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SCHAUDER ESTIMATES FOR DEGENERATE ELLIPTIC AND PARABOLIC EQUATIONS IN $\mathbb{R}^N$ WITH LIPSCHITZ DRIFT

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Abstract. We prove existence, uniqueness and Schauder estimates for the degenerate elliptic and parabolic equations (E) and (NHCP) in $\mathbb{R}^N$ associated to the degenerate Kolmogorov operator (K) defined below.

We are concerned in this paper with the properties in Hölder spaces of the degenerate Kolmogorov operator $\mathcal{K}$ defined on a smooth function $u:(x,y)\in\mathbb{R}^n\times\mathbb{R}^n\to u(x,y)\in\mathbb{R}$ by

$$\mathcal{K}u := \tfrac{1}{2}\Delta_x u + F(x,y).D_x u + x.D_y u, \tag{K}$$

where $F:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is assumed to be of class C^3 with bounded derivatives up to the third order, and $D_x u$, $D_y u$ denote the gradient of u with respect to x and y respectively. The main features of this operator are its degeneracy in y and its unbounded drift. The purpose of this paper is to prove existence and uniqueness of a solution of the elliptic equation

$$\lambda - \mathcal{K}u = f \qquad \text{in } \mathbb{R}^{2n}, \tag{E}$$

and of the nonhomogeneous Cauchy problem

$$\begin{cases} \partial_t u = \mathcal{K}u + g & \text{in } (0,+\infty)\times\mathbb{R}^{2n} \\ u(0,\cdot) = f & \text{in } \mathbb{R}^{2n} \end{cases} \tag{NHCP}$$

associated with $\mathcal{K}$, to give estimates in the sup-norm of the spatial derivatives of the solutions, and to prove Schauder estimates.

According to Freidlin [10], the operator $\mathcal{K}$ arises, for example, in the study of the motion $y(t)\in\mathbb{R}^n$ of a particle of mass one subject to a force field

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$F(y, y')$ perturbed by a noise. Indeed, y satisfies the stochastic differential equation

$$y''(t) = F(y(t), y'(t)) + W'(t), \qquad y(0) = y_0,\ y'(0) = x_0, \qquad (0.1)$$

where $(W(t))_t$ is the standard Brownian motion in $\mathbb{R}^n$. Setting $x(t) = y'(t)$, (0.1) is equivalent to the system

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} F(x,y) \\ x(t) \end{pmatrix} + \begin{pmatrix} W'(t) \\ 0 \end{pmatrix} \\ (x(0), y(0)) = (x_0, y_0) \end{cases}$$

and $\mathcal{K}$ is then the Kolmogorov (or Dynkin) operator associated with this diffusion.

Concerning the study of $\mathcal{K}$ in Lebesgue spaces, with respect to some invariant measure naturally associated to $\mathcal{K}$, we refer to Farkas-Lunardi [8] and Da Prato-Lunardi [7] [6], where regularity and dissipativity results are proved in the case where F is a gradient perturbation, in y, of a linear term.

The study in Hölder spaces of $\mathcal{K}$, and more generally of degenerate operators with unbounded coefficients, is the subject of intense research activity. It is well-known (see e.g. Lunardi [15], Bertoldi-Lorenzi [2] and references therein) that the parabolic problem associated with a uniformly elliptic operator admits, under some growth assumptions on its coefficients, a unique classical solution, and that the associated semigroup enjoys nice smoothing properties that can be used to prove Schauder estimates. The situation is much more intricate in the case of degenerate operators. A typical example of such an operator is the so-called Ornstein-Uhlenbeck operator $\mathcal{K}'$, defined on smooth functions $\varphi \in C^2(\mathbb{R}^N)$ by

$$\mathcal{K}'\varphi = \tfrac{1}{2} Tr(QD^2\varphi) + Bx \cdot D\varphi,$$

where the matrix Q is symmetric nonnegative and the matrix B is such that the hypoellipticity (in the sense of Hörmander [11]) condition $\det Q_t > 0$ holds, where

$$Q_t = \int_0^t e^{sB'} Q e^{sB^*} ds.$$

Using an explicit representation formula (based on Gaussian measures) for the semigroup $T(t)$ associated to $\mathcal{K}'$, Lunardi [16] proved a priori estimates for the space derivatives of $T(t)$, existence and uniqueness of a distributional solution of the equations (E) and (NHCP) associated to $\mathcal{K}'$, and Schauder estimates for strong solution in some Hölder spaces (defined by means of a distance not equivalent to the Euclidean one in the degenerate case). Using a modification of the Bernstein method [1], Lorenzi [12] [13] proved a priori estimates for the space derivatives of $T(t)$ in anisotropic Hölder spaces and then deduced existence and uniqueness of a distributional solution for the equations (E) and (NHCP) associated with $\mathcal{K}'$, as well as Schauder estimates. Da Prato [4] dealt with the operator obtained from $\mathcal{K}'$ by perturbing the drift B by a suitable smooth and bounded function, still assuming the hypoellipticity condition. He then obtained, still using the Bernstein method, a priori estimates for the first-order space derivatives of $T(t)$. Our purpose here is to extend their results to the operator $\mathcal{K}$ defined by (K). Notice that $\mathcal{K}$ doesn't satisfy in general the Hörmander condition of hypoellipticity on commutators of vector fields.

From now on, $\mathcal{K}$ will refer to the operator defined by (K), assuming that $F \in C^3(\mathbb{R}^{2n})$ has bounded first-, second- and third-order derivatives. We first consider the homogeneous Cauchy problem

$$\begin{cases} \partial_t u = \mathcal{K}u & \text{in } (0,+\infty)\times\mathbb{R}^{2n} \\ u(0,\cdot) = f & \text{in } \mathbb{R}^{2n} \end{cases} \tag{HCP}$$

and prove the following theorem:

Theorem 0.1. *If $f \in C_b(\mathbb{R}^{2n})$, the homogeneous Cauchy problem (HCP) admits a unique classical solution u_f such that $u_f(t,\cdot)$ belongs to $C_b^3(\mathbb{R}^{2n})$ for each $t > 0$. The associated semigroup $(T(t))_{t\geq 0}$, defined by $T(t)f(\cdot) = u_f(t,\cdot)$, satisfies the estimates given in steps 2.1 and 2.2 below (with T_ϵ replaced by T).*

These estimates can also be obtained by probabilistic methods using Malliavin calculus (see Priola [20]—see also Cerrai [3] for a probabilistic point of view on the Kolmogorov operator). In the case where F is linear, we recover the estimates obtained by Lorenzi [13].

The estimates stated in the above theorem allow us to prove existence and uniqueness of a weak solution of (E) and (NHCP) as well as Schauder estimates. We refer to the next section for the definition of the functional spaces involved in the following theorems.

Theorem 0.2. *For every $\lambda > 0$ and $f \in C_b(\mathbb{R}^{2n})$, equation (E) has a weak solution u. Moreover, if $f \in \mathcal{C}^{\theta,\theta/3}(\mathbb{R}^n \times \mathbb{R}^n)$, with $\theta \in (0,1)$, $u \in \mathcal{C}^{2+\theta,(2+\theta)/3}(\mathbb{R}^n \times \mathbb{R}^n)$ and it is the only weak solution of (E) which is bounded and continuous and whose Laplacian in x is bounded continuous. Moreover, we have the Schauder estimate*

$$\|u\|_{\mathcal{C}^{2+\theta,(2+\theta)/3}} \leq C\|f\|_{\mathcal{C}^{\theta,\theta/3}}, \tag{0.2}$$

where C is a positive constant independent of u and f.

Theorem 0.3. *If $f \in C_b(\mathbb{R}^{2n})$ and $g \in C_b([0,+\infty) \times \mathbb{R}^{2n})$, (NHCP) admits a weak solution u, which is given by the usual variation-of-constants formula*

$$u(t,x,y) = T(t)f(x,y) + \int_0^t \left(T(t-s)g(s,\cdot)\right)(x,y)ds.$$

If moreover $f \in \mathcal{C}^{2+\theta,(2+\theta)/3}(\mathbb{R}^n \times \mathbb{R}^n)$ and $g \in B([0,T], \mathcal{C}^{\theta,\theta/3}(\mathbb{R}^n \times \mathbb{R}^n))$ for some $\theta \in (0,1)$ and $T > 0$, then $u \in B([0,T], \mathcal{C}^{2+\theta,(2+\theta)/3}(\mathbb{R}^n \times \mathbb{R}^n))$ and is the only solution of (NHCP) on $[0,T]$ which is in $C_b([0,T] \times \mathbb{R}^n)$ and whose Laplacian in x is also in $C_b([0,T] \times \mathbb{R}^n)$. The following Schauder estimate holds:

$$\sup_{0 \le t \le T} \|u(t,\cdot,\cdot)\|_{\mathcal{C}^{2+\theta,(2+\theta)/3}} \le C\|f\|_{\mathcal{C}^{2+\theta,(2+\theta)/3}} + C \sup_{0 \le t \le T} \|g(t,\cdot,\cdot)\|_{\mathcal{C}^{\theta,\theta/3}}. \tag{0.3}$$

The paper is organized as follows. We recall the maximum principle available for $\mathcal{K}$ and we fix some notation concerning the functional spaces used in this paper in the first section. The second one deals with the proof of Theorem 0.1, whereas Theorems 0.2 and 0.3 are proved in the last section. The proofs rely strongly on techniques used in Lorenzi [12], [13] and Lunardi [14], [15], [16], [17], which we extend to our more general situation, and we sometimes only sketch the proofs of intermediate claims for the reader's convenience, referring for more details to these papers. Nevertheless, the estimates we get in steps 2.1 and 2.2, which are the keys of the proofs of these theorems, are new.

1. Maximum principle and functional spaces

1.1. **The maximum principle.** F being Lipschitz, there exists $\lambda > 0$ such that

$$\sup_{(x,y)\in\mathbb{R}^{2n}} \left(\mathcal{K}\phi(x,y) - \lambda\phi(x,y)\right) < \infty,$$

where $\phi(x,y) = |x|^2 + |y|^2$. As a consequence, according to Lorenzi ([12] Proposition 2.7 and Remark 3.3), $\mathcal{K}$ satisfies the following maximum principle:

Proposition 1.1. *Let u be a bounded classical solutions of problem (NHCP) in $(0,T) \times \mathbb{R}^{2n}$ with $f \in C_b(\mathbb{R}^{2n})$ and $g \in C((0,T) \times \mathbb{R}^{2n})$. If $g \le 0$ (respectively $g \ge 0$) on $(0,T) \times \mathbb{R}^{2n}$ then, for any $0 \le t \le T$,*

$$\|u(t,\cdot,\cdot)\|_\infty \le \|f\|_\infty \quad \left(\textit{respectively } \inf_{(x,y)\in\mathbb{R}^{2n}} u(t,x,y) \ge \inf_{(x,y)\in\mathbb{R}^{2n}} f(x,y)\right).$$

In particular, if $g \equiv 0$ *then, for any* $0 \le t \le T$, $\|u(t,\cdot,\cdot)\|_\infty \le \|f\|_\infty$.

All the uniqueness assertions concerning classical solutions of equations will follow from this maximum principle.

1.2. **Functional spaces.** $C_b(\mathbb{R}^N)$ denotes the space of bounded continuous functions on $\mathbb{R}^N$ endowed with the sup-norm $\|\cdot\|_\infty$. For $k = [k] + \{k\} \in [0,+\infty)$, with $[k]$ an integer and $\{k\} \in [0,1)$, $C^k(\mathbb{R}^N)$ is the subspace of $C_b(\mathbb{R}^N)$ consisting of the functions which are $[k]$-times continuously differentiable such that their $[k]$-th order derivatives are Hölder continuous of order $\{k\}$ and which satisfy

$$\|u\|_{C^k} := \sum_{|\alpha|\le[k]} \|D^\alpha u\|_\infty + \sum_{|\alpha|=[k]} \sup_{a,b\in\mathbb{R}^N, a\neq b} \frac{|D^\alpha u(a) - D^\alpha u(b)|}{|a-b|^{\{k\}}} < \infty.$$

For $k = [k]^- + \{k\}^+ \in [0,+\infty)$, with $[k]^-$ an integer and $\{k\}^+ \in (0,1]$, the Zygmund space $\mathcal{C}^k(\mathbb{R}^N)$ is the subspace of $C_b(\mathbb{R}^N)$ consisting of the functions which are in $BUC^{[k]^-}(\mathbb{R}^N)$ (i.e., the functions which are bounded uniformly continuous together with their $[k]^-$-first derivatives), and such that

$$\|u\|_{\mathcal{C}^k} := \|u\|_{C^{[k]^-}} + \sum_{|\alpha|=[k]^-} \sup_{a,b\in\mathbb{R}^N, a\neq b} \frac{\left|D^\alpha u(a) + D^\alpha u(b) - 2D^\alpha u\left(\frac{a+b}{2}\right)\right|}{|a-b|^{\{k\}^+}} < \infty.$$

It is known (see e.g. [21]) that $C^k(\mathbb{R}^N) = \mathcal{C}^k(\mathbb{R}^N)$ if k is not an integer (with equivalence of the norms), whereas $C^k(\mathbb{R}^N)$ is continuously embedded into $\mathcal{C}^k(\mathbb{R}^N)$ otherwise. Eventually, given $\eta, \theta \ge 0$ and an interval $I \subset [0,+\infty)$, we define $\mathcal{C}^{\eta,\theta}(\mathbb{R}^n \times \mathbb{R}^n)$ and $B(I, \mathcal{C}^{\eta,\theta}(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$u \in \mathcal{C}^{\eta,\theta}(\mathbb{R}^n \times \mathbb{R}^n) \Leftrightarrow \forall\, x, y \in \mathbb{R}^n,\ u(x,\cdot) \in \mathcal{C}^\theta(\mathbb{R}^n),\ u(\cdot,y) \in \mathcal{C}^\eta(\mathbb{R}^n),\ \textit{and}$$
$$\|u\|_{\mathcal{C}^{\eta,\theta}} := \sup_{x\in\mathbb{R}^n} \|u(x,\cdot)\|_{\mathcal{C}^\theta} + \sup_{y\in\mathbb{R}^n} \|u(\cdot,y)\|_{\mathcal{C}^\eta} < \infty,$$

$$u \in B(I, \mathcal{C}^{\eta,\theta}(\mathbb{R}^n \times \mathbb{R}^n)) \Leftrightarrow \forall\, t \in I,\ u(t,\cdot,\cdot) \in \mathcal{C}^{\eta,\theta}(\mathbb{R}^n \times \mathbb{R}^n),\ \textit{and}$$
$$\|u\|_{B(I,\mathcal{C}^{\eta,\theta}(\mathbb{R}^n\times\mathbb{R}^n))} := \sup_{t\in I} \|u(t,\cdot,\cdot)\|_{\mathcal{C}^{\eta,\theta}} < \infty.$$

Given a compact set $K \subset \mathbb{R}^n \times \mathbb{R}^n$ and $h \in (0,1)$, we also consider the following functional spaces:

$$u \in B(I, C^k(K)) \Leftrightarrow \|u\|_{B(I,C^k(K))} := \sup_{t\in I} \|u(t,\cdot)\|_{C^k(K)} < \infty$$

$$u \in Lip(I, C^k(K)) \Leftrightarrow \exists C > 0, \forall s,t \in I,\ \|u(t,\cdot) - u(s,\cdot)\|_{C^k(K)} \le C|t-s|$$

$$u \in C^h(I, C^k(K)) \Leftrightarrow \exists C > 0, \forall s,t \in I,\ \|u(t,\cdot) - u(s,\cdot)\|_{C^k(K)} \le C|t-s|^h.$$

2. Proof of theorem 0.1

We approximate $\mathcal{K}$ by the uniformly elliptic operator $\mathcal{K}_\epsilon$ defined by

$$\mathcal{K}_\epsilon u := \mathcal{K}u + \frac{\epsilon}{2}\Delta_y u = \frac{1}{2}\Delta_x u + \frac{\epsilon}{2}\Delta_y + F(x,y)\cdot D_x u + x\cdot D_y u.$$

It is well-known (see [15] or [18]) that the homogeneous Cauchy problem associated with $\mathcal{K}_\epsilon$, namely

$$\begin{cases} \partial_t u = \mathcal{K}_\epsilon u & \text{in } (0,+\infty)\times\mathbb{R}^{2n} \\ u(0,\cdot) = f & \text{in } \mathbb{R}^{2n}, \end{cases}$$

has a unique bounded classical solution for any $f \in C_b(\mathbb{R}^{2n})$. Moreover the associated semigroup $(T_\epsilon(t))_t$ is contractive, in view of the maximum principle, and maps $C_b(\mathbb{R}^{2n})$ to $C^3(\mathbb{R}^{2n})$ (see e.g. [15] or [12]). The purpose of the first two steps of the proof is to obtain, for small time, estimates of the space derivatives of $T_\epsilon(t)f$ independent of ϵ. These estimates allow us then to extract some sequence $\epsilon_n \to 0$ such that $T(t)f := \lim_n T_{\epsilon_n}(t)f$ is a classical solution of (HCP). The last step of the proof deals with the existence of continuous third-order space derivatives for $T(t)f$.

Step 2.1. *For any $\omega > 0$ there exist constants $C_\omega > 0$ and $T > 0$ independent of ϵ such that for any ϵ small, $t \in (0,T]$ and $f \in C_b(\mathbb{R}^{2n})$,*

$$\begin{cases} \|D_x T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t} t^{-1/2}\|f\|_\infty \\ \|D_y T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t} t^{-3/2}\|f\|_\infty. \end{cases} \tag{2.1}$$

$$\begin{cases} \|D_{xx} T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t} t^{-1}\|f\|_\infty \\ \|D_{xy} T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t} t^{-2}\|f\|_\infty. \\ \|D_{yy} T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t} t^{-3}\|f\|_\infty. \end{cases} \tag{2.2}$$

$$\begin{cases} \|D_{xxx} T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t} t^{-3/2}\|f\|_\infty \\ \|D_{xxy} T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t} t^{-5/2}\|f\|_\infty. \\ \|D_{xyy} T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t} t^{-7/2}\|f\|_\infty. \\ \|D_{yyy} T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t} t^{-9/2}\|f\|_\infty. \end{cases} \tag{2.3}$$

Proof. We write $u = u_\epsilon$. For $\alpha, \beta > 0$ and $\gamma < 0$ to be fixed later, we consider the function $z^1 : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$z^1 = u^2 + \alpha t|D_x u|^2 + \beta t^3 |D_y u|^2 + \gamma t^2 D_x u D_y u.$$

We then have

$$z^1_t - \mathcal{K}_\epsilon z^1 = g_\epsilon = g^1_\epsilon + g^2_\epsilon$$

with

$$g^1_\epsilon = (\alpha-1)|D_x u|^2 + \left(-\epsilon + (3\beta+\gamma)t^2\right)|D_y u|^2 + 2(\gamma+\alpha)tD_x u D_y u$$
$$+ 2\alpha t\partial_i F_j \partial_i u \partial_j u + \gamma t^2 \partial^i F_j \partial_i u \partial_j u + \gamma t^2 \partial_i F_j \partial^i u \partial_j u + 2\beta t^3 \partial^i F_j \partial^i u \partial_j u,$$
$$g^2_\epsilon = -\alpha t|D_{xx}u|^2 - \left(\epsilon\alpha + \beta t^2\right)t|D_{xy}u|^2 - \epsilon\beta t^3|D_{yy}u|^2$$
$$-\gamma t^2 D_{xx}u D_{xy}u - \epsilon\gamma t^2 D_{yy}u D_{xy}u,$$

where $\partial^n_m = \frac{\partial^{|m|+|n|}}{\partial x^m \partial y^n}$ for every multi-index m, n and $|D_{xx}u|^2 = \sum_{ij}(\partial_{ij}u)^2$, $D_{xx}uD_{xy}u = \partial_{ij}u\partial^j_i u$ (we use the Einstein's summation convention: if an index appears twice in a term, we sum over it). By Young's inequality and for $0 \le t \le 1$,

$$\begin{aligned}
\left|\gamma t^2 \partial_i F_j \partial^i u \partial_j u\right| &\le O(t^3)|D_y u|^2 + O(t)|D_x u|^2,\\
\left|\gamma t^2 D_{xx}u D_{xy}u\right| &\le \frac{\gamma\eta}{2}t^3|D_{xy}u|^2 + \frac{\gamma}{2\eta}t|D_{xx}u|^2,\\
\left|\epsilon\gamma t^2 D_{yy}u D_{xy}u\right| &\le \frac{\epsilon\gamma\tilde\eta}{2}t^3|D_{yy}u|^2 + \frac{\epsilon\gamma}{2\tilde\eta}t|D_{xy}u|^2,
\end{aligned} \tag{2.4}$$

for some $\eta, \tilde\eta > 0$ to be fixed later. Here and in the sequel, an expression like $O(t^k)$, k an integer, denotes a function of t depending on the parameters α, $\beta, \ldots$ such that for every choice of these parameters, we have an estimate of the form $|O(t^k)| \le Ct^k$, t small, for some positive constant C. Letting $\gamma = -\alpha$, we thus get that

$$\begin{aligned}
g^1_\epsilon &\le \Big\{\alpha - 1 + O(t)\Big\}|D_x u|^2 + \Big\{3\beta - \alpha + O(t)\Big\}t^2|D_y u|^2,\\
g^2_\epsilon &\le \Big\{-1 + \frac{1}{2\eta}\Big\}\alpha t|D_{xx}u|^2 + \epsilon\Big\{-\beta + \frac{\tilde\eta\alpha}{2}\Big\}t^3|D_{yy}u|^2,\\
&\quad + \Big\{\Big(-1 + \frac{1}{2\tilde\eta}\Big)\epsilon\alpha + \Big(-\beta + \frac{\eta\alpha}{2}\Big)t^2\Big\}t|D_{xy}u|^2.
\end{aligned} \tag{2.5}$$

We now choose $0 < \alpha < 1$, $\frac{\alpha}{4} < \beta < \frac{\alpha}{3}$ and $\eta = \tilde\eta = \frac{1}{2}$. With such a choice of the parameters, we get the existence of a $T > 0$ independent of ϵ such that $g_\epsilon \le 0$ in $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ for any ϵ. The maximum principle (cf. Proposition 1.1) and the continuity of Du at $t = 0$ (cf. Lorenzi [12] Proposition 2.7, Theorem 2.13) then give $\|z^1(t,\cdot)\|_\infty \le \|f\|^2_\infty$ for all $t \in [0,T]$. Independently, since the lowest eigenvalue λ_{min} of the quadratic form $q(x,y) = \alpha|x|^2 + \beta|y|^2 + \gamma xy$ is positive for $\gamma = -\alpha$ and $\beta > \alpha/4$, we have for any $x, y \in \mathbb{R}^n$ and $t > 0$ that

$$q_1(t,x,y) := \alpha t|x|^2 + \beta t^3|y|^2 + \gamma t^2 xy = q_1(\tilde x, \tilde y) \ge \lambda_{min}(|\tilde x|^2 + |\tilde y|^2)$$

$$\geq \lambda_{min}(t|x|^2 + t^2|y|^2),$$

where $\tilde{x} = \sqrt{t}x$, $\tilde{y} = t^{3/2}y$. We deduce that for any $t \in [0, T]$ and ϵ,

$$t^{\frac{1}{2}}\|D_x T_\epsilon(t)f\|_\infty + t^{\frac{3}{2}}\|D_y T_\epsilon(t)f\|_\infty \leq K\|f\|_\infty,$$

where the constant K is independent of ϵ. Given $\omega > 0$, let $C_\omega > 0$ be such that $C_\omega e^{\omega t} t^{-1/2} \geq KT^{-1/2}$ for all $t \geq T$. Then, since T_ϵ is a contractive semigroup, we have for any $t \geq T$ that

$$\begin{aligned}\|D_x T_\epsilon(t)f\|_\infty = \|D_x T_\epsilon(T)T_\epsilon(t-T)f\|_\infty &\leq KT^{-1/2}\|T_\epsilon(t-T)f\|_\infty \\ &\leq C_\omega e^{\omega t} t^{-1/2}\|f\|_\infty,\end{aligned}$$

which is (2.1).

To prove (2.2), we consider the function

$$z^2 = z^1 + \lambda t^2|D_{xx}u|^2 + \mu t^4|D_{xy}u|^2 + \nu t^6|D_{yy}u|^2 + \theta t^5 D_{yy}uD_{xy}u$$

for some parameters $\alpha, \beta, \lambda, \mu, \nu > 0$, $\gamma, \theta < 0$ to be fixed later. We have

$$z_t^2 - \mathcal{K}_\epsilon z^2 = g_\epsilon = g_\epsilon^1 + g_\epsilon^2 + g_\epsilon^3 + g_\epsilon^4,$$

where

$$\begin{aligned}g_\epsilon^3 =& 2\lambda t|D_{xx}u|^2 + 4\mu t^3|D_{xy}u|^2 + (6\nu + \theta)t^5|D_{yy}u|^2 + 4\lambda t^2 D_{xx}uD_{xy}u \\ &+ (5\theta + 2\mu)t^4 D_{yy}uD_{xy}u + 2\lambda t^2\partial_{ij}u\left\{\partial_{ij}F_k\partial_k u + 2\partial_i F_k\partial_{jk}u\right\} \\ &+ 2\mu t^4\partial_i^j u\left\{\partial_i^j F_k\partial_k u + \partial_i F_k\partial_k^j u + \partial^j F_k\partial_{ik}u\right\} \\ &+ \theta t^5\partial^{ij}u\left\{\partial_i^j F_k\partial_k u + \partial_i F_k\partial_k^j u + \partial^j F_k\partial_{ik}u\right\} \\ &+ \left\{2\nu t^6\partial^{ij}u + \theta t^5\partial_i^j u\right\}\left\{\partial^{ij}F_k\partial_k u + \partial^i F_k\partial_k^j u + \partial^j F_k\partial_k^i u\right\}, \\ g_\epsilon^4 =& -\lambda t^2|D_{xxx}u|^2 - (\epsilon\lambda + \mu t^2)t^2|D_{xxy}u|^2 - (\epsilon\mu + \nu t^2)t^4|D_{xyy}u|^2 \\ &- \epsilon\nu t^6|D_{yyy}u|^2 - \theta t^5 D_{xyy}uD_{xxy}u - \epsilon\theta t^5 D_{xyy}uD_{yyy}u,\end{aligned}$$

with, as for the second-order space derivatives, $|D_{xxx}u|^2 = \sum_{ijk}(\partial_{ijk}u)^2$ and $D_{xyy}uD_{xxy}u = \partial_i^{jk}u\partial_{ij}^k u$. We first write that

$$\begin{aligned}&2\lambda t^2\partial_{ij}u\left\{\partial_{ij}F_k\partial_k u + 2\partial_i F_k\partial_{jk}u\right\} \\ &+ 2\mu t^4\partial_i^j u\left\{\partial_i^j F_k\partial_k u + \partial_i F_k\partial_k^j u + \partial^j F_k\partial_{ik}u\right\} \\ &+ \left\{2\nu t^6\partial^{ij}u + \theta t^5\partial_i^j u\right\}\left\{\partial^{ij}F_k\partial_k u + \partial^i F_k\partial_k^j u + \partial^j F_k\partial_k^i u\right\} \\ &= O(t)|D_x u|^2 + O(t^2)|D_{xx}u|^2 + O(t^4)|D_{xy}u|^2 + O(t^6)|D_{yy}u|^2\end{aligned}$$

and

$$\theta t^5\partial^{ij}u\left\{\partial_i^j F_k\partial_k u+\partial_i F_k\partial_k^j u+\partial^j F_k\partial_{ik}u\right\}$$
$$\leq O(t)|D_xu|^2+O(t^4)|D_{xy}u|^2+O(t^6)|D_{yy}u|^2+O(t^2)D_{xx}u|^2,$$

so that

$$g_\epsilon^3=O(t)|D_xu|^2+\{2\lambda+O(t)\}\,t|D_{xx}u|^2+\{6\nu+\theta+O(t)\}\,t^5|D_{yy}u|^2$$
$$+\{4\mu+O(t)\}\,t^3|D_{xy}u|^2+4\lambda t^2D_{xx}uD_{xy}u+(5\theta+2\mu)t^4D_{yy}uD_{xy}u.$$

We then get with (2.5) and the definition of g_ϵ^2 with $\gamma=-\alpha$ that

$$g_\epsilon^1+g_\epsilon^2+g_\epsilon^3\leq\{\alpha-1+O(t)\}\,|D_xu|^2+\{3\beta-\alpha+O(t)\}\,t^2|D_yu|^2$$
$$+\{2\lambda-\alpha+O(t)\}\,t|D_{xx}u|^2+\left\{-\epsilon\beta t^3+(6\nu+\theta)t^5+O(t^6)\right\}|D_{yy}u|^2$$
$$+\left\{-\epsilon\alpha t+(4\mu-\beta)t^3+O(t^4)\right\}|D_{xy}u|^2$$
$$+(4\lambda+\alpha)t^2D_{xx}uD_{xy}u+\left\{\epsilon\alpha t^2+(5\theta+2\mu)t^4\right\}D_{yy}uD_{xy}u.$$

For $\hat\eta>0$ to be chosen later,

$$\left|(4\lambda+\alpha)t^2D_{xx}uD_{xy}u\right|\leq\frac{4\lambda+\alpha}{2\hat\eta}t|D_{xx}u|^2+\frac{4\lambda+\alpha}{2}\hat\eta t^3|D_{xy}u|^2.$$

Hence, using (2.4) with $\tilde\eta=1/2$ and $\beta>\frac{\alpha}{4}$, and assuming that $5\theta+2\mu=0$, we obtain

$$g_\epsilon^1+g_\epsilon^2+g_\epsilon^3\leq\{\alpha-1+O(t)\}\,|D_xu|^2+\{3\beta-\alpha+O(t)\}\,t^2|D_yu|^2 \tag{2.6}$$
$$+\left\{2\lambda-\alpha+\frac{4\lambda+\alpha}{2\hat\eta}+O(t)\right\}t|D_{xx}u|^2+\{6\nu+\theta+O(t)\}\,t^5|D_{yy}u|^2$$
$$+\left\{4\mu-\beta+\frac{4\lambda+\alpha}{2}\hat\eta+O(t)\right\}t^3|D_{xy}u|^2.$$

Moreover, for $\delta>0$ to be fixed later,

$$g_\epsilon^4\leq\Big(\frac{|\theta|}{2\delta}-\mu\Big)t^4|D_{xxx}u|^2+\Big(\frac{\delta|\theta|}{2}-\nu\Big)\epsilon t^6|D_{yyy}u|^2$$
$$+\Big\{\Big(\frac{|\theta|}{2\delta}-\mu\Big)\epsilon+\Big(\frac{\delta|\theta|}{2}-\nu\Big)t^2\Big\}t^4|D_{xyy}u|^2$$
$$=\Big(\frac{1}{5\delta}-1\Big)\mu t^4|D_{xxx}u|^2+\Big(\frac{\delta\mu}{5}-\nu\Big)\epsilon t^6|D_{yyy}u|^2$$
$$+\Big\{\Big(\frac{1}{5\delta}-1\Big)\mu\epsilon+\Big(\frac{\delta\mu}{5}-\nu\Big)t^2\Big\}t^4|D_{xyy}u|^2.$$

Independently, the quadratic form $q_2(x,y)=\mu|x|^2+\nu|y|^2+\theta xy$ is positive if and only if $\mu>0$ and $\nu>\frac{\theta^2}{4\mu}$. We now claim that we can choose the parameters such that they satisfy

$$\begin{cases} 0<\alpha<1, \quad \frac{\alpha}{4}<\beta<\frac{\alpha}{3}, \quad 2\lambda-\alpha+\frac{4\lambda+\alpha}{2\hat\eta}<0 \\ 4\mu-\beta+\frac{4\lambda+\alpha}{2}\hat\eta<0,\ 6\nu+\theta<0,\ \nu>\frac{\theta^2}{4\mu},\ |\theta|<\min\left\{2\mu\delta,\frac{2\nu}{\delta}\right\}. \end{cases} \tag{2.7}$$

Indeed, a possible choice is

$$\delta\in(1/5,1/4),\ \hat\eta=\frac{1}{2}+\alpha^2,\ \beta=\frac{\alpha}{3}-\alpha^2,\ \lambda=\alpha^4,\ \mu=\alpha^2,\ \nu=\frac{\mu}{20}$$

for α small. For such a choice of the parameters, we get the existence of $T>0$ such that $g_\epsilon\le 0$ in $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ for any $\epsilon>0$. We conclude as previously.

To prove (2.3), we consider

$$\begin{aligned} z^3 = {} & z^2+\rho t^3|D_{xxx}u|^2+\sigma t^5|D_{xxy}u|^2+\tau t^7|D_{xyy}u|^2+\chi t^9|D_{yyy}u|^2 \\ & +\omega t^8 D_{xyy}uD_{yyy}u \end{aligned}$$

for some parameters $\alpha,\beta,\lambda,\mu,\nu,\rho,\sigma,\tau,\chi>0$, $\gamma,\theta,\omega<0$ to be fixed later. We have

$$z^3_t-K_\epsilon z^3=g_\epsilon=g^1_\epsilon+g^2_\epsilon+g^3_\epsilon+g^4_\epsilon+g^5_\epsilon+g^6_\epsilon,$$

where

$$\begin{aligned} g^5_\epsilon = {} & 3\rho t^2|D_{xxx}u|^2+5\sigma t^4|D_{xxy}u|^2+7\tau t^6|D_{xyy}u|^2+(\omega+9\chi)t^8|D_{yyy}u|^2 \\ & +(8\omega+2\tau)t^7D_{xyy}uD_{yyy}u+6\rho t^3D_{xxx}uD_{xxy}u+4\sigma t^5D_{xxy}uD_{xyy}u \\ & +2\rho t^3\partial_{ijk}u\left\{\partial_{ijk}F_l\partial_l u+3\partial_{jk}F_l\partial_{il}u+3\partial_iF_l\partial_{jkl}u\right\} \\ & +2\sigma t^5\partial^k_{ij}u\left\{\partial^k_{ij}F_l\partial_l u+\partial_{ij}F_l\partial^k_l u+\partial^kF_l\partial_{ijl}u+2\partial^k_iF_l\partial_{jl}u+2\partial_iF_l\partial^k_{jl}u\right\} \\ & +\left(2\tau t^7\partial^{jk}_i u+\omega t^8\partial^{ijk}u\right) \\ & \qquad\times\left\{\partial^{jk}_iF_l\partial_l u+\partial^{jk}F_l\partial_{il}u+\partial_iF_l\partial^{jk}_l u+2\partial^j_iF_l\partial^k_l u+2\partial^jF_l\partial^k_{il}u\right\} \\ & +\left(2\chi t^9\partial^{ijk}u+\omega t^8\partial^{jk}_i u\right) \\ & \qquad\times\left\{\partial^{ijk}F_l\partial_l u+\partial^{jk}F_l\partial^i_l u+\partial^iF_l\partial^{jk}_l u+2\partial^{ij}F_l\partial^k_l u+2\partial^jF_l\partial^{ik}_l u\right\}, \\ g^6_\epsilon = {} & -\rho t^3|D_{xxxx}u|^2-(\epsilon\rho t^3+\sigma t^5)|D_{xxxy}u|^2-(\epsilon\sigma t^5+\tau t^7)|D_{xxyy}u|^2 \\ & -(\epsilon\tau t^7+\chi t^9)|D_{xyyy}u|^2-\epsilon\chi t^9|D_{yyyy}u|^2 \\ & -\omega t^8D_{xxyy}uD_{xyyy}u-\epsilon\omega t^8D_{xyyy}uD_{yyyy}u. \end{aligned}$$

In view of Young's inequality and assuming that $\tau + 4\omega = 0$, we have

$$\begin{aligned} g_\epsilon^5 &= (3\rho + O(t))t^2|D_{xxx}u|^2 + (5\sigma + O(t))t^4|D_{xxy}u|^2 \\ &+ (7\tau + O(t))t^6|D_{xyy}u|^2 + (\omega + 9\chi + O(t))t^8|D_{yyy}u|^2 \\ &+ 6\rho t^3 D_{xxx}uD_{xxy}u + 4\sigma t^5 D_{xxy}uD_{xyy}u \\ &+ O(t)|D_x u|^2 + O(t^2)|D_{xx}u|^2 + O(t^4)|D_{xy}u|^2. \end{aligned}$$

Hence, with (2.6), assuming that $\beta > \alpha/4$ and $5\theta + 2\mu = 0$,

$$\begin{aligned} & g_\epsilon^1 + g_\epsilon^2 + g_\epsilon^3 + g_\epsilon^4 + g_\epsilon^5 + g_\epsilon^6 = \\ & \{\alpha - 1 + O(t)\}|D_x u|^2 + \{3\beta - \alpha + O(t)\}t^2|D_y u|^2 \\ & + \left\{2\lambda - \alpha + \frac{4\lambda + \alpha}{2\hat{\eta}} + O(t)\right\}t|D_{xx}u|^2 + \{6\nu + \theta + O(t)\}t^5|D_{yy}u|^2 \\ & + \left\{4\mu - \beta + \frac{4\lambda + \alpha}{2}\hat{\eta} + O(t)\right\}t^3|D_{xy}u|^2 + \{3\rho - \lambda + O(t)\}t^2|D_{xxx}u|^2 \\ & + \{5\sigma - \mu + O(t)\}t^4|D_{xxy}u|^2 + \left\{-\epsilon\mu t^4 + (7\tau - \nu)t^6 + O(t^7)\right\}|D_{xyy}u|^2 \\ & + \left\{-\epsilon\nu t^6 + (\omega + 9\chi)t^8 + O(t)\right\}|D_{yyy}u|^2 \\ & - \epsilon\theta t^5 D_{xyy}uD_{yyy}u + 6\rho t^3 D_{xxx}uD_{xxy}u \\ & + (4\sigma - \theta)t^5 D_{xyy}uD_{xxy}u - \tau t^7|D_{xxyy}u|^2 - (\epsilon\tau t^7 + \chi t^9)|D_{xyyy}u|^2 \\ & - \epsilon\chi t^9|D_{yyyy}u|^2 - \omega t^8 D_{xxyy}uD_{xyyy}u - \epsilon\omega t^8 D_{xyyy}uD_{yyyy}u. \end{aligned}$$

Moreover, for some $\bar{\eta},\ \hat{\delta},\ \tilde{\delta} > 0$ to be fixed later,

$$\begin{aligned} \left|(4\sigma - \theta)t^5 D_{xyy}uD_{xxy}u\right| &\le \frac{4\sigma + |\theta|}{2\bar{\eta}}t^4|D_{xxy}u|^2 + \frac{4\sigma + |\theta|}{2}\bar{\eta}t^6|D_{xyy}u|^2, \\ \left|6\rho t^3 D_{xxx}uD_{xxy}u\right| &\le 3\rho t^2|D_{xxx}u|^2 + 3\rho t^4|D_{xxy}u|^2, \\ \left|\epsilon\theta t^5 D_{xyy}uD_{yyy}u\right| &\le \frac{\epsilon|\theta|}{2\hat{\delta}}t^4|D_{xyy}u|^2 + \frac{\epsilon|\theta|}{2}\hat{\delta}t^6|D_{yyy}u|^2, \\ \left|\omega t^8 D_{xxyy}uD_{xyyy}u\right| &\le \frac{|\omega|}{2\tilde{\delta}}t^7|D_{xxyy}u|^2 + \frac{|\omega|}{2}\tilde{\delta}t^9|D_{xyyy}u|^2, \\ \left|\epsilon\omega t^8 D_{xyyy}uD_{yyyy}u\right| &\le \frac{\epsilon|\omega|}{2\tilde{\delta}}t^7|D_{xyyy}u|^2 + \frac{\epsilon|\omega|}{2}\tilde{\delta}t^9|D_{yyyy}u|^2. \end{aligned}$$

Hence,

$$\begin{aligned} & g_\epsilon^1 + g_\epsilon^2 + g_\epsilon^3 + g_\epsilon^4 + g_\epsilon^5 + g_\epsilon^6 = \\ & \quad \{\alpha - 1 + O(t)\}|D_x u|^2 + \{3\beta - \alpha + O(t)\}t^2|D_y u|^2 \end{aligned}$$

$$
\begin{aligned}
&+\Big\{2\lambda-\alpha+\frac{4\lambda+\alpha}{2\hat\eta}+O(t)\Big\}t|D_{xx}u|^2+\Big\{6\nu+\theta+O(t)\Big\}t^5|D_{yy}u|^2\\
&+\Big\{4\mu-\beta+\frac{4\lambda+\alpha}{2}\hat\eta+O(t)\Big\}t^3|D_{xy}u|^2+\{6\rho-\lambda+O(t)\}\,t^2|D_{xxx}u|^2\\
&+\Big\{5\sigma-\mu+\frac{4\sigma+|\theta|}{2\bar\eta}+3\rho+O(t)\Big\}\,t^4|D_{xxy}u|^2\\
&+\Big\{\epsilon\Big(\frac{1}{5\hat\delta}-1\Big)\mu+\Big(7\tau-\nu+\frac{4\sigma+|\theta|}{2}\bar\eta\Big)t^2+O(t^3)\Big\}t^4|D_{xyy}u|^2\\
&+\Big\{\epsilon\Big(\frac{\mu\hat\delta}{5}-\nu\Big)+\Big(\omega+9\chi\Big)t^2+O(t^3)\Big\}t^6|D_{yyy}u|^2+\Big\{\frac{1}{8\tilde\delta}-1\Big\}\tau t^7|D_{xxyy}u|^2\\
&+\Big\{\epsilon\Big(\frac{1}{8\tilde\delta}-1\Big)\tau+\Big(\frac{\tau\tilde\delta}{8}-\chi\Big)t^2\Big\}t^7|D_{xyyy}u|^2+\epsilon\Big(\frac{\tau\tilde\delta}{8}-\chi\Big)t^9|D_{yyyy}u|^2.
\end{aligned}
$$

We now choose the parameters such that they satisfy (2.7) and also

$$
\begin{cases}
6\rho<\lambda, \quad 5\sigma-\mu+\frac{4\sigma+|\theta|}{2\bar\eta}+3\rho<0\\
7\tau-\nu+\frac{4\sigma+|\theta|}{2}\bar\eta<0, \quad \omega+9\chi<0\\
\frac{5\nu}{\mu}>\hat\delta>\frac{1}{5},\ \frac{8\chi}{\tau}>\tilde\delta>\frac{1}{8},\ \chi>\frac{\omega^2}{4\tau}.
\end{cases}
\tag{2.8}
$$

A possible choice is for example

$$
\begin{aligned}
&\beta=\frac{\alpha}{3}-\alpha^2,\ \lambda=\alpha^4,\ \mu=\alpha^2,\ \nu=\frac{\alpha^2}{16},\ \rho=\alpha^5,\ \sigma=\tau=\alpha^3,\\
&\chi=\frac{\alpha^3}{40},\ \hat\eta=\frac{1}{2}+\alpha^2,\ \bar\eta=\frac{1}{4},\ \hat\delta\in\left(\frac{1}{5},\frac{5}{16}\right),\ \tilde\delta\in\left(\frac{1}{8},\frac{1}{5}\right)
\end{aligned}
$$

for α small. For such a choice of the parameters, we get the existence of a small $T>0$ independent of ϵ such that $g_\epsilon\le 0$ in $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ for any ϵ. We then conclude as before. □

The next step shows that we can improve estimates (2.1), (2.2), and (2.3) when f is more regular:

Step 2.2. *If $f\in C_b^1(\mathbb{R}^{2n})$, estimates* (2.1), (2.2), *and* (2.3) *become*

$$
\|D_xT_\epsilon(t)f\|_\infty,\ \|D_yT_\epsilon(t)f\|_\infty\le C_\omega e^{\omega t}\|f\|_{C^1},
\tag{2.9}
$$

$$
\begin{cases}
\|D_{xx}T_\epsilon(t)f\|_\infty,\ \|D_{xy}T_\epsilon(t)f\|_\infty\le C_\omega e^{\omega t}t^{-\frac{1}{2}}\|f\|_{C^1},\\
\|D_{yy}T_\epsilon(t)f\|_\infty\le C_\omega e^{\omega t}t^{-\frac{3}{2}}\|f\|_{C^1},
\end{cases}
\tag{2.10}
$$

$$\begin{cases} \|D_{xxx}T_\epsilon(t)f\|_\infty,\ \|D_{xxy}T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t}t^{-1}\|f\|_{C^1}, \\ \|D_{xyy}T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t}t^{-2}\|f\|_{C^1}, \\ \|D_{yyy}T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t}t^{-3}\|f\|_{C^1}. \end{cases} \tag{2.11}$$

If $f \in C_b^2(\mathbb{R}^{2n})$, we have

$$\|DT_\epsilon(t)f\|_\infty,\ \|D^2T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t}\|f\|_{C^2}, \tag{2.12}$$

$$\begin{cases} \|D_{xxx}T_\epsilon(t)f\|_\infty,\ \|D_{xxy}T_\epsilon(t)f\|_\infty,\ \|D_{xyy}T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t}t^{-\frac{1}{2}}\|f\|_{C^2}, \\ \|D_{yyy}T_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t}t^{-\frac{3}{2}}\|f\|_{C^2}. \end{cases} \tag{2.13}$$

If $f \in C_b^3(\mathbb{R}^{2n})$, we have for any $k = 1, 2, 3$,

$$\|D^kT_\epsilon(t)f\|_\infty \le C_\omega e^{\omega t}\|f\|_{C^3}. \tag{2.14}$$

Proof. Since the proof of these inequalities is analogous to what we did previously, we only sketch it for (2.10). We consider the function

$$\begin{aligned} z_\epsilon = {} & u^2 + \alpha|D_xu|^2 + \alpha|D_yu|^2 - \alpha D_xuD_yu + \lambda t|D_{xx}u|^2 + \mu t|D_{xy}u|^2 \\ & + \nu t^3|D_{yy}u|^2 + \theta t^2 D_{yy}uD_{xy}u, \end{aligned}$$

where $u \equiv u_\epsilon$ and the coefficients $\alpha,\ \lambda,\ \mu,\ \nu > 0$, $\theta < 0$ will be chosen later. We have $\partial_t z - \mathcal{K}_\epsilon = g_\epsilon^2 + g_\epsilon^2 + g_\epsilon^3 + g_\epsilon^4$, where

$$\begin{aligned} g_\epsilon^1 = & -|D_xu|^2 - (\alpha+\epsilon)|D_yu|^2 + 2\alpha D_xuD_yu \\ & + \alpha(2\partial_iF_j - \partial^iF_j)\partial_iu\partial_ju + \alpha(2\partial^iF_j - \partial_iF_j)\partial^iu\partial_ju, \\ g_\epsilon^2 = & -\alpha|D_{xx}u|^2 - (1+\epsilon)\alpha|D_{xy}u|^2 - \epsilon\alpha|D_{yy}u|^2 + \alpha D_{xx}uD_{xy}u \\ & + \epsilon\alpha D_{xy}uD_{yy}u, \\ g_\epsilon^3 = {} & \lambda|D_{xx}u|^2 + \mu|D_{xy}u|^2 + (3\nu+\theta)t^2|D_{yy}u|^2 + 2(\mu+\theta)tD_{xy}uD_{yy}u \\ & + 2\lambda t\partial_{ij}u\left(\partial_{ij}F_k\partial_ku + 2\partial_iF_k\partial_{jk}u + 2\partial_j^iu\right) \\ & + (2\mu t\partial_i^ju + \theta t^2\partial^{ij}u)(\partial_i^jF_k\partial_ku + \partial_iF_k\partial_k^ju + \partial^jF_k\partial_{ik}u) \\ & + (2\nu t^3\partial^{ij}u + \theta t^2\partial_i^ju)(\partial^{ij}F_k\partial_ku + 2\partial^iF_k\partial_k^ju), \\ g_\epsilon^4 = & -\lambda t|D_{xxx}u|^2 - (\mu+\epsilon\lambda)t|D_{xxy}u|^2 - (\nu t^2+\epsilon\mu)t|D_{xyy}u|^2 \\ & - \epsilon\nu t^3|D_{yyy}u|^2 - \theta t^2D_{xyy}uD_{xxy}u - \epsilon\theta t^2D_{xyy}uD_{yyy}u. \end{aligned}$$

We let $\theta = -\mu$ and estimate these terms as follows:

$$\begin{aligned} g_\epsilon^1 &= \left\{-1 + O(\sqrt{\alpha})\right\}|D_xu|^2 + \alpha\left\{-1 + O(\sqrt{\alpha})\right\}|D_yu|^2, \\ g_\epsilon^2 &\le -\frac{\alpha}{2}|D_{xx}u|^2 - \frac{\alpha}{2}(1+\epsilon)|D_{xy}u|^2 - \frac{\epsilon\alpha}{2}|D_{yy}u|^2, \end{aligned}$$

$$\begin{aligned} g_\epsilon^3 \leq & O(t)|D_x u|^2 + \{\lambda + O(t)\}|D_{xx}u|^2 + \{\mu + O(t)\}|D_{xy}u|^2 \\ & + \{3\nu - \mu + O(t)\} t^2 |D_{yy}u|^2, \\ g_\epsilon^4 \leq & \left\{\frac{1}{2\delta} - 1\right\} \mu t |D_{xxy}u|^2 + \left\{\frac{\delta\mu}{2} - \nu\right\} \epsilon t^3 |D_{yyy}u|^2 \\ & + \left\{\left(\frac{\delta\mu}{2} - \nu\right) t^2 + \epsilon\left(\frac{1}{2\delta} - 1\right)\mu\right\} t |D_{xyy}u|^2, \end{aligned}$$

for some $\delta > 0$. Hence,

$$\begin{aligned} & g_\epsilon^1 + g_\epsilon^2 + g_\epsilon^3 + g_\epsilon^4 \\ \leq & \left\{-1 + O(\sqrt{\alpha}) + O(t)\right\}|D_x u|^2 + \alpha\left\{-1 + O(\sqrt{\alpha})\right\}|D_y u|^2 \\ & + \left\{\lambda - \frac{\alpha}{2} + O(t)\right\}|D_{xx}u|^2 + \left\{\mu - \frac{\alpha}{2} + O(t)\right\}|D_{xy}u|^2 \\ & + t^2\left\{3\nu - \mu + O(t)\right\}|D_{yy}u|^2 \\ & + \left\{\frac{1}{2\delta} - 1\right\}\mu t|D_{xxy}u|^2 + \left\{\frac{\delta\mu}{2} - \nu\right\}\epsilon t^3|D_{yyy}u|^2 \\ & + \left\{\left(\frac{\delta\mu}{2} - \nu\right)t^3 + \epsilon\left(\frac{1}{2\delta} - 1\right)\mu t\right\}|D_{xyy}u|^2. \end{aligned}$$

If we choose $\lambda = \mu = \alpha/4$, $\nu = 2\mu/7 = \alpha/14$ and $\delta = 1/2$, we see that for α small, the quadratic form $(x, y) \to \mu|x|^2 + \nu|y|^2 + \theta xy$ is positive definite and that there exists $T > 0$ small, independent of ϵ, such that $g_\epsilon^1 + g_\epsilon^2 + g_\epsilon^3 + g_\epsilon^4 \leq 0$ in $[0, T] \times \mathbb{R}^{2n}$ for any ϵ small. We conclude as previously. □

We can now prove existence and uniqueness for the homogeneous Cauchy problem (HCP):

Step 2.3. *For any $f \in C_b(\mathbb{R}^{2n})$, the Cauchy problem (HCP) admits a unique solution u_f.*

Proof. Fix $f \in C_b(\mathbb{R}^{2n})$ and set $u_\epsilon = T_\epsilon(\cdot)f$. Let $0 < T_0 < T$, $I = [T_0, T]$ and $K \subset \mathbb{R}^n \times \mathbb{R}^n$ compact. Since $\partial_t u_\epsilon = \mathcal{K}_\epsilon u_\epsilon$, we see from the previous step that the sequence (u_ϵ) is bounded in $B(I, C^3(K))$ and belongs to $Lip(I, C(K))$. It then follows from ([14], Proposition 1.1.4(i) and Corollary 1.2.19) that (u_ϵ) is bounded in $C^{(1-\alpha)/3}(I, C^{2+\alpha}(K))$ and that $(\partial_t u_\epsilon)$ is bounded in $C^{(1-\alpha)/3}(I, C(K))$. Hence the sequences $(\partial_t u_\epsilon)$ and $(D^\beta u_\epsilon)$ for $|\beta| = 0, 1, 2$ are uniformly bounded and equicontinuous in $I \times K$. As a consequence, there exists $u_f \in C^{1,2}((0, +\infty) \times \mathbb{R}^{2n})$ such that, up to a subsequence, $\partial_t u_\epsilon \to \partial_t u_f$ and $D^\beta u_\epsilon \to D^\beta u_f$, $|\beta| = 0, 1, 2$, uniformly in any compact subset of $(0, +\infty) \times \mathbb{R}^{2n}$. In particular, $\partial_t u_f = \mathcal{K} u_f$. It remains to show that u_f satisfies the initial condition. This can be done exactly as in Lorenzi [12]: we first prove the claim for $f \in C_c^2(\mathbb{R}^{2n})$ using the formula ([18], Proposition 4.3)

$$T_\epsilon(t)f(x) - f(x) = \int_0^t (T_\epsilon(s)\mathcal{K}_\epsilon f)(x)ds;$$

we then extend the result by density to a function $f \in C_c(\mathbb{R}^{2n})$, and eventually prove it for $f \in C_b(\mathbb{R}^{2n})$ by a localization argument. We refer the reader to [12] for a detailed proof. The uniqueness of u_f follows from the maximum principle. □

According to the previous step, we can consider the contractive semigroup $(T(t))_{t\geq 0}$ defined on $C_b(\mathbb{R}^{2n})$ by $T(t)f = u_f(t,\cdot)$. Since $T_\epsilon(\cdot)f \to T(\cdot)f$ in $C^{1,2}_{loc}((0,+\infty)\times\mathbb{R}^{2n})$, $(T(t))_{t\geq 0}$ satisfies estimates (2.1) and (2.2). As in [13], we can also prove that

Step 2.4. *If* $(f_n) \subset C_b(\mathbb{R}^{2n})$ *converges to* $f \in C_b(\mathbb{R}^{2n})$ *in* $C_{loc}(\mathbb{R}^{2n})$*, then* $T(\cdot)f_n \to T(\cdot)f$ *in* $C_{loc}([0,+\infty)\times\mathbb{R}^{2n})$ *and also in* $C^{1,2}_{loc}((0,+\infty)\times\mathbb{R}^{2n})$.

Indeed, we can for the moment prove only that $T(\cdot)f_n \to T(\cdot)f$ in $C_{loc}([0,+\infty)\times\mathbb{R}^{2n})$ and also in $C^{1,1}_{loc}((0,+\infty)\times\mathbb{R}^{2n})$. The result of the next step allows us to redo the proof of this result and to see that we have the convergence in $C^{1,2}_{loc}((0,+\infty)\times\mathbb{R}^{2n})$ too.

We can now prove that

Step 2.5. $(T(t))$ *satisfies* (2.3).

Proof. The proof follows that of Theorem 3.5 in [12]. We write it down completely for the reader's convenience. Let $f \in C_b^3(\mathbb{R}^{2n})$ and $u_\epsilon = T_\epsilon(\cdot)f$, $u = T(\cdot)f$. We first prove that u has a derivative of third order. We fix $h \leq n$ and consider the operator τ_k^h defined on $C_b(\mathbb{R}^{2n})$ by

$$\tau_k^h\psi(x,y) = \frac{\psi(x+ke_h,y)-\psi(x,y)}{k},$$

where $e_1,\ldots,e_n$ is the standard basis of $\mathbb{R}^n$. Given $R>0$, we localize by considering a cut-off function $\eta_R \in C_c^\infty(B_0(R))$ such that $0 \leq \eta_R \leq 1$ and $\eta_R \equiv 1$ in $B_0(R/2)$, where $B_0(R)$ is the ball of radius R centered at 0, and set $v_{\epsilon,R} = \eta_R u_\epsilon$, $v^h_{\epsilon,k,R} = \tau_k^h(v_{\epsilon,R})$. Then $v^h_{\epsilon,k,R}$ is a classical solution of

$$\begin{cases} \partial_t v^h_{\epsilon,k,R} = \mathcal{K}_\epsilon v^h_{\epsilon,k,R} + g^h_{\epsilon,k,R} \; in \; (0,+\infty)\times\mathbb{R}^{2n} \\ v^h_{\epsilon,k,R}(0,x) = \tau_k^h(\eta_R f)(x,y), \end{cases}$$

where (we sum over i but not over k and h)

$$g^h_{\epsilon,k,R}(t,x,y) = (\tau^h_k F_i)(x,y)(\partial_i v_{\epsilon,R})(t,x+ke_h,y) + (\partial^h v_{\epsilon,R})(t,x+ke_h,y)$$
$$- \tau^h_k(u_\epsilon \mathcal{K}_\epsilon \eta_R) - \partial_i \eta_R(x+ke_h,y)\tau^h_k \partial_i u_\epsilon(t,x,y) - \partial_i u_\epsilon(t,x,y)\tau^h_k \partial_i \eta_R$$
$$- \epsilon \partial^i \eta_R(x+ke_h,y)\tau^h_k \partial^i u_\epsilon(t,x,y) - \epsilon \partial^i u_\epsilon(t,x,y)\tau^h_k \partial^i \eta_R(t,x,y).$$

According to [19], Theorem 3.5, for $t>0$ and $(x,y)\in\mathbb{R}^{2n}$,

$$v^h_{\epsilon,k,R}(t,x,y) = T_\epsilon(t)\left(\tau^h_k(\eta_R f)\right)(x,y) + \int_0^t \left(T_\epsilon(t-s)g^h_{\epsilon,k,R}(s,\cdot)\right)(x,y)\,ds. \tag{2.15}$$

Since $\tau^h_k(\eta_R f)\in C_b(\mathbb{R}^{2n})$, the first term on the right-hand side converges to $T(t)\left(\tau^h_k(\eta_R f)\right)(x,y)$ as $\epsilon\to 0$ for any $t>0$ and $(x,y)\in\mathbb{R}^{2n}$. If we denote by $g^h_{k,R}$ the function we get by replacing respectively ϵ and u_ϵ by 0 and u in the definition of $g^h_{\epsilon,k,R}$, and recalling that η_R has compact support, we have that $g^h_{\epsilon,k,R}\to g^h_{k,R}$ as $\epsilon\to 0$ uniformly in $I\times\mathbb{R}^{2n}$ for any $I\subset(0,+\infty)$ compact. If $K\subset\mathbb{R}^{2n}$ is compact, we then have for any $s,t>0$, using the fact that T_ϵ is a contractive semigroup, that

$$\|T_\epsilon(t)g^h_{\epsilon,k,R}(s,\cdot) - T(t)g^h_{k,R}(s,\cdot)\|_{L^\infty(K)}$$
$$\le \|g^h_{\epsilon,k,R}(s,\cdot) - g^h_{k,R}(s,\cdot)\|_{C(\mathbb{R}^{2n})} + \|T_\epsilon(t)g^h_{k,R}(s,\cdot) - T(t)g^h_{k,R}(s,\cdot)\|_{L^\infty(K)},$$

which converge to 0 as $\epsilon\to 0$ in view of Step 2.3. Moreover, in view of (2.9) and for ϵ small,

$$\|T_\epsilon(t-s)g^h_{\epsilon,k,R}(s,\cdot)\|_{L^\infty(K)} \le \|g^h_{\epsilon,k,R}(s,\cdot)\|_{L^\infty(K)} \le C$$

where the constant C is independent of ϵ. We can thus apply the dominated convergence theorem to pass to the limit $\epsilon\to 0$ in (2.15). We then get for any $t>0$, $x,y\in\mathbb{R}^n$ that

$$v^h_{k,R}(t,x,y) = T(t)\left(\tau^h_k(\eta_R f)\right)(x,y) + \int_0^t \left(T(t-s)g^h_{k,R}(s,\cdot)\right)(x,y)\,ds. \tag{2.16}$$

We now want to pass to the limit $k\to 0$ in the above expression. First $\tau^h_k(\eta_R f)\to\partial_h(\eta_R f)$ in $C^0_{loc}(\mathbb{R}^{2n})$ and, in view of Step 2.3, $g^h_{k,R}(s,\cdot)\to g^h_R$ in $C^0_{loc}((0,+\infty),\mathbb{R}^{2n})$, where

$$g^h_R = \partial_h F_i \partial_i v_R + \partial^h v_R - \partial_h(uK\eta_R) - \partial_i\eta_R\partial_{ih}u - \partial_i u\partial_{ih}\eta_R$$

with $v_R=\eta_R u$. Using Step 2.4 and by dominated convergence (see (2.12)—$f\in C^2(\mathbb{R}^{2n})$), we can thus take the limit $k\to 0$ in (2.16) to get that for any

$t > 0$ and any $(x, y) \in \mathbb{R}^n$,

$$\partial_h(\eta_R u)(t,x,y) = T(t)\left(\partial_h(\eta_R f)\right)(x,y) + \int_0^t \Big(T(t-s)g_R^h(s,\cdot)\Big)(x,y)\;ds. \tag{2.17}$$

Since $\eta_R f \in C_b^1(\mathbb{R}^{2n})$, for each $t > 0$ the first member in the right-hand side belongs to $C_b^2(\mathbb{R}^{2n})$. Independently, interpolating (2.12) and (2.14) (since $f \in C_b^3$), we see that, for any $\alpha \in (0,1)$ and any $T > 0$, $(T_\epsilon(\cdot)f)_\epsilon$ is bounded in $B((0,T), C^{2+\alpha}(\mathbb{R}^{2n}))$. We deduce that $T(\cdot)f \in B((0,T), C^{2+\alpha}(\mathbb{R}^{2n}))$ and then that $g_R^h \in B((0,T), C^\alpha(\mathbb{R}^{2n}))$. Moreover, interpolating (2.2) and (2.10), we get that for any $\psi \in C_b^{3/4}(\mathbb{R}^{2n})$,

$$\|D_{xx}T(t)\psi\|_\infty,\ \|D_{xy}T(t)\psi\|_\infty \le \frac{C}{t^{7/8}}\|\psi\|_{C^{3/4}(\mathbb{R}^{2n})}.$$

Choosing $\alpha = 3/4$, we thus get, by the dominated convergence theorem applied to (2.17), that for any $i, j = 1, \dots, n$, $\partial_{ijh}u$ and $\partial_{ih}^j u$ exist and are continuous in $(0,+\infty) \times \mathbb{R}^{2n}$. We get in the same way the existence and continuity of $\partial_i^{jh}u$ for any i, j, h. The first estimate in (2.3) implies that the function $D_{xx}T_\epsilon(t)f$ is Lipschitz in x with Lipschitz constant not exceeding $Ce^{\omega t}t^{-3/2}\|f\|_\infty$. Hence $D_{xx}T(t)f$ is also Lipschitz with the same estimate for the Lipschitz constant. Since $D_{xxx}T(t)f$ exists, we deduce that it satisfies estimate (2.3). We see in the same way that $T(t)$ satisfies all the estimates (2.3), (2.11), (2.13) and (2.14) except those concerning $D_{yyy}T(t)$.

To prove the same result for $D_{yyy}T(t)$, we take $h \in \{1, \dots, n\}$ arbitrary, so that

$$\partial^h(\eta_R u)(t,x,y) = T(t)\Big(\partial^h(\eta_R f)\Big)(x,y) + \int_0^t \Big(T(t-s)g_R^h(s,\cdot)\Big)(x,y)\;ds, \tag{2.18}$$

$$g_R^h = \partial^h F_i \partial_i v_R - \partial^h(u\mathcal{K}\eta_R) - \partial_i\eta_R\partial_i^h u - \partial_i u \partial_i^h \eta_R,$$

with $v_R = \eta_R u$. As before, the first member in the right-hand side of (2.18) belongs to $C_b^2(\mathbb{R}^{2n})$ for any $t > 0$. We are going to prove that $g_R^h \in B((0,T), C^{3/2}(\mathbb{R}^{2n}))$. As a consequence of what we just did, we have $g_R^h \in B((0,T), C^1(\mathbb{R}^{2n}))$. By interpolation, we have for any $\theta \in (0,1)$ and $\psi \in C_b^1(\mathbb{R}^{2n})$,

$$\|D_{xx}T(t)\psi\|_{C^\theta} \le \frac{Ce^{\omega t}}{t^{(1-\theta)/2}}\|\psi\|_{C^1},\ \|D_{xy}T(t)\psi\|_{C^\theta} \le \frac{Ce^{\omega t}}{t^{(1+3\theta)/2}}\|\psi\|_{C^1}.$$

Hence, by the dominated convergence theorem applied to (2.18), we see that $D_{xxy}(\eta_R u), D_{xyy}(\eta_R u) \in B((0,T), C^\theta(\mathbb{R}^{2n}))$ for any $\theta \in (0, 1/3)$ and $T > 0$. As a consequence, $g_R^h \in B((0,T), C^{1+\theta}(\mathbb{R}^{2n}))$ for any $\theta \in (0,1/3)$ and $T > 0$. We use this argument once more. By interpolation,

$$\|D_{xx}T(t)\psi\|_\infty,\ \|D_{xy}T(t)\psi\|_\infty \leq \frac{Ce^{\omega t}}{t^{(1-\theta)/2}}\|\psi\|_{C^{1+\theta}},$$

$$\|D_{xx}T(t)\psi\|_{C^1} \leq \frac{Ce^{\omega t}}{t^{(2-\theta)/2}}\|\psi\|_{C^{1+\theta}},\ \|D_{xy}T(t)\psi\|_{C^1} \leq \frac{Ce^{\omega t}}{t^{(4-3\theta)/2}}\|\psi\|_{C^{1+\theta}}.$$

Interpolating these inequalities we get

$$\|D_{xx}T(t)\psi\|_{C^{1/2}} \leq \frac{Ce^{\omega t}}{t^{(3-2\theta)/4}}\|\psi\|_{C^{1+\theta}},\ \|D_{xy}T(t)\psi\|_{C^{1/2}} \leq \frac{Ce^{\omega t}}{t^{(5-4\theta)/4}}\|\psi\|_{C^{1+\theta}}.$$

We now choose $\theta \in (1/4, 1/3)$, so that $(5 - 4\theta)/4 < 1$, and we deduce that $D_{xy}(\eta_R u)$, and thus g_R^h, belongs to $B((0,T), C^{3/2}(\mathbb{R}^{2n}))$ for ant $T > 0$. We can now conclude that $D_{yyy}(\eta_R u)$ exists and is continuous by using the interpolation inequality $\|D_{yy}T(t)\psi\|_\infty \leq Ce^{\omega t}t^{-3/4}\|\psi\|_{C^{3/2}}$.

We thus proved that $u(t,\cdot) \in C_b^3(\mathbb{R}^{2n})$ satisfies the appropriate estimates under the assumptions that $f \in C_b^3(\mathbb{R}^{2n})$. By density, we still have that $u(t,\cdot) \in C_b^3(\mathbb{R}^{2n})$ for $f \in BUC^k(\mathbb{R}^{2n})$ for any $k = 0, 1, 2$. For a general $f \in C^k(\mathbb{R}^{2n})$, it suffices to write $T(t)f = T(t/2)T(t/2)f$ and remark that $T(t/2)f \in C_b^2(\mathbb{R}^{2n}) \subset BUC^1(\mathbb{R}^{2n})$. Moreover, the appropriate estimates for the derivatives of third order of u are still valid since we know that $D^2T(t)f$ is Lipschitz with the appropriate Lipschitz constant (coming from the estimates for $D^3T_\epsilon(t)f$) as we explained above. □

3. Proof of theorems 0.2 and 0.3

Before beginning the proof of Theorems 0.2 and 0.3, we need some estimates on the behavior of $(T(t))_{t\geq 0}$ between the anisotropic spaces $\mathcal{C}^{3\alpha,\alpha}(\mathbb{R}^{2n})$ and $\mathcal{C}^{3\beta,\beta}(\mathbb{R}^{2n})$. We first prove that

Step 3.1. *for any $\omega > 0$ there exists a constant C_ω such that for any $t > 0$ and $f \in C_b^{3,1}(\mathbb{R}^n \times \mathbb{R}^n)$,*

$$\|T(t)f\|_{C^{3,1}} \leq C_\omega e^{\omega t}\|f\|_{C^{3,1}}.$$

Proof. For $\alpha, \lambda, \mu, \nu, \rho, \tau, \chi, \psi, \omega > 0$, $\theta, \sigma, \phi, \upsilon < 0$ to be fixed later, we consider the function $z \equiv z_\epsilon$ defined by

$$\begin{aligned} z &= u^2 + \alpha|D_x u|^2 + \alpha|D_y u|^2 - \alpha D_x u D_y u + \lambda|D_{xx}u|^2 + \mu t|D_{xy}u|^2 \\ &\quad + \nu t^3|D_{yy}u|^2 + \theta t^2 D_{xy}uD_{yy}u + \rho|D_{xxx}u|^2 + \sigma t D_{xxx}uD_{xxy}u \end{aligned}$$

$$+\tau t^2 D_{xxx}uD_{xyy}u + \phi t^3 D_{xxx}uD_{yyy}u + \chi t^2|D_{xxy}u|^2 + \psi t^4|D_{xyy}u|^2$$
$$+\omega t^6|D_{yyy}u|^2 + \upsilon t^5 D_{xyy}uD_{yyy}u,$$

where $u \equiv u_\epsilon = T_\epsilon(\cdot)f$. Then $\partial_t z - \mathcal{K}_\epsilon z = g_1 + g_2 + g_3 + g_4 + g_5$ with

$$g_1 = -|D_x u|^2 + \alpha(2\partial_j F_i - \partial^j F_i)\partial_i u\partial_j u - (\alpha+\epsilon)|D_y u|^2$$
$$+ \alpha(2\partial^j F_i - \partial_j F_i)\partial^j u\partial_i u + 2\alpha D_x u D_y u,$$
$$g_2 = -\alpha|D_{xx}u|^2 + (\mu-\alpha)|D_{xy}u|^2 + (3\nu+\theta)t^2|D_{yy}u|^2$$
$$+ (4\lambda-\alpha)D_{xx}uD_{xy}u + 2(\mu+\theta)tD_{xy}uD_{yy}u$$
$$+ 2\lambda\partial_{ij}u(\partial_{ij}F_k\partial_k + 2\partial_i F_k\partial_{jk})$$
$$+ (2\mu t\partial_i^j u + \theta t^2\partial^{ij}u)(\partial_i^j F_k\partial_k u + \partial_i F_k\partial_k^j u + \partial^j F_k\partial_{ik}u)$$
$$+ (2\nu t^3\partial^{ij}u + \theta t^2\partial_i^j u)(\partial^{ij}F_k\partial_k u + \partial^i F_k\partial_k^j u + \partial^j F_k\partial_k^i u)$$
$$- \epsilon\alpha(|D_{xy}u|^2 + |D_{yy}u|^2 - D_{xy}uD_{yy}u),$$
$$g_3 = -\lambda|D_{xxx}u|^2 + (-\epsilon\lambda + (2\chi-\mu)t)|D_{xxy}u|^2$$
$$+ (-\epsilon\mu t + (4\psi-\nu)t^3)|D_{xyy}u|^2$$
$$+ (6\omega t^5 - \epsilon\nu t^3)|D_{yyy}u|^3 - \theta t^2 D_{xxy}uD_{xyy}u + (5\upsilon t^4 - \epsilon\theta t^2)D_{xyy}uD_{yyy}u$$
$$+ D_{xxx}u(\sigma D_{xxy}u + 2\tau tD_{xyy}u + 3\phi t^2 D_{yyy}u)$$
$$+ (2\rho\partial_{ijk}u + \sigma t\partial_{ij}^k u + \tau t^2\partial_i^{jk}u + \phi t^3\partial^{ijk})$$
$$\times (\partial_{ijk}F_l\partial_l u + \partial_{jk}F_l\partial_{il}u + \partial_{ik}F_l\partial_{jl}u + \partial_{ij}F_l\partial_{kl}u$$
$$+ \partial_k F_l\partial_{ijl}u + \partial_j F_l\partial_{ikl}u + \partial_i F_l\partial_{jkl}u + \partial_{ij}^k u + \partial_{ik}^j u + \partial_{jk}^i u)$$
$$+ (\sigma t\partial_{ijk}u + 2\chi t^2\partial_{ij}^k u)(\partial_{ij}^k F_l\partial_l u + 2\partial_j^k F_l\partial_{il}u + \partial^k F_l\partial_{ijl}u + \partial_{ij}F_l\partial_l^k u$$
$$+ 2\partial_j F_l\partial_{il}^k u + 2\partial_i^{jk}u)$$
$$+ (\tau t^2\partial_{ijk}u + 2\psi t^4\partial_i^{jk}u + \upsilon t^5\partial^{ijk}u)$$
$$\times (\partial_i^{jk}F_l\partial_l u + 2\partial_i^j F_l\partial_l^k u + \partial_i F_l\partial_l^{jk}u + \partial^{jk}F_l\partial_{il}u + 2\partial^j F_l\partial_{il}^k u + \partial^{ijk}u)$$
$$+ (\phi t^3\partial_{ijk}u + 2\omega t^6\partial^{ijk}u + \upsilon t^5\partial_i^{jk}u)$$
$$\times (\partial^{ijk}F_l\partial_l u + 2\partial^{ij}F_l\partial_l^k u + \partial^{jk}F_l\partial_l^i u + 2\partial^j F_l\partial_l^{ik}u + \partial^i F_l\partial_l^{jk}u),$$
$$g_4 = -\rho|D_{xxxx}u|^2 - \chi t^2|D_{xxxy}u|^2 - \psi t^4|D_{xxyy}u|^2 - \omega t^6|D_{xyyy}u|^2$$
$$- \sigma tD_{xxxx}uD_{xxxy}u - \tau t^2\partial_{il}^{jk}u\partial_{ijkl}u - \phi t^3\partial_{ijkl}u\partial_l^{ijk}u - \upsilon t^5\partial_{il}^{jk}u\partial_l^{ijk}u,$$
$$g_5 = -\epsilon\rho|D_{xxxy}u|^2 - \epsilon\chi t^2|D_{xxyy}u|^2 - \epsilon\psi t^4|D_{xyyy}u|^2 - \epsilon\omega t^6|D_{yyyy}u|^2$$

$$- \epsilon\upsilon t^5 \partial_i^{jkl} u \partial^{ijkl} u - \epsilon \partial_{ijk}^l u[\sigma t \partial_{ij}^{kl} u + \tau t^2 \partial_i^{jkl} u + \phi t^3 \partial^{ijkl} u].$$

We have

$$g_1 \leq \{-1 + O(\sqrt{\alpha})\}|D_x u|^2 + \alpha\{-1 + O(\sqrt{\alpha})\}|D_y u|^2, \tag{3.1}$$

and, assuming that $\theta + \mu = 0$ and $\lambda,\ \mu = o(\alpha)$,

$$\begin{aligned} g_2 = {} & (O(t) + o(\alpha))|D_x u|^2 + \{(3\nu + \theta) + O(t)\}t^2 |D_{yy} u|^2 \\ & - \left(\frac{\alpha}{2}(1 + o(1)) + O(t)\right)\left(|D_{xx} u|^2 + |D_{xy} u|^2\right). \end{aligned} \tag{3.2}$$

We now estimate g_3. Assuming that $\rho, \sigma, \tau, \phi = o(\lambda)$, we first write that

$$\begin{aligned} & (2\rho\partial_{ijk} u + \sigma t \partial_{ij}^k u + \tau t^2 \partial_i^{jk} u + \phi t^3 \partial^{ijk}) \\ & \quad \times (\partial_{ijk} F_l \partial_l u + \partial_{jk} F_l \partial_{il} u + \partial_{ik} F_l \partial_{jl} u + \partial_{ij} F_l \partial_{kl} u \\ & \qquad + \partial_k F_l \partial_{ijl} u + \partial_j F_l \partial_{ikl} u + \partial_i F_l \partial_{jkl} u + \partial_{ij}^k u + \partial_{ik}^j u + \partial_{jk}^i u) \\ & = 6\rho D_{xxx} u D_{xxy} u + 3\sigma t |D_{xxy} u|^2 + 3\tau t^2 |D_{xyy} u||D_{xxy} u| + 3\phi t^3 D_{xxy} u D_{yyy} u \\ & \quad + o(\lambda)|D_{xxx} u|^2 + o(\alpha)|D_x u|^2 + o(\alpha)|D_{xx} u|^2 + O(t^2)|D_{xxy} u|^2 \\ & \quad + O(t^4)|D_{xyy} u|^2 + O(t^6)|D_{yyy} u|^2, \\ & (\sigma t \partial_{ijk} u + 2\chi t^2 \partial_{ij}^k u) \\ & \quad \times (\partial_{ij}^k F_l \partial_l u + 2\partial_j^k F_l \partial_{il} u + \partial^k F_l \partial_{ijl} u + \partial_{ij} F_l \partial_l^k u + 2\partial_j F_l \partial_{il}^k u + 2\partial_i^{jk} u) \\ & = 2\sigma t D_{xxx} u D_{xyy} u + 4\chi t^2 D_{xxy} u D_{xyy} u + O(t)|D_x u|^2 + O(t) D_{xx} u|^2 \\ & \quad + O(t) D_{xxx} u|^2 + O(t) D_{xy} u|^2 + O(t^2)|D_{xxy} u|^2 + o(\lambda)|D_{xxx} u|^2, \\ & (\tau t^2 \partial_{ijk} u + 2\psi t^4 \partial_i^{jk} u + \upsilon t^5 \partial^{ijk} u) \\ & \quad \times (\partial_i^{jk} F_l \partial_l u + 2\partial_i^j F_l \partial_l^k u + \partial_i F_l \partial_l^{jk} u + \partial^{jk} F_l \partial_{il} u + 2\partial^j F_l \partial_{il}^k u + \partial^{ijk} u) \\ & = \tau t^2 D_{xxx} u D_{yyy} u + 2\psi t^4 D_{xyy} u D_{yyy} u + \upsilon t^5 |D_{yyy} u|^2 + O(t)|D_x u|^2 \\ & \quad + O(t)|D_{xy} u|^2 + O(t)|D_{xx} u|^2 + O(t)|D_{xxx} u|^2 + O(t^2)|D_{xxy} u|^2 \\ & \quad + O(t^4)|D_{xyy} u|^2 + O(t^6)|D_{yyy} u|^2, \\ & (\phi t^3 \partial_{ijk} u + 2\omega t^6 \partial^{ijk} u + \upsilon t^5 \partial_i^{jk} u) \\ & \quad \times (\partial^{ijk} F_l \partial_l u + 2\partial^{ij} F_l \partial_l^k u + \partial^{jk} F_l \partial_l^i u + 2\partial^j F_l \partial_l^{ik} u + \partial^i F_l \partial_l^{jk} u) \\ & = O(t)|D_{xxx} u|^2 + O(t)|D_x u|^2 + O(t) D_{xy} u|^2 + O(t^4)|D_{xyy} u|^2 \\ & \quad + O(t^6)|D_{yyy} u|^2. \end{aligned}$$

Hence,

$$
\begin{aligned}
g_1 + g_2 + g_3 &\le \{-1 + O(\sqrt{\alpha}) + O(t)\}|D_x u|^2 + \alpha\{-1 + O(\sqrt{\alpha})\}|D_y u|^2 \\
&- \left(\frac{\alpha}{2}(1 + o(1)) + O(t)\right)\left(|D_{xx}u|^2 + |D_{xy}u|^2\right) + t^2\{3\nu + \theta + O(t)\}|D_{yy}u|^2 \\
&+ \{-\lambda + o(\lambda) + O(t)\}|D_{xxx}u|^2 + \{-\mu + 2\chi + 3\sigma + O(t)\}t|D_{xxy}u|^2 \\
&+ \{4\psi - \nu + O(t)\}t^3|D_{xyy}u|^2 + \{6\omega + \upsilon + O(t)\}t^5|D_{yyy}u|^2 \\
&+ (6\rho + \sigma)D_{xxx}uD_{xxy}u + 2(\sigma + \tau)tD_{xxx}uD_{xyy}u \\
&+ (3\phi + \tau)t^2D_{xxx}uD_{yyy}u + (3\tau - \theta + 4\chi)t^2|D_{xxy}u||D_{xyy}u| \\
&+ 3\phi t^3D_{xxy}uD_{yyy}u + (5\upsilon + 2\psi)t^4D_{xyy}uD_{yyy}u \\
&- \epsilon\left(\mu t|D_{xyy}u|^2 + \nu t^3|D_{yyy}u|^2 + \theta t^2D_{xyy}uD_{yyy}u\right).
\end{aligned}
$$

Assuming that $\sigma = -6\rho$, $\tau = 6\rho$, $\phi = -2\rho$, $\nu \ge \frac{\mu}{4}$ and $5\upsilon + 2\psi = 0$, this becomes

$$
\begin{aligned}
g_1 + g_2 + g_3 &\le \{-1 + O(\sqrt{\alpha}) + O(t)\}|D_x u|^2 + \alpha\{-1 + O(\sqrt{\alpha})\}|D_y u|^2 \\
&- \left(\frac{\alpha}{2}(1 + o(1)) + O(t)\right)\left(|D_{xx}u|^2 + |D_{xy}u|^2\right) + t^2\{3\nu - \mu + O(t)\}|D_{yy}u|^2 \\
&+ \{-\lambda + o(\lambda) + O(t)\}|D_{xxx}u|^2 + \{-\mu + 2\chi - 18\rho + O(t)\}t|D_{xxy}u|^2 \\
&+ \{4\psi - \nu + O(t)\}t^3|D_{xyy}u|^2 + \{6\omega + \upsilon + O(t)\}t^5|D_{yyy}u|^2 \\
&+ (18\rho + \mu + 4\chi)t^2|D_{xxy}u||D_{xyy}u| - 6\rho t^3D_{xxy}uD_{yyy}u.
\end{aligned}
$$

For $\delta > 0$ to be fixed later, we have

$$
\begin{aligned}
&\left|(18\rho + \mu + 4\chi)t^2|D_{xxy}u||D_{xyy}u|\right| \\
&\le \frac{18\rho + \mu + 4\chi}{2\delta}t|D_{xxy}u|^2 + \frac{1}{2}(18\rho + \mu + 4\chi)\delta t^3|D_{xyy}u|^2
\end{aligned}
$$

and also

$$
\left|-6\rho t^3D_{xxy}uD_{yyy}u\right| \le 3\rho t|D_{xxy}u|^2 + 3\rho t^5|D_{yyy}u|^2.
$$

Hence,

$$
\begin{aligned}
g_1 + g_2 + g_3 &\le \{-1 + O(\sqrt{\alpha}) + O(t)\}|D_x u|^2 + \alpha\{-1 + O(\sqrt{\alpha})\}|D_y u|^2 \\
&- \left(\frac{\alpha}{2}(1 + o(1)) + O(t)\right)\left(|D_{xx}u|^2 + |D_{xy}u|^2\right) \\
&+ t^2\{3\nu - \mu + O(t)\}|D_{yy}u|^2 + \{-\lambda + o(\lambda) + O(t)\}|D_{xxx}u|^2 \\
&+ t\{-\mu + 2\chi - 15\rho + \frac{18\rho + \mu + 4\chi}{2\delta} + O(t)\}|D_{xxy}u|^2
\end{aligned}
$$

$$+ t^3\{4\psi - \nu + \frac{1}{2}(18\rho + \mu + 4\chi)\delta + O(t)\}|D_{xyy}u|^2$$
$$+ t^5\{6\omega + \upsilon + 3\rho + O(t)\}|D_{yyy}u|^2.$$

We thus choose the coefficients such that

$$\begin{cases} 3\nu - \mu < 0, \qquad -\mu + 2\chi - 15\rho + \frac{18\rho+\mu+4\chi}{2\delta} < 0, \\ 4\psi - \nu + \frac{1}{2}(18\rho + \mu + 4\chi)\delta < 0, \qquad 6\omega + \upsilon + 3\rho < 0. \end{cases} \tag{3.3}$$

Independently, the conditions for the quadratic forms defining z to be positive at $t = 1$ read as

$$\begin{cases} \nu > \theta^2/4\mu = \mu/4, \quad \chi > \sigma^2/4\rho = 9\rho, \quad \psi > \frac{\tau^2}{4\rho} + \frac{\sigma^2\tau^2}{16\rho^2(\chi-\sigma^2/\rho)} = \frac{9\rho\chi}{\chi-9\rho}, \\ \omega > \rho + \frac{9\rho^2}{\chi-9\rho} + \frac{(\upsilon+6\rho+\frac{54\rho^2}{\chi-9\rho})^2}{4\psi-36\rho-\frac{36^2\rho^2}{4(\chi-9\rho)}} = \frac{4\rho\psi\chi+\upsilon(\upsilon\chi-9\upsilon\rho+12\chi\rho)}{4(\psi\chi-9\rho\psi-9\rho\chi)}. \end{cases} \tag{3.4}$$

We can choose for example

$$\lambda = \mu = \alpha^2,\ \nu = \tfrac{2}{7}\alpha^2,\ \rho = \alpha^6,\ \psi = \chi = \alpha^3,\ \omega = \tfrac{3}{50}\alpha^3,\ \delta \in \left(\tfrac{1}{2}, \tfrac{4}{7}\right)$$

for α small. With that choice, we have, for α small, that $g_1 + g_2 + g_3 \le 0$ in $[0, T] \times \mathbb{R}^{2n}$ with T independent of ϵ. It remains to check that, with the same choice of parameters, $g_4, g_5 \le 0$ in $[0, T'] \times \mathbb{R}^{2n}$ with T' independent of ϵ. We have

$$\begin{aligned} g_4 = &- \alpha^6|D_{xxxx}u|^2 - \alpha^3t^2|D_{xxxy}u|^2 - \alpha^3t^4|D_{xxyy}u|^2 - \frac{3}{50}\alpha^3t^6|D_{xyyy}u|^2 \\ &+ 6\alpha^6tD_{xxxx}uD_{xxxy}u - 6\alpha^6t^2D_{xxxx}uD_{xxyy}u + 2\alpha^6t^3D_{xxxx}uD_{xyyy}u \\ &+ \frac{2}{5}\alpha^3t^5D_{xxyy}uD_{xyyy}u. \end{aligned}$$

Writing

$$\begin{aligned} \left|6\alpha^6tD_{xxxx}uD_{xxxy}u\right| &\le \frac{3\alpha^6}{\delta}|D_{xxxx}u|^2 + 3\alpha^6\delta t^2|D_{xxxy}u|^2, \\ \left|6\alpha^6t^2D_{xxxx}uD_{xxyy}u\right| &\le \frac{3\alpha^6}{\delta}|D_{xxxx}u|^2 + 3\alpha^6\delta t^4|D_{xxyy}u|^2, \\ \left|2\alpha^6t^3D_{xxxx}uD_{xyyy}u\right| &\le \frac{\alpha^6}{\delta}|D_{xxxx}u|^2 + \alpha^6\delta t^6|D_{xyyy}u|^2, \\ \left|\frac{2}{5}\alpha^3t^5D_{xxyy}uD_{xyyy}u\right| &\le \frac{\alpha^3}{5\tilde{\delta}}|D_{xxyy}u|^2 + \frac{1}{5}\alpha^3\tilde{\delta}t^6|D_{xyyy}u|^2, \end{aligned}$$

for $\delta, \tilde{\delta} > 0$ to be chosen later, we obtain

$$g_4 \leq \alpha^6 \left(-1 + \frac{7}{\delta}\right) |D_{xxxx}u|^2 + \alpha^3(-1 + 3\alpha^3\delta)t^2|D_{xxxy}u|^2$$
$$+ \alpha^3 \left(-1 + \frac{1}{5\tilde{\delta}} + 3\alpha^3\delta\right) t^4|D_{xxyy}u|^2 + \alpha^3 \left(-\frac{3}{50} + \frac{\tilde{\delta}}{5} + \tilde{\delta}\alpha^3\right) t^6|D_{xyyy}u|^2.$$

Choosing $\delta = 14$ and $\tilde{\delta} = 1/4$, we see that, for α small, $g_4 \leq 0$ in $[0, T'] \times \mathbb{R}^{2n}$ with T' independent of ϵ. The same holds for g_5 since we can write in the same way that

$$\epsilon^{-1} g_5 \leq \alpha^6 \left(-1 + \frac{7}{\delta}\right) |D_{xxxy}u|^2 + \alpha^3(-1 + 3\alpha^3\delta)t^2|D_{xxyy}u|^2$$
$$+ \alpha^3 \left(-1 + \frac{1}{5\tilde{\delta}} + 3\alpha^3\delta\right) t^4|D_{xyyy}u|^2 + \alpha^3 \left(-\frac{3}{50} + \frac{\tilde{\delta}}{5} + \tilde{\delta}\alpha^3\right) t^6|D_{yyyy}u|^2,$$

and we choose as before $\delta = 14$ and $\tilde{\delta} = 1/4$ to get the result.

We eventually have that $\partial_t z - \mathcal{K}_\epsilon z = g_1 + g_2 + g_3 + g_4 + g_5 \leq 0$ in $[0, T'] \times \mathbb{R}^{2n}$ with T' independent of ϵ. We conclude by using the maximum principle and the semigroup property as in the proof of step 2.1 that for any $\omega > 0$ there exists a constant C_ω independent of ϵ such that for any ϵ small, $t > 0$ and $f \in C_b^{3,1}(\mathbb{R}^n \times \mathbb{R}^n)$,

$$\|T_\epsilon(t) f\|_{C^{3,1}} \leq C_\omega e^{\omega t} \|f\|_{C^{3,1}}.$$

□

It follows from Step 2.1 that

$$\|T(t)\|_{L(C^0, C^{3,1})} \leq C e^{\omega t} t^{-3/2}.$$

The interpolation equality

$$(C_b(\mathbb{R}^{2n}), C_b^{3,1}(\mathbb{R}^{2n}))_{\theta,\infty} = \mathcal{C}_b^{3\theta,\theta}(\mathbb{R}^{2n}) \qquad \forall\, \theta \in (0,1)$$

proved in ([13], Lemma 5.1; see also [9], Lemma 2 in Appendix C.2 with $p = q_0 = q_1 = q = \infty$) then implies, as in ([13], Proposition 5.4), that for every $\omega > 0$, there exists $C > 0$ such that for every $t > 0$ and $0 < \alpha \leq \beta \leq 1$,

$$\|T(t)\|_{L(\mathcal{C}^{3\alpha,\alpha}, \mathcal{C}^{3\beta,\beta})} \leq C e^{\omega t} t^{-3(\beta-\alpha)/2}. \tag{3.5}$$

Since $T(t)$ is not strongly continuous neither in $C_b(\mathbb{R}^{2n})$ nor in $BUC(\mathbb{R}^{2n})$ as shown in [5], we cannot define its infinitesimal generator. Nevertheless, as in much of the quoted literature, we can associate with $(T(t))$ an operator

A, which will play the role of generator as follows. For $\lambda > 0$, we consider the operator $R(\lambda)$ on $X := C_b(\mathbb{R}^{2n})$ defined by

$$R(\lambda)f(x,y) = \int_0^\infty e^{-\lambda t}(T(t)f)(x,y)dt. \tag{3.6}$$

Since the semigroup $(T(t))$ is contractive, the integral is well defined and $R(\lambda)$ is continuous from X to X with norm $\|R(\lambda)\| \leq 1/\lambda$. Moreover, $R(\lambda)$ satisfies the resolvent identity and it is one-to-one because, for every $(x,y) \in \mathbb{R}^{2n}$, $(R(\lambda)f)(x,y)$ is the Laplace transform of the function $t \to (T(t)f)(x,y)$, which is equal to $f(x)$ for $t = 0$. Therefore (see e.g. [22], Theorem VIII.4.1), there exists a closed operator $A : D(A) \to X$ such that

$$D(A) = \text{Range of } R(\lambda)\ \forall\ \lambda > 0, \text{ and } R(\lambda) = R(\lambda, A).$$

We now remark that if $f = R(\lambda)\phi \in D(A)$ with $\phi \in C^{\theta,\theta/3}(\mathbb{R}^n \times \mathbb{R}^n)$ for some $\theta > 1$, then $Af = \mathcal{K}f$. Indeed, fix $\eta \in (3, 2+\theta)$ not an integer. By (3.5) with $\alpha = \theta/3$ and $\beta = \eta/3$,

$$\|(T(t)\phi)\|_{\mathcal{C}^{\eta,\eta/3}} \leq Ce^{\omega t}t^{-(\eta-\theta)/2}\|\phi\|_{C^{\theta,\theta/3}}$$

with $(\eta - \theta)/2 < 1$. Moreover $\mathcal{C}^{\eta,\eta/3} = C^{\eta,\eta/3} \subset C^{2,1}$ since η is not integer. Choosing $\omega \in (0, \lambda)$, we can thus apply the dominated convergence theorem to get

$$\begin{aligned} \mathcal{K}f(x,y) &= \int_0^\infty e^{-\lambda t}(\mathcal{K}T(t)\phi)(x,y)dt = \int_0^\infty e^{-\lambda t}(\partial_t T(t)\phi)(x,y)dt \\ &= -\phi(x,y) + \lambda f(x,y). \end{aligned}$$

Hence $(\lambda - \mathcal{K})f = \phi = (\lambda - A)f$, which gives $\mathcal{K}f = Af$. More generally, we can characterize $D(A)$ and assert that $A = \mathcal{K}$ on $D(A)$ as in Lorenzi [13]. We first have that given any $f \in C_b^2(\mathbb{R}^{2n})$ such that $\mathcal{K}f \in C_b(\mathbb{R}^{2n})$, there holds

$$T(t)\mathcal{K}f = \mathcal{K}T(t)f. \tag{3.7}$$

The proof given in ([12], Lemma 4.4) consists in remarking that (3.7) holds with $\mathcal{K}_\epsilon$ instead of $\mathcal{K}$ and then passing to the limit. We can then deduce from (3.7), as in ([12], Proposition 4.5), that

Step 3.2.

$$\begin{aligned} & f \in D(A) \iff \\ & \exists\ (f_n) \subset C_b^2(\mathbb{R}^{2n}),\ g \in C_b(\mathbb{R}^{2n})\ s.t.\ f_n \to f,\ \mathcal{K}f_n \to g\ in\ C_{loc}(\mathbb{R}^{2n}), \\ & \sup_n\ (\|f_n\|_\infty + \|\mathcal{K}f_n\|_\infty) < \infty. \end{aligned} \tag{3.8}$$

Moreover, $\mathcal{K} = A$ *on* $D(A)$.

We can now prove Theorem 0.2:

Step 3.3. *Proof of Theorem* 0.2.

Proof. We fix $\lambda > 0$ and $f \in C_b(\mathbb{R}^{2n})$, and we are going to prove that $u := R(\lambda)f$ is a weak solution of (E). Let $f_n \subset C_b^2(\mathbb{R}^{2n})$ be bounded in $C_b(\mathbb{R}^{2n})$ and converging to f in $C_{loc}(\mathbb{R}^{2n})$. Then $u_n := R(\lambda)f_n \in C_b^2(\mathbb{R}^{2n}) \cap D(A)$ and, as above, $u_n \to u$ in $C_{loc}(\mathbb{R}^{2n})$. Since $\mathcal{K} = A$ on $D(A)$, we thus have that u_n is a classical solution of $\lambda u_n - \mathcal{K} u_n = f_n$. Hence, for any $\phi \in C_c^\infty(\mathbb{R}^{2n})$,

$$\int_{\mathbb{R}^{2n}} f_n \phi dx = \int_{\mathbb{R}^{2n}} u_n (\lambda - \mathcal{K}^*)\phi \, dx.$$

Passing to the limit in this inequality gives the result.

Now if $f \in C^{\theta,\theta/3}(\mathbb{R}^{2n})$ for some $\theta \in (0,1)$, it follows from estimates (3.5) and the arguments of Lunardi ([17], Theorem 2.1) that u satisfies the Schauder estimates (0.2). Indeed, given $\eta \in (\theta, 1)$, we are going to prove that

$$\begin{aligned} u \in &(C^{\eta,\eta/3}(\mathbb{R}^n \times \mathbb{R}^n), C^{2+\eta,(2+\eta)/3}(\mathbb{R}^n \times \mathbb{R}^n))_{1-(\eta-\theta)/2,\infty} \\ &= \mathcal{C}^{2+\theta,(2+\theta)/3}(\mathbb{R}^n \times \mathbb{R}^n). \end{aligned} \tag{3.9}$$

We recall that if $Y_2 \subset Y_1$ are Banach spaces then the interpolation space $(Y_1, Y_2)_{\gamma,\infty}$ is defined by

$$(Y_1, Y_2)_{\gamma,\infty} = \{u \in Y_1 : \|u\|_{\gamma,\infty} := \sup_{0<\xi<1} \xi^{-\gamma} \mathcal{K}(\xi, u) < \infty\},$$

where

$$\mathcal{K}(\xi, u) = \inf_{\begin{cases} a \in Y_1, b \in Y_2 \\ u = a + b \end{cases}} \|a\|_{Y_1} + \xi \|b\|_{Y_2}.$$

For a given $\xi > 0$, we write $u = a + b$ with

$$\begin{aligned} a(x,y) &= \int_0^\xi e^{-\lambda t} (T(t)f)(x,y) dt, \\ b(x,y) &= \int_\xi^{+\infty} e^{-\lambda t} (T(t)f)(x,y) dt. \end{aligned}$$

In view of (3.5) and Lemma 3.5 in Lunardi [15], we have $a \in C^{\eta,\eta/3}(\mathbb{R}^n \times \mathbb{R}^n)$ and $b \in C^{2+\eta,(2+\eta)/3}(\mathbb{R}^n \times \mathbb{R}^n)$ with

$$\|a\|_{C^{\eta,\eta/3}} \leq C \int_0^\xi t^{-(\eta-\theta)/2} dt \|f\|_{C^{\theta,\theta/3}} \leq C\xi^{1-(\eta-\theta)/2}\|f\|_{C^{\theta,\theta/3}},$$

and, choosing $\omega < \lambda$,

$$\|b\|_{C^{2+\eta,(2+\eta)/3}} \leq C \int_\xi^{+\infty} t^{-1-(\eta-\theta)/2} dt \|f\|_{C^{\theta,\theta/3}} \leq C\xi^{-(\eta-\theta)/2}\|f\|_{C^{\theta,\theta/3}}.$$

Hence,

$$\mathcal{K}(\xi, u) \leq C\xi^{1-(\eta-\theta)/2}\|f\|_{C^{\theta,\theta/3}},$$

which gives (3.9) and (0.2).

We now prove that $v = 0$ is the unique weak solution of $\lambda u - \mathcal{K}u = 0$ that belongs to $C_b(\mathbb{R}^{2n})$ and such that $\Delta_x v \in C_b(\mathbb{R}^{2n})$. We adapt to our situation the arguments of Lorenzi [13]. It suffices to prove that $v \in D(A)$ (since then $v := R(\lambda)f$ for some $f \in C_b(\mathbb{R}^{2n})$ and thus, in view of what we just did, $0 = \lambda v - \mathcal{K}v = f$ weakly). We are going to prove that the $\hat{v}_m$'s defined by

$$\hat{v}_m = T(1/m)v$$

approximate v as needed in (3.8). According to Theorem 0.1, $\hat{v}_m \in C_b^3(\mathbb{R}^{2n})$, $\hat{v}_m \to v$ in $C_{loc}(\mathbb{R}^{2n})$ and $\|\hat{v}_m\|_\infty \leq \|v\|_\infty$ for any m. We now prove that

$$\mathcal{K}\hat{v}_m = T(1/m)\mathcal{K}v, \tag{3.10}$$

which will clearly give the desired result. To use (3.7), we approximate v by $v_\epsilon \in C_b^2(\mathbb{R}^{2n})$ defined by convolution by $v_\epsilon := v \star \rho_\epsilon$, $\rho_\epsilon = \epsilon^{-2n}\rho(\epsilon^{-1}x)$, where ρ is some smooth, nonnegative function with compact support in the unit ball of $\mathbb{R}^{2n}$ and of norm 1 in $L^1(\mathbb{R}^{2n})$. It is standard that $v_\epsilon \to v$ and $\Delta_x v_\epsilon \to \Delta_x v$ in $C_{loc}(\mathbb{R}^{2n})$ as $\epsilon \to 0$. Assuming for the moment that $\mathcal{F}v_\epsilon \to \mathcal{F}v$ in $C_{loc}(\mathbb{R}^{2n})$, where $\mathcal{F}$ is the formal operator defined by

$$\mathcal{F}\phi := FD_x\phi + xD_y\phi, \tag{3.11}$$

we eventually get that $\mathcal{K}v_\epsilon \to \mathcal{K}v$ in $C_{loc}(\mathbb{R}^{2n})$. In view of (3.7),

$$T(1/m)\mathcal{K}v_\epsilon = \mathcal{K}T(1/m)v_\epsilon.$$

Passing to the limit in this equality using step 2.4 gives (3.10). It remains to prove that $\mathcal{F}v_\epsilon \to \mathcal{F}v$ in $C_{loc}(\mathbb{R}^{2n})$. Let $(v^n) \subset C_b^1(\mathbb{R}^{2n})$ be such that $v^n \to u$ in $C_{loc}(\mathbb{R}^{2n})$ and $v_\epsilon^n = v^n \star \rho_\epsilon$. Then

$$\mathcal{F}v_\epsilon^n - \mathcal{F}v^n \star \rho_\epsilon = f_1^n + f_2^n + f_3^n$$

with, since $v^n_\epsilon \to v^n$ in $C_{loc}(\mathbb{R}^{2n})$ as $\epsilon \to 0$,

$$\mathcal{F}v^n_\epsilon(z) = F_i(z)\int_{B_0(\epsilon)} u(z-\xi)\partial_i\rho_\epsilon(\xi)d\xi + x_i\int_{B_0(\epsilon)} u(z-\xi)\partial^i\rho_\epsilon(\xi)d\xi$$
$$\to \mathcal{F}v_\epsilon,$$
$$\mathcal{F}v^n \star \rho_\epsilon \to \mathcal{F}v\star\rho_\epsilon \text{ (by performing integration by parts)},$$
$$f^n_1(z) = \int_{B_0(\epsilon)} (F_i(z) - F_i(z-\xi))v^n(z-\xi)\partial_i\rho_\epsilon(\xi)d\xi$$
$$\to f_1(z) := \int_{B_0(\epsilon)} (F_i(z) - F_i(z-\xi))v(z-\xi)\partial_i\rho_\epsilon(\xi)d\xi$$
$$f^n_2(z) = ((v^n\partial_i F_i)\star\rho_\epsilon)(z) \to f_2(z) := ((v\partial_i F_i)\star\rho_\epsilon)(z)$$
$$f^n_3(z) = \int_{B_0(\epsilon)} \xi_i v^n(z-\xi)(\partial^i\rho_\epsilon)(\xi)d\xi$$
$$\to f_3(z) := \int_{B_0(\epsilon)} \xi_i v^n(z-\xi)(\partial^i\rho_\epsilon)(\xi)d\xi.$$

All the previous convergences are locally uniform in $z=(x,y)$ as $n\to+\infty$. Hence,

$$\mathcal{F}v_\epsilon - \mathcal{F}v\star\rho_\epsilon = f_1+f_2+f_3.$$

We have (all the convergences are locally uniform in $z=(x,y)$ as $\epsilon\to 0$)

$$f_1(z) = \int_{B_0(\epsilon)} (F_i(z) - F_i(z-\xi))v(z-\xi)\partial_i\rho_\epsilon(\xi)d\xi$$
$$= \epsilon^{-1}\int_{B_0(1)} (F_i(z) - F_i(z-\epsilon\xi))v(z-\epsilon\xi)\partial_i\rho(\xi)d\xi$$
$$\to v(z)\partial_k F_i(z)\int_{B_0(1)}\xi_k\partial_i\rho(\xi)d\xi + v(z)\partial^k F_i(z)\int_{B_0(1)}\xi^k\partial_i\rho(\xi)d\xi$$
$$= -v(z)\partial_i F_i(z),$$
$$f_2(z) = ((v\partial_i F_i)\star\rho_\epsilon)(z) \to v(z)\partial_i F_i(z),$$
$$f_3(z) = \int_{B_0(\epsilon)}\xi_i v(z-\xi)(\partial^i\rho_\epsilon)(\xi)d\xi = \int_{B_0(1)}\xi_i v(z-\epsilon\xi)\partial^i\rho(\xi)d\xi$$
$$\to v(z)\int_{B_0(1)}\xi_i\partial^i\rho(\xi)d\xi = 0.$$

Since $\mathcal{F}v = \mathcal{K}v - \Delta_x v \in C_b(\mathbb{R}^{2n})$, $\mathcal{F}v\star\rho_\epsilon \to \mathcal{F}v$ in $C_{loc}(\mathbb{R}^{2n})$, and we eventually get that $\mathcal{F}v_\epsilon \to \mathcal{F}v$ in $C_{loc}(\mathbb{R}^{2n})$. □

Step 3.4. *Let* $f \in C_b(\mathbb{R}^{2n})$ *and* $g \in C([0,+\infty) \times \mathbb{R}^{2n})$ *such that* $g \in B([0,T], C^2(\mathbb{R}^{2n}))$ *for* $T > 0$. *Then*

$$u(t,x,y) := (T(t)f)(x,y) + \int_0^t \left(T(t-s)g(s,\cdot)\right)(x,y)ds$$

is the only bounded classical solution to (NHCP).

Proof. According to the previous section,

$$T(\cdot)f \in C([0,+\infty) \times \mathbb{R}^{2n}) \cap C^{1,3}((0,+\infty) \times \mathbb{R}^{2n}).$$

It thus suffices to check the regularity of

$$v(t,x,y) = \int_0^t \left(T(t-s)g(s,\cdot)\right)(x,y)ds. \tag{3.12}$$

Since $(T(t))$ is contractive, $v \in C([0,+\infty) \times \mathbb{R}^{2n})$. According to (2.12), $v(t,\cdot) \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$ for any $t > 0$. Independently, for $(x,y) \in \mathbb{R}^{2n}$ given, the map $t \to (T(t-s)g(s,\cdot))(x,y)$ belongs to $C^1((s,+\infty))$ with $\partial_t(T(t-s)g(s,\cdot))(x,y) = \mathcal{K}(T(t-s)g(s,\cdot))(x,y)$. Moreover

$$\|\mathcal{K}(T(t-s)g(s,\cdot))\|_{C(\mathbb{R}^{2n})} \le C\|(T(t-s)g(s,\cdot))\|_{C^2(\mathbb{R}^{2n})} \le C. \tag{3.13}$$

Hence, according to the dominated convergence theorem, v is C^1 in t with

$$\begin{aligned}
\partial_t v(t,x) &= \int_0^t \mathcal{K}(T(t-s)g(s,\cdot))(x,y)ds + g(t,x) \\
&= \mathcal{K}\int_0^t (T(t-s)g(s,\cdot))(x,y)ds + g(t,x)
\end{aligned}$$

(the last equality comes from (3.13) and the dominated convergence theorem).

We eventually get that $v \in C([0,+\infty) \times \mathbb{R}^{2n}) \cap C^{1,2}((0,+\infty) \times \mathbb{R}^{2n})$ with $\partial_t v = \mathcal{K}v$. Hence $u \in C([0,+\infty) \times \mathbb{R}^{2n}) \cap C^{1,2}((0,+\infty) \times \mathbb{R}^{2n})$ with

$$\partial_t u = \mathcal{K}T(t)f + \mathcal{K}v + g = \mathcal{K}u + g.$$

□

The proof of Theorem 0.3 follows now the same scheme as the proof of Theorem 0.2.

Step 3.5. *Proof of Theorem* 0.3.

Proof. Let $f \in C_b(\mathbb{R}^{2n})$ and $g \in C_b([0,\infty) \times \mathbb{R}^{2n})$ and

$$u(t,x,y) = (T(t)f)(x,y) + v(t,x,y), \tag{3.14}$$

where v is defined by (3.12). We are going to show that u is a weak solution of (NHCP). We already know from the proof of the previous step that $u \in C([0,\infty) \times \mathbb{R}^{2n})$. Let $g_n \in C_b^{1,2}([0,\infty) \times \mathbb{R}^{2n})$ be such that $g_n \to g$ in $C_{loc}([0,\infty) \times \mathbb{R}^{2n})$. We denote by v_n the function v with g replaced by g_n, and by u_n the function defined by (3.14) with v replaced by v_n. Then, according to step 3.4, u_n is a classical solution of (NHCP) with g replaced by g_n. Moreover, $v_n \to v$ uniformly in $[0,T] \times K$ for any $T > 0$ and $K \subset \mathbb{R}^{2n})$ compact. Indeed, since $T(t)$ is order-preserving,

$$\begin{aligned}
&\sup_{(t,z)\in[0,T]\times K} |v_n(t,z) - v(t,z)| \\
&= \sup_{(t,z)\in[0,T]\times K} \left| \int_0^t (T(s)(g_n(t-s,\cdot) - g(t-s,\cdot)))(z)ds \right| \\
&\leq \sup_{z\in K} \int_0^T |T(s)(\sup_{r\in[0,T]} |g_n(r,\cdot) - g(r,\cdot)|)(z)|ds,
\end{aligned}$$

and we pass to the limit using step 2.4. We can now write that for any $\phi \in C_c^\infty((0,+\infty) \times \mathbb{R}^{2n})$,

$$\begin{aligned}
\int_{(0,+\infty)\times\mathbb{R}^{2n}} g_n\phi \; dt\, dx\, dy &= \int_{(0,+\infty)\times\mathbb{R}^{2n}} (\partial_t u_n - \mathcal{K}u_n)\phi \; dt\, dx\, dy \\
&= \int_{(0,+\infty)\times\mathbb{R}^{2n}} u_n(\partial_t - \mathcal{K}^*)\phi \; dt\, dx\, dy,
\end{aligned}$$

where $\mathcal{K}^*$ is the formal adjoint of $\mathcal{K}$ defined on a smooth function ψ by

$$\mathcal{K}^*\psi = \Delta_x\psi - FD_x\psi - (divF)\psi - xD_y\psi. \tag{3.15}$$

Passing to the limit in this equality shows that u is a weak solution of (NHCP).

The Schauder estimate (0.3) is proved as in Lunardi [17]. Indeed, if $f \in C^{2+\theta,2+\theta/3}(\mathbb{R}^n \times \mathbb{R}^n)$, then, in view of (3.5), we have

$$\sup_{0\leq t\leq T} \|T(t)f\|_{C^{2+\theta,2+\theta/3}} \leq C\|f\|_{C^{2+\theta,2+\theta/3}}. \tag{3.16}$$

We now estimate v. For any $(t,\xi) \in [0,T] \times (0,1)$, we consider the functions a and b defined by

$$a(x,y) = \int_{(t-\xi)^+}^t (T(t-s)g(s,\cdot))(x,y)ds$$

and

$$b(x,y)=\int_0^{(t-\xi)^+}(T(t-s)g(s,\cdot))(x,y)ds,$$

where $(t-\xi)^+=\max(t-\xi,0)$. Then $u(t,\cdot)=a+b$ and, for every $\eta\in(\theta,1)$, we have $a\in C^{\eta,\eta/3}(\mathbb{R}^n\times\mathbb{R}^n)$ and $b\in C^{2+\eta,(2+\eta)/3}(\mathbb{R}^n\times\mathbb{R}^n)$ with, according to (3.5),

$$\begin{aligned}
\|a\|_{C^{\eta,\eta/3}} &\le C\int_{(t-\xi)^+}^{t}\frac{ds}{(t-s)^{\frac{\eta-\theta}{2}}}\sup_{0\le t\le T}\|g\|_{C^{\theta,\theta/3}}\\
&\le C\xi^{1-\frac{\eta-\theta}{2}}\sup_{0\le t\le T}\|g\|_{C^{\theta,\theta/3}},\\
\|b\|_{C^{2+\eta,(2+\eta)/3}} &\le C\int_0^{(t-\xi)^+}\frac{ds}{(t-s)^{1-\frac{\eta-\theta}{2}}}\sup_{0\le t\le T}\|g\|_{C^{\theta,\theta/3}}\\
&\le C\xi^{-\frac{\eta-\theta}{2}}\sup_{0\le t\le T}\|g\|_{C^{\theta,\theta/3}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
v(t,\cdot)\in&\left(C^{\eta,\eta/3}(\mathbb{R}^n\times\mathbb{R}^n),C^{2+\eta,(2+\eta)/3}(\mathbb{R}^n\times\mathbb{R}^n)\right)_{1-\frac{\eta-\theta}{2},\infty}\\
&=C^{2+\theta,(2+\theta)/3}(\mathbb{R}^n\times\mathbb{R}^n)
\end{aligned}$$

with

$$\begin{aligned}
\|v(t,\cdot)\|_{C^{2+\theta,(2+\theta)/3}} &\le C\sup_{0<\xi<1}\frac{\|a\|_{C^{\eta,\eta/3}}+\xi\|b\|_{C^{2+\eta,(2+\eta)/3}}}{\xi^{1-\frac{\eta-\theta}{2}}}\\
&\le C\sup_{0\le t\le T}\|g\|_{C^{\theta,\theta/3}}
\end{aligned}$$

and the constant C is independent of t. This inequality and (3.16) give (0.3).

Let w be a weak solution of (NHCP) on $[0,T]$ with $f=g=0$ such that $w,\Delta_x w\in C_b([0,T]\times\mathbb{R}^{2n})$. We want to prove that $w\equiv 0$. We briefly sketch the proof of this result for the reader's convenience and refer to Lorenzi [13] for more details. We first extend w to $\mathbb{R}\times\mathbb{R}^{2n}$ in a smooth way by setting $w(t,\cdot)=0$ for $t<0$, and $w(t,\cdot)=w(T,\cdot)$ for $t\ge T$. To apply the uniqueness assertion of step 3.4, we then regularize w by convolution by considering

$$w_\epsilon(t,z)=(w\star(\rho_\epsilon\phi_\epsilon))(t,z)=\int_{\mathbb{R}\times\mathbb{R}^{2n}}w(t-s,z-\xi)\rho_\epsilon(s)\phi_\epsilon(\xi)ds\,d\xi$$

with $\rho_\epsilon(s)=\epsilon^{-1}\rho(\epsilon^{-1}s)$, $\phi_\epsilon(\xi)=\epsilon^{-2n}\phi(\epsilon^{-2n}\xi)$, where ρ and ϕ are some nonnegative, smooth function with compact support in the unit ball of $\mathbb{R}$

and $\mathbb{R}^{2n}$ respectively and of L^1-norm 1. For $t_0 > 0$ given, the function $\chi_\epsilon := w_\epsilon(\cdot + t_0, \cdot)$ is a classical solution of

$$\begin{cases} \partial_t \chi_\epsilon(t,z) - \mathcal{K}\chi_\epsilon(t,z) = g_\epsilon(t+t_0,z) \text{ in } [0,T]\times\mathbb{R}^{2n}, \\ \chi_\epsilon(0,\cdot) = w_\epsilon(t_0,\cdot) \end{cases}$$

where $g_\epsilon := \partial_t w_\epsilon - \mathcal{K} w_\epsilon \in C^{1,2}([0,T],\mathbb{R}^{2n})$. In view of step 3.4,

$$w_\epsilon(t+t_0,z) = \chi_\epsilon(t,z) = (T(t)w_\epsilon(t_0,\cdot))(z) + \int_0^t (T(t_s)g_\epsilon(s+t_0,\cdot))(z)ds.$$

Assuming for the moment that $g_\epsilon \to 0$ in $C_{loc}((0,T)\times\mathbb{R}^{2n})$, we thus get by passing to the limit, using step 2.4, that

$$w(t+t_0,\cdot) = T(t)w(t_0,\cdot) \text{ in } [0,T-t_0]\times\mathbb{R}^{2n}.$$

Letting $t_0 \to 0$ gives $w \equiv 0$. It remains to prove that $g_\epsilon \to 0$ in $C_{loc}((0,T)\times\mathbb{R}^{2n})$. Since $\Delta_x w_\epsilon \to \Delta_x w$ in $C_{loc}((0,T)\times\mathbb{R}^{2n})$, it suffices to prove that $\partial_t w_\epsilon - \mathcal{F}w_\epsilon \to \partial_t w - \mathcal{F}w \in C_b([0,T]\times\mathbb{R}^{2n})$ in $C_{loc}((0,T)\times\mathbb{R}^{2n})$, $\mathcal{F}$ being defined in (3.11). We do this exactly in the same way as in the proof of step 3.3. We first prove that

$$\begin{aligned} &\{\partial_t w_\epsilon(t,z) - \mathcal{F}w_\epsilon(t,z)\} - \{(\partial_t w \star (\rho_\epsilon\phi_\epsilon))(t,z) - (\mathcal{F}w \star (\rho_\epsilon\phi_\epsilon))(t,z)\} \\ &= \int_{\mathbb{R}\times\mathbb{R}^{2n}} (F_i(z-\xi) - \mathcal{F}_i(z))w(t-s,z-\xi)\rho_\epsilon(s)\partial_i\phi_\epsilon(\xi)ds\,d\xi \\ &\quad - \int_{\mathbb{R}\times\mathbb{R}^{2n}} \xi_i w(t-s,z-\xi)\rho_\epsilon(s)\partial^i\phi_\epsilon(\xi)ds\,d\xi - ((u\partial_i F_i)\star(\rho_\epsilon\phi_\epsilon))(t,z) \end{aligned}$$

and then pass to the limit $\epsilon \to 0$. □

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