been considered in [2,5] where it is proved that $\lambda_1(\Omega)$ is attained as soon as

$$\lambda_1(\Omega) < 1. \tag{5}$$

We will provide an alternative proof of this result. Notice that according to [2,5], the large inequality in (5) always holds. We refer the reader to [2] for the derivation of the Euler equation satisfied by a minimizer. According to [19], $\lambda = 1$ is the best first constant in the embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$ in the sense that for any $\epsilon > 0$ there exists $B_\epsilon > 0$ such that for any $u \in BV(\Omega)$,

$$\int_{\partial\Omega} |u| \mathrm{d}H^{N-1} \le (1+\epsilon) \int_\Omega |\nabla u| + B_\epsilon \int_\Omega |u| \mathrm{d}x, \tag{6}$$

and 1 is the lowest constant such that such an inequality holds for any $\epsilon > 0$ and any $u \in BV(\Omega)$. The inequality (5) is then the usual condition ensuring that $\lambda_1(\Omega)$ is attained when dealing with a critical problem (see e.g. [3,7]).

Our first result provides a local geometric condition on Ω for (5) to hold. Before stating it, we need a definition. We say that a point $x \in \partial\Omega$ is a *"good point"* if the curvature of $\partial\Omega$ at x is big enough, or more precisely if the principal curvatures $\lambda_1, \ldots, \lambda_{N-1}$ of $\partial\Omega$ at x are all positive and satisfy $H = \sum_{i=1}^{N-1} \lambda_i > 1$, and if the graph of $\partial\Omega$ around x is close to the parabola $y \to (1/2)\sum \lambda_i y_i^2$ when considered in a local coordinate system such that $x = 0$ and the unit outward normal derivative at 0 of $\partial\Omega$ is $(0, \ldots, 0, 1)$ (see (12) for a precise statement).

The result is the following:

Theorem 1. *If there exists a "good point" $x \in \partial\Omega$, then (5) holds.*

Similarly, we can also prove that (5) holds when a part of $\partial\Omega$ is close to a convex cone of vertex $x \in \partial\Omega$ and angle in $(0, \pi/2)$, that is a non-flat cone, since in that case the "curvature" of $\partial\Omega$ at x is infinite.

It is well known that for $p > 1$, the trace embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ is continuous and compact. In particular the best constant $\lambda_p(\Omega)$ for this embedding, namely

$$\lambda_p(\Omega) = \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p + |u|^p \mathrm{d}x}{\int_{\partial\Omega} |u|^p \mathrm{d}H^{N-1}},$$

is attained by some positive u_p normalized by $\int_{\partial\Omega} u_p^p \mathrm{d}H^{N-1} = 1$. To show the existence of an extremal for $\lambda_1(\Omega)$, the authors of [2] approached $\lambda_1(\Omega)$ via $\lambda_p(\Omega)$. They proved that

$$\lambda_p(\Omega) \to \lambda_1(\Omega) \quad \text{as } p \to 1, \tag{7}$$

and also that

Theorem 2. *If $\lambda_1(\Omega) < 1$, there exists a nonnegative function $u \in BV(\Omega)$ normalized by $\int_{\partial\Omega} |u| \mathrm{d}H^{N-1} = 1$, which attains the infimum in the definition of $\lambda_1(\Omega)$, and such that*

$$u_p \to u \quad \text{in } L^1(\partial\Omega) \quad \text{and} \quad \int_\Omega |\nabla u_p|^p \mathrm{d}x \to \int_\Omega |\nabla u|$$

as $p \to 1$.

We will give a short proof of this result, different from the one provided in [2,5].

As an immediate corollary, we have that

Corollary 1. *If $\partial\Omega$ has a "good point", then $\lambda_1(\Omega)$ is attained.*

As an application of Theorem 1, we study a shape optimization problem related to $\lambda_1(\Omega)$. Given $\alpha \in (0, |\Omega|)$, where $|\Omega|$ denotes the volume of Ω, and a measurable subset $A \subset \Omega$ of volume α, we first consider the minimization problems

$$\lambda_{1,A} = \inf_{\begin{cases} u \in BV(\Omega), u \not\equiv 0 \text{ on } \partial\Omega \\ u = 0 \quad \text{in } A \end{cases}} \frac{\int_\Omega |\nabla u| + |u| \mathrm{d}x}{\int_{\partial\Omega} |u| \mathrm{d}H^{N-1}},$$

and

$$\lambda_{p,A} = \inf_{\begin{cases} u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega) \\ u = 0 \quad \text{in } A \end{cases}} \frac{\int_\Omega |\nabla u|^p + |u|^p \mathrm{d}x}{\int_{\partial\Omega} |u|^p \mathrm{d}H^{N-1}}.$$

It is easily seen that $\lambda_{p,A}$, $p > 1$, is attained. As regards $\lambda_{1,A}$, we have, in the same spirit as what we had for $\lambda_1(\Omega)$, that

Theorem 3. *If*

$$\lambda_{1,A} < 1,$$

there exists an extremal for $\lambda_{1,A}$. Moreover this inequality holds as soon as there exists a good point $x \in \partial\Omega$ such that $A \cap B_x(r) = \emptyset$ for some $r > 0$.

Remark that $\lambda_{p,A}$, $p \geq 1$, does not change if we modify A on a set of Lebesgue measure zero. To give a meaning to $\lambda_{p,A}$, $p > 1$, when $|A| = 0$, the authors of [10] modified $\lambda_{p,A}$ by considering $W_A^{1,p}(\Omega) := \overline{C_c^\infty(\bar{\Omega} \setminus A)}$ in place of $W^{1,p}(\Omega)$. In the case $p = 1$, we introduce in a similar way in place of $BV(\Omega)$ the set $BV_A(\Omega)$ of the functions $u \in BV(\Omega)$ that can be approximated by a sequence $u_\epsilon \in C_c^\infty(\bar{\Omega} \setminus A)$ in the sense that $u_\epsilon \to u$ in $L^1(\Omega)$ and $\int_\Omega |\nabla u_\epsilon| \to \int_\Omega |\nabla u|$. We can then prove as in [8] that $BV_A(\Omega) = BV(\Omega)$ if and only if $\mathrm{cap}_1(A) = 0$, where $\mathrm{cap}_1(A)$ denotes the 1-capacity of A defined by

$$\mathrm{cap}_1(A) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|, u \in BV(\mathbb{R}^n), A \subset \mathrm{int}\{u \geq 1\} \right\}.$$

In the case where A is compact, the coarea formula implies that $\mathrm{cap}_1(A) = \inf |\partial\omega|$ where the infimum is taken over all the smooth open subsets $\omega \subset \mathbb{R}^n$ containing A (see [18]). We consider the minimization problem

$$\lambda'_{1,A} = \inf_{u \in BV_A(\Omega),\, u \not\equiv 0 \text{ on } \partial\Omega} \frac{\int_\Omega |\nabla u| + |u| \mathrm{d}x}{\int_{\partial\Omega} |u| \mathrm{d}H^{N-1}}.$$

Then $\lambda_{1,A} \le \lambda'_{1,A}$ with equality when $\mathrm{cap}_1(A) = 0$. If $\mathrm{cap}_1(A) > 0$, both cases $\lambda_{1,A} = \lambda'_{1,A}$ and $\lambda_{1,A} < \lambda'_{1,A}$ can occur. For example if a part of the boundary of $\Omega \subset \mathbb{R}^2$ has curvature big enough (e.g. like a smooth version of the set $Q_{\delta,\eta}$ defined below next to Theorem 6), then $\lambda_1(\Omega)$ will be attained by some χ_C where $C \subsetneq \Omega$. Then if we put a small curve A in the interior of $\Omega \setminus C$, $\chi_C \in BV_A(\Omega)$ and thus $\lambda_\emptyset = \lambda_{1,A} = \lambda'_{1,A}$. In contrast, if $\Omega \subset \mathbb{R}^2$ is a ball such that $|\partial\Omega| = |\Omega|$, then we know that $\lambda_1(\Omega)$ is attained only by the $\mu\chi_\Omega$, $\mu \in \mathbb{R}$. Then if A is a small segment inside Ω, $\lambda_{1,A} < \lambda'_{1,A}$.

We now want to minimize $\lambda_{p,A}$, $p \ge 1$, when A runs over all the measurable subsets of Ω of volume α, i.e. we look at the following shape optimization problem:

$$\lambda_p(\alpha) = \inf_{A\subset\Omega, |A|=\alpha} \lambda_{p,A}$$

for $p \ge 1$ and $\alpha \in (0, |\Omega|)$.

The optimization problem $\lambda_p(\alpha)$, $p > 1$, has been considered recently. The existence of an optimal set has been established in [10], and its regularity investigated in [11] for $p = 2$. The optimization problem $\lambda_p(\alpha)$ with a critical exponent has been considered in [9]. Such problems of optimal design appear in several branches of applied mathematics, especially for the case $p = 2$, for example in problems of minimization of the energy stored in the design under a prescribed loading. We refer the reader to [4] for more details.

We prove the following relation between $\lambda_p(\alpha)$ and $\lambda_1(\alpha)$:

Theorem 4. *We have*

$$\limsup_{p\to 1} \lambda_p(\alpha) \le \lambda_1(\alpha). \tag{8}$$

Moreover, if there exists a good point $x \in \partial\Omega$, then

$$\lim_{p\to 1} \lambda_p(\alpha) = \lambda_1(\alpha). \tag{9}$$

The proof of this theorem gives the existence of an extremal $u \in BV(\Omega)$ for $\lambda_1(\alpha)$ but, since we can only prove that $|\{u = 0\}| \ge \alpha$ and not $|\{u = 0\}| = \alpha$, we cannot assert the existence of an optimal hole A such that $\lambda_1(\alpha) = \lambda_{1,A}$. However if we consider the following modified optimal design problem:

$$\tilde\lambda_1(\alpha) = \inf_{\begin{cases} u \in BV(\Omega),\, u \not\equiv 0 \text{ on } \partial\Omega \\ |\{u=0\}| = \alpha \end{cases}} \frac{\int_\Omega |\nabla u| + |u| \mathrm{d}x}{\int_{\partial\Omega} |u| \mathrm{d}H^{N-1}}, \tag{10}$$

we can prove that

Theorem 5. *If there exists a good point $x \in \partial\Omega$, then $\tilde\lambda_1(\alpha)$ is attained by some u. In particular $\{u = 0\}$ is an optimal hole for $\tilde\lambda_1(\alpha)$.*

It follows from [10] that $\lambda_p(\alpha) = \tilde\lambda_p(\alpha)$, $p > 1$, where $\tilde\lambda_p(\alpha)$ is defined by

$$\tilde\lambda_p(\alpha) = \inf_{\begin{cases} u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega) \\ |\{u=0\}| = \alpha \end{cases}} \frac{\int_\Omega |\nabla u|^p + |u|^p \mathrm{d}x}{\int_{\partial\Omega} |u|^p \mathrm{d}H^{N-1}},$$

but for the same reason as before, we cannot establish the convergence of $\tilde\lambda_p(\alpha)$ to $\tilde\lambda_1(\alpha)$ as $p \to 1$.

Our last result concerning λ_1 is the computation of the first variation, the so-called shape derivative, of the functional $\Omega \to \lambda_1(\Omega)$. Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector-field, and $\Omega_\delta = T_\delta(\Omega)$, where T_δ is the C^1-diffeomorphism defined for δ small by

$$T_\delta(x) = x + \delta R(x).$$

We will prove that the map $\delta \to \lambda_1(\Omega_\delta)$ is continuous at $\delta = 0$, and also differentiable at $\delta = 0$ under an additional uniqueness assumption holding for example when Ω is a ball.

Remark that if we allow perturbations of the domains that are less regular, we may not have continuity of $\lambda_1(\Omega_\delta)$ as the following example shows. Let $Q = [0, 1]^N$ be the unit cube of $\mathbb{R}^N$, and let $Q_{\delta,\eta} = Q \cup A_{\delta,\eta}$ with

$$A_{\delta,\eta} = [1, 1+\eta] \times [0, \delta] \times [0, 1]^{N-2}, \quad \delta, \eta > 0.$$

Then taking χ_A as a test-function for estimating $\lambda_1(Q_\delta)$, we get

$$\lambda_1(Q_\delta) \le \frac{\delta + \eta\delta}{C\eta} \to 0$$

as $\delta \to 0$ if $\eta \gg \delta$. This shows that, even if $|Q_\delta \Delta Q| \to 0$ or $Q_\delta \to Q$ in Hausdorff distance, we do not have continuity of $\lambda_1(Q_\delta)$. Indeed $\lambda_1(Q_\delta) \to 0 \ne \lambda_1(Q)$.

Shape analysis is the subject of an intense research activity. We refer the reader to for example [14] for an introduction to this field. To the best of the author's knowledge, the shape analysis of a problem involving the L^1-norm of the gradient has only been considered up to know in [13,21] where the authors deal with the best constant for the embedding of $W^{1,1}(\Omega)$ into $L^1(\Omega)$.

Our result is the following:

Theorem 6. *We have*

$$\lambda_1(\Omega_\delta) \to \lambda_1(\Omega)$$

as $\delta \to 0$. Moreover, if we assume that $\lambda_1(\Omega) < 1$ and that there exists a unique nonnegative extremal $u \in BV(\Omega)$ for $\lambda_1(\Omega)$ normalized by $\int_{\partial\Omega} u \, dH^{N-1} = 1$, then $u = |A \cap \partial\Omega|^{-1}\chi_A$ for some set of finite perimeter $A \subset \bar{\Omega}$, and the map $\delta \to \lambda_1(\Omega_\delta)$ is differentiable at $\delta = 0$ with

$$\frac{d}{d\delta}\lambda_1(\Omega_\delta)_{|\delta=0} = \int_{\bar{\Omega}} \{f(\nu)\chi_{\partial^* A\cap\Omega} - \lambda_1(\Omega) f(\vec{n})\chi_{A\cap\partial\Omega} - (R, \nu)\chi_{\partial^* A}\} \frac{dH^{N-1}}{|A \cap \partial\Omega|}, \tag{11}$$

where $f(X) = \operatorname{div} R - (X; DR.X)$, $X \in \mathbb{R}^n$, ν is the Radon–Nikodym derivative of ∇u with respect to $|\nabla u|$, $\vec{n}$ is the unit outward normal to $\partial\Omega$, and $\partial^* A$ is the reduced boundary of A (see [1,8,22]).

In the particular case where Ω is such that $\lambda_1(\Omega) < 1$ and Ω is its own unique eigenset (i.e. $A = \bar{\Omega}$), formula (11) writes as

$$\frac{d}{d\delta}\lambda_1(\Omega_\delta)_{|\delta=0} = \int_{\partial\Omega} \{(R, \vec{n}) - \lambda_1(\Omega)(\operatorname{div} R - (\vec{n}; DR.\vec{n}))\} \frac{dH^{N-1}}{|\partial\Omega|}.$$

Denoting by div_g the divergence operator of the manifold $(\partial\Omega, g)$, where g is the metric induced by the Euclidean metric on $\partial\Omega$, by H the mean curvature of $\partial\Omega$ (i.e. the sum of the principal curvatures of $\partial\Omega$), and by $R_{\partial\Omega}$ the tangential part of R, we have (see [12]):

$$\operatorname{div} R - (\vec{n}; DR.\vec{n}) = \operatorname{div}_g R_{\partial\Omega} + H(R, \vec{n})$$

We thus get according to Green' formula that

$$\frac{d}{d\delta}\lambda_1(\Omega_\delta)_{|\delta=0} = \int_{\partial\Omega} (1 - \lambda_1(\Omega)H)(R, \vec{n}) \frac{dH^{N-1}}{|\partial\Omega|}.$$

The paper is organized as follows. We prove Theorems 1–5 in the following section and Theorem 6 in the last one.

1. Proof of Theorems 1–4

1.1. Proof of Theorem 1

Let $x_0 \in \partial\Omega$ be a "good point". By taking an appropriate coordinate system, we can assume that $x_0 = 0$ and that there exists $r > 0$ such that

$$B_r \cap \Omega = \{(y, t) \in B_r, \ t > \rho(y)\}$$
$$B_r \cap \partial\Omega = \{(y, t) \in B_r, \ t = \rho(y)\}$$

where $y = (y_1, \dots, y_{N-1}) \in \mathbb{R}^{N-1}$, B_r is the Euclidean ball centered at the origin and of radius r, and

$$\rho(y) = \frac{1}{2}|y|_\lambda^2(1 + O(|y|^\alpha))$$

for some $\alpha > 0$, with

$$|y|_\lambda^2 = \sum_{i=1}^{N-1} \lambda_i y_i^2,$$

where the λ_i's are the principal curvatures of $\partial\Omega$ at 0. We assume that α is such that as $\epsilon \to 0$,

$$|\{y \in \mathbb{R}^{N-1}, \rho(y) \le \epsilon^2/2\}\Delta\{y \in \mathbb{R}^{N-1}, |y|_\lambda \le \epsilon\}| = o(\epsilon^{N+1}), \tag{12}$$

where $A\Delta B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of the sets $A, B \subset \mathbb{R}^{N-1}$ and $|A|$ the volume of A. A sufficient condition for (12) to hold is $\alpha > 2$.

We consider the test-functions

$$u_\epsilon(y, t) = \chi_{\Omega\cap\{0\le t\le\epsilon^2/2\}}(y, t).$$

Assume for the moment that the following asymptotic developments hold:

$$\int_\Omega |\nabla u_\epsilon| = b_{N-1}^\lambda \epsilon^{N-1} + o(\epsilon^{N+1}), \tag{13}$$

$$\int_\Omega |u_\epsilon| \mathrm{d}y\mathrm{d}t = \frac{\omega_{N-2}^\xi}{2(N+1)(N-1)\sqrt{\prod \lambda_i}} \epsilon^{N+1} + o(\epsilon^{N+1}), \tag{14}$$

and

$$\int_{\partial\Omega} |u_\epsilon| \mathrm{d}H^{N-1} = \epsilon^{N-1} b_{N-1}^\lambda + \frac{\omega_{N-2}^\xi \sum \lambda_i}{2(N-1)(N+1)\sqrt{\prod \lambda_i}} \epsilon^{N+1} + o(\epsilon^{N+1}), \tag{15}$$

where $b_{N-1}^\lambda = |\{y \in \mathbb{R}^{N-1}, |y|_\lambda \le 1\}|$ and $\omega_{N-2}^\xi = |\{y \in \mathbb{R}^{N-1}, \sum y_i^2 = 1\}|$. It then follows that

$$\begin{aligned} \lambda_1 &\le \frac{\int_\Omega |\nabla u_\epsilon| + \int_\Omega |u_\epsilon| \mathrm{d}x}{\int_{\partial\Omega} |u_\epsilon| \mathrm{d}H^{N-1}} \\ &= 1 + \frac{\omega_{N-2}^\xi}{2(N-1)(N+1)b_{N-1}^\lambda \sqrt{\prod \lambda_i}} \left\{1 - \sum \lambda_i\right\} \epsilon^2 + o(\epsilon^2), \end{aligned}$$

from which we deduce Theorem 1.

We now prove (13)–(15). In view of (12),

$$\begin{aligned} \int_\Omega |\nabla u_\epsilon| &= |\{\rho(y) \le \epsilon^2/2\}| = |\{|y|_\lambda \le \epsilon\}| + o(\epsilon^{N+1}) \\ &= \epsilon^{N-1} b_{N-1}^\lambda + o(\epsilon^{N+1}) \end{aligned}$$

which proves (13). We now prove (14). We first note that

$$\begin{aligned} \int_\Omega |u_\epsilon| \mathrm{d}y\mathrm{d}t &= \int_{\{\rho(y)\le\epsilon^2/2\}} \left(\int_{\rho(y)}^{\epsilon^2/2} \mathrm{d}t \right) \mathrm{d}y \\ &= \frac{\epsilon^2}{2} |\{|y|_\lambda \le \epsilon\}| - \int_{\{|y|_\lambda\le\epsilon\}} \frac{1}{2}|y|_\lambda^2(1 + O(|y|^\alpha))\mathrm{d}y + o(\epsilon^{N+1}) \\ &= \frac{b_{N-1}^\lambda}{2}\epsilon^{N+1} - \frac{\epsilon^{N+1}}{2} \int_{\{|y|_\lambda\le 1\}} |y|_\lambda^2 \mathrm{d}y + o(\epsilon^{N+1}). \end{aligned}$$

Denoting by b^{ξ}_{N-1} (resp. ω^{ξ}_{N-2}) the volume of the unit ball (resp. the unit sphere) of $\mathbb{R}^{N-1}$ for the usual Euclidean metric ξ, we have

$$b^{\lambda}_{N-1} = \frac{b^{\xi}_{N-1}}{\sqrt{\prod \lambda_i}} = \frac{\omega^{\xi}_{N-2}}{(N-1)\sqrt{\prod \lambda_i}},$$

and, by the coarea formula,

$$\begin{aligned}\int_{\{|y|_\lambda \le 1\}} |y|^2_\lambda \mathrm{d}y &= \frac{1}{\sqrt{\prod \lambda_i}} \int_0^1 \left(\int_{\{|y|_\xi = t\}} |y|^2_\xi \mathrm{d}H^{N-2} \right) \mathrm{d}t \\ &= \frac{\omega^{\xi}_{N-2}}{(N+1)\sqrt{\prod \lambda_i}}.\end{aligned}$$

Hence

$$\int_\Omega |u_\epsilon| \mathrm{d}y\mathrm{d}t = \frac{\omega^{\xi}_{N-2}}{2(N+1)(N-1)\sqrt{\prod \lambda_i}} \epsilon^{N+1} + o(\epsilon^{N+1})$$

which is (14). Eventually, to prove (15), we write that

$$\begin{aligned}\int_{\partial\Omega} |u_\epsilon| \mathrm{d}H^{N-1} &= \int_{\{\rho(y) \le \epsilon^2/2\}} \sqrt{1 + |\nabla\rho|^2} \mathrm{d}y \\ &= \int_{\{|y|_\lambda \le \epsilon\}} \sqrt{1 + |\nabla\rho|^2} \mathrm{d}y + o(\epsilon^{N+1}) \\ &= \int_{\{|y|_\lambda \le \epsilon\}} \left(1 + \frac{1}{2} \sum \lambda_i^2 y_i^2 + o(|y|^2_\lambda) \right) \mathrm{d}y + o(\epsilon^{N+1}) \\ &= \epsilon^{N-1} b^{\lambda}_{N-1} + \frac{\epsilon^{N+1}}{2} \int_{\{|y|_\lambda \le 1\}} \sum \lambda_i^2 y_i^2 \mathrm{d}y + o(\epsilon^{N+1})\end{aligned}$$

with, using the symmetry of the sphere and then the coarea formula,

$$\begin{aligned}\int_{\{|y|_\lambda \le 1\}} \sum \lambda_i^2 y_i^2 \mathrm{d}y &= \frac{\sum \lambda_i}{\sqrt{\prod \lambda_i}} \int_{\{|y|_\xi \le 1\}} y_i^2 \mathrm{d}y \\ &= \frac{\sum \lambda_i}{(N-1)\sqrt{\prod \lambda_i}} \int_{\{|y|_\xi \le 1\}} |y|^2_\xi \mathrm{d}y \\ &= \frac{\omega^{\xi}_{N-2} \sum \lambda_i}{(N-1)(N+1)\sqrt{\prod \lambda_i}}.\end{aligned}$$

Hence

$$\int_{\partial\Omega} |u_\epsilon| \mathrm{d}H^{N-1} = \epsilon^{N-1} b^{\lambda}_{N-1} + \frac{\omega^{\xi}_{N-2} \sum \lambda_i}{2(N-1)(N+1)\sqrt{\prod \lambda_i}} \epsilon^{N+1} + o(\epsilon^{N+1})$$

which is (15).

We now assume that, at a point $x \in \partial\Omega$, Ω is close to the cone $C_\omega = \{\lambda\omega, \lambda \ge 0\}$, where ω is a subset of the unit sphere of $\mathbb{R}^N$, in the sense that

$$\begin{aligned}&|\epsilon^{-1}(\Omega - x) \cap B_0(1)| \sim |C_\omega \cap B_0(1)|, \\ &|\epsilon^{-1}\partial(\Omega - x) \cap B_0(1)| \sim |\partial C_\omega \cap B_0(1)|, \\ &|\epsilon^{-1}(\Omega - x) \cap \partial B_0(1)| \sim |C_\omega \cap \partial B_0(1)|\end{aligned}$$

as $\epsilon \to 0$. Using $u_\epsilon = \chi_{\Omega\cap B_x(\epsilon)}$ as a test-function, we have

$$\int_\Omega u_\epsilon \mathrm{d}x = |\Omega \cap B_x(\epsilon)| \sim \epsilon^N |C_\omega \cap B_0(1)|,$$
$$\int_{\partial\Omega} u_\epsilon \mathrm{d}\sigma = |\partial\Omega \cap B_x(\epsilon)| \sim \epsilon^{N-1} |\partial C_\omega \cap B_0(1)|,$$
$$\int_{\partial\Omega} |\nabla u_\epsilon| = |\Omega \cap \partial B_x(\epsilon)| \sim \epsilon^{N-1} |C_\omega \cap \partial B_0(1)| = \epsilon^{n-1}|\omega|,$$

with

$$|\partial C_\omega \cap B_0(1)| = \int_0^1 |\partial(r\omega)|\mathrm{d}r = \frac{|\partial\omega|}{N-1},$$

and thus

$$\lambda_1 \le \frac{|\omega|}{|\partial C_\omega \cap B_0(1)|} + O(\epsilon) = \frac{(N-1)|\omega|}{|\partial\omega|} + O(\epsilon).$$

Hence if $(N-1)|\omega| < |\partial\omega|$, we get (5). In the particular case where ω is a spherical cap, i.e. the intersection of $\partial B_0(1)$ with a half-space H^+ defined by an affine hyperplane H, in such a way that C_ω is convex of angle $\alpha \in (0, \pi/2]$, we can get in a similar way that

$$\lambda_1 \lesssim \frac{(N-1)|H \cap B_0(1)|}{|H \cap \partial B_0(1)|} = \frac{(N-1)\sin^{N-1}(\alpha) b^{\xi}_{N-1}}{\sin^{N-2}(\alpha)\omega^{\xi}_{N-2}}$$
$$= \sin(\alpha).$$

Hence if $\epsilon^{-1}(\Omega - x)$ is asymptotically close to the cone C_ω with angle $\alpha \in (0, \pi/2)$, (5) holds.

1.2. Proof of Theorem 2

We adapt to our case the argument of [6]. In view of (7), the sequence $(\lambda_p)_{p>1}$ is bounded, from which it follows that the sequence $(\|u_p\|_{W^{1,p}})$ is bounded, and eventually that the sequence (u_p) is bounded in $BV(\Omega)$. In particular, there exists $u \in BV(\Omega)$ such that, up to a subsequence, $u_p \to u$ strongly in $L^q(\Omega)$ for all $q < N/(N-1)$ and a.e. In particular, $u \ge 0$ a.e. According to [16] (see also [5]) and in view of (6), there exist a subset $I \subset \mathbb{N}$, a sequence of points $(x_i)_{i\in I} \subset \partial\Omega$ and sequences of positive reals $(\mu_i)_{i\in I}$, $(\nu_i)_{i\in I}$, and two measures μ and ν, with supp $\nu \subset \partial\Omega$, such that

$$\begin{cases} |\nabla u_p|^p \mathrm{d}x \rightharpoonup \mu \ge |\nabla u| + \sum_{i\in I} \nu_i \delta_{x_i}, \\ |u_p|^p \mathrm{d}H^{N-1} \rightharpoonup \nu = |u|\mathrm{d}H^{N-1} + \sum_{i\in I} \nu_i \delta_{x_i}. \end{cases} \tag{16}$$

Let $\sigma_p = |\nabla u_p|^{p-2}\nabla u_p$. Given $q \in [1, +\infty)$, it is easily seen, using Hölder's inequality, that (σ_p) is bounded in $L^q(\Omega)$ for p small enough. Hence there exists $\sigma \in \cap_{q\ge1} L^q(\Omega)$ such that $\sigma_p \to \sigma$ weakly in $L^q(\Omega)$ for every $q > 1$. Notice that $\sigma \in L^\infty(\Omega)$ with $\|\sigma\|_\infty \le 1$. Indeed for any $\psi \in C_c^\infty(\Omega, \mathbb{R}^n)$, we have

$$\left|\int_\Omega \sigma\psi \mathrm{d}x\right| = \lim_{p\to1} \left|\int_\Omega \sigma_p\psi \mathrm{d}x\right| \le \lim_{p\to1} \|\nabla u_p\|_p^{p-1}\|\psi\|_p = \int_\Omega |\psi|\mathrm{d}x.$$

Passing to the limit in the Euler equation for u_p, namely

$$\int_\Omega \sigma_p \nabla\psi \mathrm{d}x + \int_\Omega u_p^{p-1}\psi \mathrm{d}x = \lambda_p(\Omega)\int_{\partial\Omega} u_p^{p-1}\psi \mathrm{d}H^{N-1}, \quad \forall\psi \in W^{1,p}(\bar{\Omega}), \tag{17}$$

we get, in view of (7), that

$$\begin{cases} -\mathrm{div}\,\sigma + 1 = 0 & \text{in } \Omega \\ \sigma.\vec{n} = \lambda_1(\Omega) & \text{on } \partial\Omega, \end{cases} \tag{18}$$

where $\vec{n}$ is the unit outward normal to $\partial\Omega$. Let $\phi \in C^\infty(\bar{\Omega})$. Passing to the limit in (17) with $\psi = u_p\phi$, using (7), we obtain

$$\int_\Omega \phi \mathrm{d}\mu + \int_\Omega u\sigma\nabla\phi \mathrm{d}x + \int_\Omega u\phi \mathrm{d}x = \lambda_1(\Omega)\int_{\partial\Omega}\phi \mathrm{d}\nu. \tag{19}$$

According to the definition of the measure $\sigma\nabla u$, defined weakly by integration by parts (see [6]), and in view of (18), we have

$$\begin{aligned}\int_\Omega u\sigma\nabla\phi \mathrm{d}x &= \int_\Omega \mathrm{div}\,(\phi u\sigma)\mathrm{d}x - \int_\Omega \phi u(\mathrm{div}\,\sigma)\mathrm{d}x - \int_\Omega \phi(\sigma\nabla u)\\ &= \lambda_1(\Omega)\int_{\partial\Omega}\phi u \mathrm{d}H^{N-1} - \int_\Omega \phi u \mathrm{d}x - \int_\Omega \phi(\sigma\nabla u).\end{aligned} \tag{20}$$

Plugging this in (19) and using the definitions of μ and ν, we eventually get

$$\int_\Omega \phi(|\nabla u| - \sigma\nabla u) \le (\lambda_1 - 1)\int_\Omega \phi\left(\sum_{i\in I}\nu_i\delta_{x_i}\right).$$

Since $|\sigma\nabla u| \le \|\sigma\|_\infty|\nabla u| \le |\nabla u|$ and $\lambda_1 < 1$ by assumption, we deduce that $\nu_i = 0$ for all $i \in I$. In particular $\int_{\partial\Omega} u\mathrm{d}H^{N-1} = 1$. Moreover, inserting (20) into (19), we see that $\mu = \sigma\nabla u \le |\nabla u|$. Hence $\mu = |\nabla u|$.

1.3. *Proof of Theorem 3*

The proof of the first part is analogous to the proof of Theorem 2. As regards the second part, just remark that since the principal curvatures at *good points* $x \in \partial\Omega$ are positive, we have supp $u_\epsilon \subset B_x(r)$ for ϵ small, where u_ϵ is the sequence of test-functions considered in the proof of Theorem 1. Hence the u_ϵ's are also admissible test-functions for $\lambda_{1,A}$.

1.4. *Proof of Theorem 4*

We first prove (8). Given $\epsilon > 0$, let $D \subset \Omega$ measurable, $|D| = \alpha$, be such that

$$\lambda_1(D) \le \lambda_1(\alpha) + \epsilon.$$

The same arguments as were used to prove (7) show that $\lambda_p(D) \to \lambda_1(D)$ as $p \to 1$ (see [2]). Hence

$$\limsup_{p\to 1}\lambda_p(\alpha) \le \lim_{p\to 1}\lambda_p(D) = \lambda_1(D) \le \lambda_1(\alpha) + \epsilon.$$

Since ϵ is arbitrary, we deduce (8).

As regards (9), we first note that

$$\lambda_p(\alpha) = \inf_{u\in W^{1,p}(\Omega),|\{u=0\}|\ge\alpha}\frac{\int_\Omega |\nabla u|^p + |u|^p \mathrm{d}x}{\int_{\partial\Omega}|u|^p \mathrm{d}H^{N-1}}$$

and, in the same way,

$$\lambda_1(\alpha) = \inf_{u\in BV(\Omega),|\{u=0\}|\ge\alpha}\frac{\int_\Omega |\nabla u| + \int_\Omega |u|\mathrm{d}x}{\int_{\partial\Omega}|u|\mathrm{d}H^{N-1}}.$$

For $p > 1$, it is known (see [10]) that the last infimum is attained by some non-negative u_p normalized by $\int_{\partial\Omega}|u_p|^p \mathrm{d}H^{N-1} = 1$, and satisfying $|\{u_p = 0\}| = \alpha$. Independently, since there exists a *good point* $x \in \partial\Omega$, we have

$$\lambda_1(\alpha) < 1. \tag{21}$$

Indeed, let $D \subset \Omega$ be measurable of volume α and consider $D' := (D \setminus B_x(r)) \cup \bar{D}$ for a small $r > 0$ and $\bar{D} \subset \Omega$ being such that $|D'| = \alpha$ and $\bar{D} \subset \Omega \setminus B_x(r)$. Then $D' \cap B_x(r) = \emptyset$, and thus, according to Theorem 1,

$$\lambda_1(\alpha) \le \lambda_1(D') < 1,$$

as we wanted to prove. Now, as in the proof of Theorem 1 and in view of (21), we have that, along a subsequence,

$$\begin{cases} u_p^p \to u \quad \text{in } L^1(\Omega) \text{ and a.e.} \\ \int_\Omega |\nabla u_p|^p \mathrm{d}x \to \int_\Omega |\nabla u| \\ \int_{\partial\Omega} u \mathrm{d}H^{N-1} = \lim_{p\to 1} \int_{\partial\Omega} u_p^p \mathrm{d}H^{N-1} = 1 \end{cases}$$

as $p \to 1$, for some non-negative $u \in BV(\Omega)$. In particular $|\{u = 0\}| \ge \alpha$. Hence

$$\begin{aligned}\lambda_p(\alpha) &= \int_\Omega |\nabla u_p|^p \mathrm{d}x + \int_\Omega |u_p|^p \mathrm{d}x = \int_\Omega |\nabla u| + \int_\Omega |u| \mathrm{d}x + o(1) \\ &\ge \lambda_1(\alpha).\end{aligned}$$

This proves (9).

1.5. *Proof of Theorem 5*

A straightforward modification of the proof of (3) allows us to rewrite (10) as

$$\tilde{\lambda}_1(\alpha) = \inf_{\begin{cases} C \subset \bar{\Omega}, \chi_C \in BV(\mathbb{R}^n) \\ |\Omega \setminus C| = \alpha \end{cases}} \frac{|\partial C \cap \Omega| + |C|}{|C \cap \partial\Omega|}. \tag{22}$$

Let (C_n) be a minimizing sequence for this problem. As in the proof of Theorem 4, the existence of a *good point* $x \in \partial\Omega$ implies that

$$\tilde{\lambda}_1(\alpha) < 1. \tag{23}$$

In particular, for n large enough,

$$|\partial C_n \cap \Omega| + |C_n| \le 2|C_n \cap \partial\Omega| \le 2|\partial\Omega|,$$

from which we deduce that (χ_{C_n}) is bounded in $BV(\Omega)$. Hence there exists a set of finite perimeter C such that $\chi_{C_n} \to \chi_C$ in $L^1(\Omega)$ and a.e. In particular $|\Omega \setminus C| = \alpha$. Moreover, as in the proof of Theorem 6 below, we can deduce from (23) that $\int_\Omega |\nabla \chi_{C_n}| \to \int_\Omega |\nabla \chi_{C_n}|$, i.e. $|\partial C_n \cap \Omega| \to |\partial C \cap \Omega|$, and $\int_{\partial\Omega|} \chi_{C_n} \mathrm{d}H^{N-1} \to \int_{\partial\Omega|} \chi_C \mathrm{d}H^{N-1}$, i.e. $|C_n \cap \partial\Omega| \to |C_n \cap \partial\Omega|$. Hence C attains the infimum in (22), which proves Theorem 5.

2. Proof of Theorem 6

To simplify the notation, we let $\lambda = \lambda_1(\Omega)$ and $\lambda_\delta = \lambda_1(\Omega_\delta)$.

According to the change of variable formula for functions of bounded variations [12], and the change of variable formula for the boundary integral [14], we have that

$$\lambda_\delta = \inf_{u \in BV(\Omega), u \not\equiv 0 \text{ on } \partial\Omega} Q_\delta(u)$$

with

$$Q_\delta(u) = \frac{\int_\Omega |(DT_\delta)^{-1}\nu||\det DT_\delta||\nabla u| + \int_\Omega |u||\det DT_\delta| \mathrm{d}x}{\int_{\partial\Omega} |u|\,|{}^t(DT_\delta)^{-1}\vec{n}||\det DT_\delta| \mathrm{d}H^{N-1}},$$

where ν is the Radon–Nikodym derivative of ∇u with respect to $|\nabla u|$, and $\vec{n}$ is the unit outward normal to Ω. We also let $Q = Q_0$, namely

$$Q(u) = \frac{\int_\Omega |\nabla u| + \int_\Omega |u| \mathrm{d}x}{\int_{\partial\Omega} |u| \mathrm{d}H^{N-1}},$$

so that

$$\lambda_\delta = \inf_{u\in BV(\Omega), u\not\equiv 0 \text{ on } \partial\Omega} Q(u).$$

We first prove that for any $u \in BV(\Omega)$,

$$Q_\delta(u) = (1 + O(\delta))Q(u)$$

where the $O(\delta)$ is uniform in u. The continuity of $\delta \to \lambda_\delta$ at $\delta = 0$ then easily follows. Let $u \in BV(\Omega)$. Since $|\nu| = 1 |\nabla u|$-a.e., we can assume that $|\nu| = 1$ everywhere. Then

$$|(DT_\delta)^{-1}\nu| = 1 - (\nu, DR.\nu)\delta + o(\delta), \tag{24}$$

and in the same way,

$$|{}^t(DT_\delta)^{-1}\vec{n}| = 1 - (\vec{n}, DR.\vec{n})\delta + o(\delta). \tag{25}$$

We also have

$$|\det DT_\delta| = \det DT_\delta = 1 + \delta(\operatorname{div} R) + o(\delta), \tag{26}$$

all the $o(\delta)$ being uniform in $x \in \bar{\Omega}$. Since $R \in C^1(\bar{\Omega})$, we get

$$Q_\delta(u) = \frac{(1 + O(\delta)) \int_\Omega (|\nabla u| + |u| \mathrm{d}x)}{(1 + O(\delta)) \int_{\partial\Omega} |u| \mathrm{d}H^{N-1}} = (1 + O(\delta))Q(u),$$

as we wanted to prove. Theorem 6 then easily follows.

We now assume that $\lambda < 1$. Since then $\limsup_{\delta\to 0} \lambda_\delta < 1$, it follows from Theorem 2 that there exists a nonnegative extremal $v_\delta \in BV(\Omega_\delta)$ for λ_δ normalized by $\int_{\partial\Omega_\delta} v_\delta \mathrm{d}H^{N-1} = 1$. Let $u_\delta = v_\delta \circ T_\delta \in BV(\Omega)$. Then the sequence (u_δ) is bounded in $BV(\Omega)$. Indeed, according to (24) and (26), we have

$$\begin{aligned}\int_\Omega |\nabla u_\delta| + \int_\Omega u_\delta \mathrm{d}x &= \int_{\Omega_\delta} |(DT_\delta^{-1})^{-1}\nu_{v_\delta}||\det DT_\delta^{-1}||\nabla v_\delta| + \int_{\Omega_\delta} v_\delta |\det DT_\delta^{-1}| \mathrm{d}x \\ &= (1 + O(\delta)) \int_{\Omega_\delta} |\nabla v_\delta| + v_\delta \mathrm{d}x = (1 + O(\delta))\lambda_\delta \\ &= (1 + o(1))\lambda.\end{aligned}$$

There thus exists a nonnegative $u \in BV(\Omega)$ such that $u_\delta \to u$ in $L^1(\Omega)$. Moreover, as in the proof of Theorem 2,

$$|\nabla u_\delta| \rightharpoonup \mu \geq |\nabla u| + \sum_{i\in I} \nu_i \delta_{x_i},$$

$$|u_\delta| \mathrm{d}H^{N-1} \rightharpoonup \nu = |u| \mathrm{d}H^{N-1} + \sum_{i\in I} \nu_i \delta_{x_i}.$$

We can now obtain

$$\begin{aligned}\lambda &= \lim_{\delta\to 0} \lambda_\delta = \lim_{\delta\to 0} Q_\delta(v_\delta) = \lim_{\delta\to 0} (1 + O(\delta))Q(u_\delta) \geq \frac{\int_\Omega |\nabla u| + \sum_{i\in I} \nu_i + \int_\Omega u \mathrm{d}x}{\int_{\partial\Omega} u \mathrm{d}H^{N-1} + \sum_{i\in I} \nu_i} \\ &\geq \frac{\lambda \int_{\partial\Omega} u \mathrm{d}H^{N-1} + \sum_{i\in I} \nu_i}{\int_{\partial\Omega} u \mathrm{d}H^{N-1} + \sum_{i\in I} \nu_i},\end{aligned}$$

i.e. $\lambda \sum_{i\in I} \nu_i \geq \sum_{i\in I} \nu_i$. Since $\lambda < 1$, we must have $\nu_i = 0$ for all $i \in I$, so that

$$1 = \int_{\partial\Omega} v_\delta \mathrm{d}H^{N-1} = \int_{\partial\Omega} u_\delta \mathrm{d}H^{N-1} + o(1) = \int_{\partial\Omega} u \mathrm{d}H^{N-1} + o(1).$$

Using the inferior semi-continuity of the total variation, we can now write

$$\lambda = \lim \lambda_\delta = \lim Q_\delta(v_\delta) = \lim(1 + O(\delta))Q(u_\delta) \geq \frac{\int_\Omega |\nabla u| + \int_\Omega u \mathrm{d}x}{\int_{\partial\Omega} u \mathrm{d}H^{N-1}} \geq \lambda.$$

Hence u is an eigenfunction for λ and

$$\begin{aligned} &\int_\Omega |\nabla u_\delta| \to \int_\Omega |\nabla u|, \\ &\int_{\partial\Omega} u_\delta \mathrm{d}H^{N-1} \to \int_{\partial\Omega} u \mathrm{d}H^{N-1}. \end{aligned} \tag{27}$$

We now prove the formula for the derivative (11). We first get using (24)–(26) that

$$\begin{aligned} Q_\delta(u) &= \frac{\int_\Omega (1 + \delta f(\nu) + o(\delta))\, |\nabla u| + \int_\Omega (1 + \delta \operatorname{div} R + o(\delta)) u \mathrm{d}x}{\int_{\partial\Omega} (1 + \delta f(\vec{n}) + o(\delta)) u \, \mathrm{d}H^{N-1}} \\ &= \frac{\lambda + \delta \left(\int_\Omega f(\nu)|\nabla u| + u \operatorname{div} R \mathrm{d}x \right) + o(\delta)}{1 + \delta \int_{\partial\Omega} f(\vec{n}) u \mathrm{d}H^{N-1} + o(\delta)} \\ &= \lambda + \delta \left(\int_\Omega (f(\nu)|\nabla u| + u \operatorname{div} R \mathrm{d}x) - \lambda \int_{\partial\Omega} f(\vec{n}) u \mathrm{d}H^{N-1} \right) + o(\delta), \end{aligned}$$

where

$$f(X) = \operatorname{div} R - (X, DR.X), \quad X \in \mathbb{R}^n. \tag{28}$$

Hence

$$\begin{aligned} \lambda_\delta - \lambda &\leq Q_\delta(u) - \lambda \\ &= \delta \left(\int_\Omega (f(\nu)|\nabla u| + u \operatorname{div} R \mathrm{d}x) - \lambda \int_{\partial\Omega} f(\vec{n}) u \mathrm{d}H^{N-1} \right) + o(\delta). \end{aligned} \tag{29}$$

It remains to prove the opposite inequality. Letting $\nu_\delta \equiv \nu_{u_\delta}$, we obtain, using (24)–(26) and the strong convergence $u_\delta \to u$ in $L^1(\Omega)$, that

$$\begin{aligned} Q_\delta(u_\delta) &= \frac{\int_\Omega \{1 + \delta f(\nu_\delta) + o(\delta)\}\, |\nabla u_\delta| + \int_\Omega (1 + \delta \operatorname{div} R + o(\delta)) u_\delta \mathrm{d}x}{\int_{\partial\Omega} |u_\delta| \mathrm{d}H^{N-1} + \delta \int_{\partial\Omega} f(\vec{n}) u_\delta \mathrm{d}H^{N-1} + o(\delta)} \\ &= \frac{\int_\Omega (|\nabla u_\delta| + u_\delta \mathrm{d}x) + \delta \int_\Omega \{f(\nu_\delta)|\nabla u_\delta| + (\operatorname{div} R) u \mathrm{d}x\} + o(\delta)}{\int_{\partial\Omega} u_\delta \mathrm{d}H^{N-1} + \delta \int_{\partial\Omega} f(\vec{n}) u \mathrm{d}H^{N-1} + o(\delta)}. \end{aligned}$$

We can rewrite (27) as

$$\int_{\bar\Omega} |\nabla \bar{u}_\delta| \to \int_{\bar\Omega} |\nabla \bar{u}|, \tag{30}$$

where $\bar{u}_\delta$ (resp. $\bar{u}$) denotes the extension of u_δ (resp. u) to $\mathbb{R}^n \setminus \bar\Omega$ by 0. Independently, we clearly have the weak convergence of $\nabla \bar{u}_\delta$ to $\nabla \bar{u}$. We can thus apply Reshetnyak's theorem [20,17,1] to get that

$$\int_{\bar\Omega} g(x, \nu_\delta(x))|\nabla \bar{u}_\delta| \to \int_{\bar\Omega} g(x, \nu(x))|\nabla \bar{u}|$$

for any continuous function $g : \bar\Omega \times S \to \mathbb{R}$, where S denotes the unit sphere of $\mathbb{R}^n$. In particular

$$\int_\Omega f(\nu_\delta)|\nabla u_\delta| \to \int_\Omega f(\nu)|\nabla u|.$$

Hence

$$Q_\delta(u_\delta) = Q(u_\delta) + \delta \left\{ \int_\Omega (f(\nu)|\nabla u| + u \operatorname{div} R \mathrm{d}x) - \lambda \int_{\partial\Omega} f(\vec{n}) u \mathrm{d}H^{N-1} \right\} + o(\delta).$$

We now have

$$\begin{aligned} \lambda_\delta - \lambda &\geq Q_\delta(u_\delta) - Q(u_\delta) \\ &= \delta \left(\int_\Omega (f(\nu)|\nabla u| + u \operatorname{div} R \mathrm{d}x) - \lambda \int_{\partial\Omega} f(\vec{n}) u \mathrm{d}H^{N-1} \right) + o(\delta). \end{aligned} \tag{31}$$

We deduce from (29) and (31) and the uniqueness of u that the map $\delta \to \lambda_\delta$ is differentiable at $\delta = 0$ with

$$\lambda'_\delta(0) = \int_\Omega (f(\nu)|\nabla u| + u \operatorname{div} R \mathrm{d}x) - \lambda \int_{\partial\Omega} f(\vec{n}) u \mathrm{d}H^{N-1}. \tag{32}$$

As there always exists an eigenset $A \subset \bar{\Omega}$, i.e. a set of finite perimeter that attains the infimum in (3), and since u is by hypothesis the only normalized eigenfunction for λ, we have $u = |A \cap \partial\Omega|^{-1} \chi_A$. It follows from geometric measure theory that $|\nabla \chi_A| = |A \cap \partial\Omega|^{-1} H^{N-1}_{|\partial^* A}$ (see [1,8,22]). Recalling the definition (28) of f and using the Green formula for sets of finite perimeter, we can now rewrite (32) as (11).

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References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of bounded variations and free discontinuity problems, in: Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
[2] F. Andreu, J.M. Mazon, J.D. Rossi, The best constant for the Sobolev embedding form $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, Nonlinear Anal. 59 (2004) 1125–1145.
[3] T. Aubin, Equations différentielles non-linéaires et le problème de Yamabé concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976) 269–296.
[4] A. Cherkaev, E. Cherkaeva, Optimal design for uncertain loading condition, in: Homogenization, in: Ser. Adv. Math. Appl. Sci., vol. 50, World Sci. Publishing, River Edge, NJ, 1999, pp. 193–213.
[5] F. Demengel, On some nonlinear equation involving the 1-laplacian and trace map inequalities, Nonlinear Anal. 48 (2002) 1151–1163.
[6] F. Demengel, On some nonlinear partial differential equations involving the 1-Laplacian and critical Sobolev exponent, ESAIM 4 (1999) 667–686.
[7] O. Druet, E. Hebey, The AB program in geometric analysis: Sharp Sobolev inequalities and related problems, Mem. Amer. Math. Soc. 160 (2002).
[8] L.C. Evans, R.F. Gariepy, Measure theory and fine properties of functions, in: Studies in Advanced Math., CRC Press, Ann Harbor, 1992.
[9] J. Fernandez Bonder, N. Saintier, Estimates for the Sobolev trace constant with critical exponent and applications, Ann. Mat. Pura. Aplicata (in press).
[10] J. Fernández Bonder, J.D. Rossi, N. Wolanski, On the best Sobolev trace constant and extremals in domains with holes, Bull. Sci. Math. 130 (2006) 565–579.
[11] J. Fernández Bonder, J.D. Rossi, N. Wolanski, Regularity of the free boundary in an optimization problem related to the best Sobolev trace constant, SIAM J. Control Optim. 44 (5) (2006) 1614–1635.
[12] E. Giusti, Minimal surfaces and functions of bounded variation, in: Monographs in Mathematics, Birkhäuser, 1984.
[13] E. Hebey, N. Saintier, Stability and perturbations of the domain for the first eigenvalue of the 1-laplacian, Arch. Math. 86 (4) (2006) 340–351.
[14] A. Henrot, M. Pierre, Variation et optimisation de formes - une analyse géométrique, in: Mathématiques et applications, vol. 48, Springer, Berlin, New York, 2005.
[15] I.R. Ionescu, T. Lachand-Robert, Generalized Cheeger sets related to landslides, Calc. Var. Partial Differential Equations 23 (2005) 227–249.
[16] P.L. Lions, The concentration–compactness principle in the calculus of variations—the limit case part. 2, Rev. Mat. Iberoamericana 1 (2) (1985) 45–121.
[17] S. Luckhaus, L. Modica, The Gibbs–Thompson relation within the gradient theory of phase transitions, ARMA 107 (1) 71–83.
[18] V.G. Maz'ya, Sobolev Spaces, in: Springer Series in Soviet Mathematics, Berlin, New York, 1985.
[19] M. Motron, Around the best constants for the Sobolev trace map from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, Asymptot. Anal. 29 (2002) 69–90.
[20] Yu.G. Reshetnyak, Weak convergence of completely additive vector functions on a set, Siberian Math. J. 9 (1968) 1039–1045. Translated from Sibirskii Mathematicheskii Zhurnal 9 (1968) 1386–1394.
[21] N. Saintier, Shape derivative of the first eigenvalue of the 1-Laplacian (submitted for publication).
[22] W.P. Ziemer, Weakly differentiable functions. Sobolev spaces and functions of bounded variations, in: Graduate Texts in Mathematics, vol. 120, Springer-Verlag, 1989.