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Asymptotic estimates and blow-up theory for critical equations involving the p-Laplacian

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Abstract We prove the SH_1^p—theory for critical equations involving the p-Laplace operator on compact manifolds. We also prove pointwise estimates for these equations.

Let (M, g) be a smooth compact Riemannian n-manifold, and $p \in (1, n)$. We denote by $H_1^p(M)$ the standard Sobolev space of functions in L^p which are such that their gradient is also in L^p. We let $(h_\alpha)_\alpha$ be a sequence of $C^{0,\theta}$ functions on M, $0 < \theta < 1$, and consider equations like

$$(\Delta_p)_g u + h_\alpha u^{p-1} = u^{p^*-1} \tag{1}$$

where $(\Delta_p)_g u = -div_g(|\nabla u|_g^{p-2}\nabla u)$ is the p-Laplacian, $p^* = np/(n-p)$ is the critical Sobolev exponent for the embedding of the Sobolev space $H_1^p(M)$ into Lebesgue's spaces, and u is required to be positive. By standard regularity results, see Druet [8], Guedda-Véron [16] and Tolksdorf [24], $u \in C^{1,\theta}(M)$. We let $(u_\alpha)_\alpha$ be a bounded sequence in $H_1^p(M)$ of solutions of (1) in the sense that for any α,

$$(\Delta_p)_g u_\alpha + h_\alpha u_\alpha^{p-1} = u_\alpha^{p^*-1} \tag{2}$$

and $\|u_\alpha\|_{H_1^p} \le \Lambda$ where $\Lambda > 0$ is independent of α. We also assume that the h_α's converge in $C^{0,\theta}(M)$ to some limiting function h_∞ with the property that there exists $\lambda > 0$ such that for any $u \in H_1^p(M)$,

$$\int_M (|\nabla u|^p + h_\infty |u|^p)dv_g \ge \lambda \|u\|_{H_1^p}^p \tag{3}$$

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We concentrate in this paper on the study of the asymptotics of the u_α's. Such asymptotics have been intensively studied in the special case where $p = 2$. Brézis-Coron [2], Lions [20], Sacks-Uhlenbeck [21], and especially Struwe [23] for the kind of equations we are concerned with developed the H_1^2-theory in the Euclidean case. The result extends to the Riemannian case, and one gets that, up to a subsequence, the u_α's express as a solution of the limit equation, plus a finite sum of bubbles we get by rescaling fundamental solutions of the critical Euclidean equation $\Delta u = u^{2^*-1}$, plus a rest which converges to 0 in H_1^2 as $\alpha \to +\infty$. We extend this result of Struwe [23] to the more general equation (1) in the context of Riemannian manifolds, and thus develop the H_1^p-theory for such equations in the Riemannian context. One of the interesting and promising aspects in the study of the p-Laplacian is that the 1-Laplacian, involved in isoperimetric problems, can be seen as the limit as $p \to 1$ of the p-Laplacian. Such a property was used by Druet [9], [10] when proving that sharp local isoperimetric inequalities are controled by the scalar curvature, and when proving that the local version of the Cartan-Hadamard conjecture is true. The H_1^p-theory in the Euclidean case was developped by Alves [1]. However, the restrictive condition that $2 \le p < n$ was required in [1]. Our result, Theorem 0.1 below, holds for all p. The complete C^0-theory for the asymptotics of the u_α's was developed by Druet-Hebey-Robert [13] in the special case $p = 2$. The C^0-estimate we prove in this article for the general equation (1) goes back to Schoen-Zhang [22] and Druet [11] where it was proved in specific situations (in particular when the energy is minimal). Such an estimate has interesting applications. Among other possible references we refer to Druet [11]. See also Druet-Hebey [12]. In what follows, we let i_g be the injectivity radius of (M, g). Given $\delta \in (0, i_g/2)$, we let η_δ be a smooth cut-off function in $\mathbb{R}^n$ such that $\eta_\delta = 1$ in $B_0(\delta)$ and $\eta_\delta = 0$ in $\mathbb{R}^n \backslash B_0(2\delta)$. For $x \in M$, we let $\eta_{\delta,x}$ be the smooth cut-off function in M given by

$$\eta_{\delta,x}(y) = \eta_\delta\big(\exp_x^{-1}(y)\big)$$

where $\exp_x$ is the exponential map at x. As a remark, we regard $\exp_x$ as defined in $\mathbb{R}^n$. An intrinsic definition is possible if M is parallelizable. If not we let Ω_i and $\tilde{\Omega}_i$, $i = 1, \dots, N$, be open subsets of M such that for any i, $\tilde{\Omega}_i$ is parallelizable and $\overline{\Omega_i} \subset \tilde{\Omega}_i$, and such that $M = \cup \Omega_i$. The canonical exponential map gives N maps $\exp_x$ defined in $\Omega_i \times \mathbb{R}^n$, and $\exp_x$ is, depending on the situation, one of these maps. A property of $\exp_x$ that holds for any $x \in M$ should then be regarded as a property that holds for any i and any $x \in \overline{\Omega_i}$. We let u be a nonnegative nontrivial solution in $D_1^p(\mathbb{R}^n)$ of the Euclidean equation $\Delta_p u = u^{p^*-1}$, where $D_1^p(\mathbb{R}^n)$ is defined as the completion of the space of smooth functions with compact support with respect to the norm $\|u\| = \|\nabla u\|_p$. Given a converging sequence $(x_\alpha)_\alpha$ of points in M, and a sequence $(R_\alpha)_\alpha$ of positive real numbers, with the property that $R_\alpha \to +\infty$ as $\alpha \to +\infty$, we define a bubble as a sequence $(B_\alpha)_\alpha$ of functions in M defined by the equation

$$B_\alpha(x) = \eta_{\delta,x_\alpha}(x) R_\alpha^{\frac{n-p}{p}} u\big(R_\alpha \exp_{x_\alpha}^{-1}(x)\big) \tag{4}$$

We refer to the x_α's as the centers of (B_α), and to the R_α's, or R_α^{-1}'s, as the weights of $(B_\alpha)_\alpha$. We let also

$$(\Delta_p)_g u + h_\infty u^{p-1} = u^{p^*-1} \tag{5}$$

be the limit equation we get when letting $\alpha \to +\infty$ in (1). Our result states as follows:

Theorem 0.1 *Let (M, g) be a smooth compact Riemannian n-manifold, $p \in (1, n)$, and $(h_\alpha)_\alpha$ be a sequence of $C^{0,\theta}$ functions on M, $0 < \theta < 1$, which converge in $C^{0,\theta}(M)$ to some limiting function h_∞ for which (3) is true. Let $(u_\alpha)_\alpha$ be a bounded sequence in $H_1^p(M)$ of positive solutions of (1). Then there exists a nonnegative solution $u^0 \in H_1^p(M)$ of the limit equation (5), there exists $k \in \mathbb{N}$, and there exist k bubbles $(B_\alpha^i)_\alpha$, $i = 1 \dots k$, such that, up to a subsequence,*

$$u_\alpha = u^0 + \sum_{i=1}^{k} B_\alpha^i + S_\alpha$$

where $(S_\alpha)_\alpha$ is a sequence of functions in $H_1^p(M)$ such that $S_\alpha \to 0$ in $H_1^p(M)$ as $\alpha \to +\infty$. Moreover there exists a constant $C > 0$ independant of α and $x \in M$ such that for any α and any $x \in M$,

$$\left(\min_{i=1,\dots,k} d_g\left(x_\alpha^i, x\right)\right)^{\frac{n-p}{p}} \left|u_\alpha(x) - u^0(x)\right| \le C$$

where d_g is the distance with respect to the metric g, and the x_α^i's are the centers of the bubbles $(B_\alpha^i)_\alpha$.

An additional information we have on the H_1^p-decomposition in this theorem is that the energies associated to the different terms in this decomposition split. See equation (8) below for more details. Another additional information we have, concerning this time the pointwise estimate, is that

$$\lim_{R \to +\infty} \lim_{\alpha \to +\infty} \sup_{x \in M \backslash \Omega_\alpha(R)} R_\alpha^k(x)^{\frac{n-p}{p}} \left|u_\alpha(x) - u^0(x)\right| = 0$$

where, $R_\alpha^k(x) = \min_{i=1,\dots,k} d_g(x_\alpha^i, x)$, and, for $R > 0$, $\Omega_\alpha(R) = \bigcup_{i=1}^k B_{x_\alpha^i}(R\mu_\alpha^i)$. Positive solutions of the Euclidean equation $\Delta_p u = u^{p^*-1}$ have been classified by Ghoussoub and Yuan [15] (see also Damascelli - Pacella [5] and Damascelli - Pacella - Ramaswamy [6]) in the special case of positive radial solutions. We have here that a positive radial solutions u of $\Delta_p u = u^{p^*-1}$ is of the form

$$u(x) = \left(an\left(\frac{n-p}{p-1}\right)^{p-1}\right)^{\frac{n-p}{p^2}} \left(a + |x - x_0|^{\frac{p}{p-1}}\right)^{1-\frac{n}{p}}$$

for some $a > 0$ and $x_0 \in \mathbb{R}^n$.
We prove the H_1^p-decomposition of Theorem 0.1 in Sects. 1–3, and the C^0-estimate of Theorem 0.1 in Sect. 4.

1 The H_1^p-decomposition

We prove the first part of the theorem, and deal with the more general notion of Palais-Smale sequences (a perturbative extension of the notion of strong solutions). We do not assume anything about the sign of the u_α's in this section. We fix $1 < p < n$ and consider the functional I_g^α defined on $H_1^p(M)$ by

$$I_g^\alpha(u) = \frac{1}{p}\int_M |\nabla u|^p dv_g + \frac{1}{p}\int_M h_\alpha(x)|u|^p dv_g - \frac{1}{p^*}\int_M |u|^{p^*} dv_g$$

We recall that a sequence $(u_\alpha)_\alpha \subset H_1^p(M)$ is said to be a Palais - Smale (P-S) sequence for I_g^α if the following holds:

1. $I_g^\alpha(u_\alpha)$ is bounded w.r.t. α , and
2. $DI_g(u_\alpha) \to 0$ strongly in $H_1^p(M)'$ as $\alpha \to +\infty$.

Given a P-S sequence $(u_\alpha)_\alpha$ for I_g^α, we claim here that there exist $k \in \mathbb{N}$, sequences $(R_\alpha^i)_\alpha$ of positive real numbers with $R_\alpha^i \to +\infty$ as $\alpha \to +\infty$, converging sequences $(x_\alpha^i)_\alpha$ of points in M, $i = 1 \dots k$, a solution $u^0 \in H_1^p(M)$ of the limit equation

$$\Delta_p u + h_\infty |u|^{p-2}u = |u|^{p^*-2}u \ , \tag{6}$$

and k nontrivial solutions $u^i \in D_1^p(\mathbb{R}^n)$ of the Euclidean equation $\Delta_p u = |u|^{p^*-2}u$, $i = 1 \dots k$, such that up to a subsequence, the following equations hold. Namely that

$$u_\alpha = u^0 + \sum_{i=1}^k \eta_\alpha^i u_\alpha^i + o(1) \ , \text{ and} \tag{7}$$

$$I_g^\alpha(u_\alpha) = I_g^\infty(u^0) + \sum_{i=1}^k E(u^i) + o(1) \tag{8}$$

where

$$u_\alpha^i(x) = \left(R_\alpha^i\right)^{\frac{n}{p}-1} u^j\left(R_\alpha^i \exp_{x_\alpha^j}^{-1}(x)\right) ,$$

$\eta_\alpha^i = \eta_{\delta, x_\alpha^i}$, $\|o(1)\|_{H_1^p} \to 0$ as $\alpha \to +\infty$,

$$I_g^\infty(u) = \frac{1}{p}\int_M |\nabla u|^p dv_g + \frac{1}{p}\int_M h_\infty(x)|u|^p dv_g - \frac{1}{p^*}\int_M |u|^{p^*} dv_g \ ,$$

h_∞ is the limit in $C^{0,\theta}(M)$ of the h_α's, and

$$E(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p dx - \frac{1}{p^*}\int_{\mathbb{R}^n} |u|^{p^*} dx$$

It is easily checked that (7), (8) imply the first part of Theorem 0.1 if we prove in addition (see Sect. 3 below) that u^0 and the u^i's have to be nonnegative when the u_α's are nonnegative. Note here that, obviously, a bounded sequence in $H_1^p(M)$ of solutions of (1) is a P-S sequence for I_g^α. We divide the proof of (7) and (8) into several steps. Step 1.1 is very classical. We state it with no proof.

Step 1.1 *Palais-Smale sequences for I_g^α are bounded in $H_1^p(M)$.*

Step 1.1 easily follows from the definition of a P-S sequence. There is no change in the proof when passing from the $p = 2$ case (as detailed for instance in Druet-Hebey-Robert [13] or Struwe [23]) and the cases $p \neq 2$. See also Brézis-Nirenberg [4]. Step 1.2 in the proof of (7) and (8) is as follows.

Step 1.2 *Let $(u_\alpha)_\alpha$ be a P-S sequence for I_g^α such that $u_\alpha \rightharpoonup u^0$ in $H_1^p(M)$. Then u^0 is a solution of the limit equation (6).*

Proof of step 1.2. Step 1.2 is straightforward when $p = 2$, and a little bit more tricky when $p \neq 2$. Thanks to step 1.1, the u_α's are bounded in $H_1^p(M)$. Hence, by the definition of a P-S sequence,

$$\int_M |\nabla u_\alpha|_g^{p-2}\nabla u_\alpha \nabla\phi dv_g + \int_M h_\alpha |u_\alpha|^{p-2}u_\alpha \phi dv_g - \int_M |u_\alpha|^{p^*-2}u_\alpha\phi dv_g = o(1) \tag{9}$$

for all smooth functions ϕ on M. Without loss of generality, up to a subsequence, we can assume that $u_\alpha \to u^0$ almost everywhere and in L^p. By standard integration theory, we easily pass to the limit in the second and third terms in the left hand side of the equation. Then, we need to prove that

$$\int_M |\nabla u_\alpha|^{p-2}\nabla u_\alpha\nabla\phi dv_g \to \int_M |\nabla u^0|^{p-2}\nabla u^0\nabla\phi dv_g \tag{10}$$

as $\alpha \to +\infty$. We borrow ideas from Evans [14] and Demengel-Hebey [7]. We denote by Σ_α and Θ the vector fields $|\nabla u_\alpha|^{p-2}\nabla u_\alpha$ and $|\nabla u^0|^{p-2}\nabla u^0$. Then $(\Sigma_\alpha)_\alpha$ is bounded in $L^{\frac{p}{p-1}}(M)$ and we can thus assume that $(\Sigma_\alpha)_\alpha$ converges weakly in $L^{\frac{p}{p-1}}(M)$ to some vector field $\Sigma \in L^{\frac{p}{p-1}}(M)$. Let $\delta > 0$ be given. By Egoroff's theorem, there exists $E_\delta \subset M$ such that

$$\int_{M\setminus E_\delta} dv_g < \delta$$

and $(u_\alpha)_\alpha$ converges uniformly to u^0 in E_δ. As a consequence, for a given $\epsilon > 0$, we can take α sufficiently large to get that $|u_\alpha(x) - u^0(x)| < \epsilon/2$ for all $x \in E_\delta$. We define a truncation function β_ϵ by

$$\beta_\epsilon(x) = \begin{cases} x & \text{if } |x| < \epsilon \\ \frac{\epsilon x}{|x|} & \text{if } |x| \geq \epsilon \end{cases}$$

It is easily checked that

$$(\Sigma_\alpha - \Theta).\nabla(\beta_\epsilon \circ (u_\alpha - u^0)) \geq 0$$

almost everywhere in M. Indeed, since $p > 1$, the function $\phi : X \in \mathbb{R}^n \to |X|^p$ is convexe and thus ϕ' is nondecreasing in the sense that, for any $X, Y \in \mathbb{R}^n$, the equation $(|X|^{p-2}X - |Y|^{p-2}Y; X - Y) \geq 0$ holds true. Applying this equation to ∇u_α and ∇u^0, we get that $(\Sigma_\alpha - \Theta).\nabla(u_\alpha - u^0) \geq 0$. Thus, for α sufficiently large, we have that

$$\int_{E_\delta} (\Sigma_\alpha - \Theta).\nabla(u_\alpha - u^0) dv_g \leq \int_M (\Sigma_\alpha - \Theta).\nabla(\beta_\epsilon \circ (u_\alpha - u^0)) dv_g$$

Now we note that $\beta_\epsilon \circ (u_\alpha - u^0)$ converges weakly to 0 in $H_1^p(M)$ so that

$$\int_M \Theta.\nabla(\beta_\epsilon \circ (u_\alpha - u^0)) dv_g \to 0$$

We also have that for α sufficiently large,

$$\int_M \Sigma_\alpha.\nabla(\beta_\epsilon \circ (u_\alpha - u^0)) dv_g < \epsilon$$

Indeed, since $(\beta_\epsilon \circ (u_\alpha - u^0))$ is bounded in $H_1^p(M)$,

$$DI_g^\alpha(u_\alpha)(\beta_\epsilon \circ (u_\alpha - u^0)) = o(1)$$

so that

$$\int_M \Sigma_\alpha \nabla(\beta_\epsilon \circ (u_\alpha - u^0)) dv_g = o(1) + I_1 + I_2$$

where

$$\begin{aligned} |I_1| &= \left| \int_M |u_\alpha|^{p^*-2} u_\alpha (\beta_\epsilon \circ (u_\alpha - u^0)) dv_g \right| \\ &\leq \epsilon \int_M |u_\alpha|^{p^*-1} dv_g \leq C\epsilon \end{aligned}$$

and, for α sufficiently large,

$$\begin{aligned} |I_2| &= \left| \int_M h_\alpha |u_\alpha|^{p-2} u_\alpha (\beta_\epsilon \circ (u_\alpha - u^0)) dv_g \right| \\ &\leq \epsilon(\|h_\infty\|_\infty + 1) \int_M |u_\alpha|^{p-1} dv_g \leq C\epsilon \end{aligned}$$

As a consequence, we get that

$$\limsup_{\alpha \to +\infty} \int_{E_\delta} (\Sigma_\alpha - \Theta).\nabla(u_\alpha - u^0) dv_g \leq C\epsilon$$

Since $\epsilon > 0$ is arbitrary, it follows that $(\Sigma_\alpha - \Theta).\nabla(u_\alpha - u^0)$ converges to 0 in $L^1(E_\delta)$ and thus, up to a subsequence, also a.e in E_δ. We now use the fact that if a sequence $(X_\alpha)_\alpha \subset \mathbb{R}^n$ is such that

$$(|X_\alpha|^{p-2}X_\alpha - |X|^{p-2}X).(X_\alpha - X) \to 0$$

then $X_\alpha \to X$ to obtain that $\nabla u_\alpha \to \nabla u^0$ a.e in E_δ. Since $\delta > 0$ is arbitrary, this implies that ∇u_α converges to ∇u^0 a.e in M and, thus, $|\nabla u_\alpha|^{p-2}\nabla u_\alpha \to \Theta$ a.e in

M. Since $(|\nabla u_\alpha|^{p-2}\nabla u_\alpha)_\alpha$ is bounded in $L^{\frac{p}{p-1}}(M)$, we get that $(|\nabla u_\alpha|^{p-2}\nabla u_\alpha)_\alpha$ converges weakly to Θ in $L^{\frac{p}{p-1}}(M)$ and thus that $\Sigma = \Theta$. This proves (10). Returning to (9), letting $\alpha \to +\infty$, we then get that

$$\int_M |\nabla u^0|_g^{p-2}\nabla u^0\nabla\phi dv_g + \int_M h_\infty|u^0|^{p-2}u^0\phi dv_g - \int_M |u^0|^{p^*-2}u^0\phi dv_g$$

for all smooth functions ϕ on M. In particular, u^0 is a solution of the limit equation (6). This proves step 1.2.

From now on, we let I_g be the functional defined for $u \in H_1^p(M)$ by

$$I_g(u) = \frac{1}{p}\int_M |\nabla u|^p dv_g - \frac{1}{p^*}\int_M |u|^{p^*}dv_g .$$

Given a normed vector space $(E, \|\cdot\|)$, and $p > 1$, we recall that for $\theta > 0$ sufficiently small, depending only on p,

$$\begin{aligned}&\big\|\|x+y\|^{p-2}(x+y) - \|x\|^{p-2}x - \|y\|^{p-2}y\big\|\\ &\quad\le C(\|x\|^{p-1-\theta}\|y\|^\theta + \|x\|^\theta\|y\|^{p-1-\theta})\end{aligned} \tag{11}$$

and

$$\Big\| \|x+y\|^p - \|x\|^p - \|y\|^p \Big\| \le C\left(\|x\|^{p-\theta}\|y\|^\theta + \|y\|^{p-\theta}\|x\|^\theta\right) \tag{12}$$

for all x and y in E, where $C > 0$ is independent of x and y. Now step 1.3 in the proof of (7) and (8) is as follows.

Step 1.3 *Let $(u_\alpha)_\alpha$ be a P-S sequence for I_g^α such that $u_\alpha \rightharpoonup u^0$ in $H_1^p(M)$, and $v_\alpha = u_\alpha - u^0$. Then*

$$I_g(v_\alpha) = I_g^\alpha(u_\alpha) - I_g^\infty(u^0) + 0(I)$$

and (v_α) is a P-S sequence for I_g.

Proof of step 1.3. We write that

$$\begin{aligned} I_g^\alpha(u_\alpha) = {} & I_g^\alpha(u^0) + I_g(v_\alpha)\\ & + \frac{1}{p}\int_M (|\nabla(v_\alpha + u^0)|^p - |\nabla u^0|^p - |\nabla v_\alpha|^p)dv_g\\ & + \frac{1}{p}\int_M h_\alpha(|v_\alpha + u^0|^p - |u^0|^p)dv_g\\ & - \frac{1}{p^*}\int_M (|v_\alpha + u^0|^{p^*} - |v_\alpha|^{p^*} - |u^0|^{p^*})dv_g \end{aligned} \tag{13}$$

Since the embedding $H_1^p(M) \hookrightarrow L^p(M)$ is compact, we can assume that, up to a subsequence, $u_\alpha \to u^0$ in $L^p(M)$. In particular, the second integral in (13) goes to 0 as $\alpha \to +\infty$. Moreover, we can prove as in Step 1.2 that $|\nabla u_\alpha| \to |\nabla u^0|$ a.e.

Theorem 1 of [3] implies then that the first and third integrals in (13) also go to 0. We thus obtain that

$$I_g^\alpha(u_\alpha) = I_g^\alpha(u^0) + I_g(v_\alpha) + o(1)$$

Noting that $I_g^\infty(u^0) = I_g^\alpha(u^0) + o(1)$, we actually get that

$$I_g(v_\alpha) = I_g^\alpha(u_\alpha) - I_g^\infty(u^0) + o(1)$$

Now we prove that $(v_\alpha)_\alpha$ is a P-S sequence for I_g. First we write that

$$\begin{aligned} I_g(v_\alpha) &= I_g^\alpha(u_\alpha) - I_g^\infty(u^0) + o(1) \\ &= O(1) + o(1) \end{aligned}$$

so that $(I_g^\infty(v_\alpha))_\alpha$ is bounded. Then it remains to prove that $DI_g(v_\alpha) \to 0$ in $H_1^p(M)'$, namely that $DI_g(v_\alpha).\phi = o(1)\|\phi\|_{H_1^p(M)}$ for all $\phi \in H_1^p(M)$. For a given $\phi \in C^\infty(M)$,

$$\begin{aligned} & DI_g^\alpha(u_\alpha).\phi - DI_g(v_\alpha).\phi \\ &= -\int_M \Phi_\alpha \phi dv_g + \int_M |\nabla u_\alpha|^{p-2}\nabla u_\alpha \nabla\phi dv_g \\ &\quad - \int_M |\nabla v_\alpha|^{p-2}\nabla v_\alpha \nabla\phi dv_g + \int_M h_\alpha |u_\alpha|^{p-2}u_\alpha \phi dv_g \\ &\quad - \int_M |u^0|^{p^*-2}u^0\phi dv_g \end{aligned} \tag{14}$$

where

$$\Phi_\alpha = |v_\alpha + u^0|^{p^*-2}(v_\alpha + u^0) - |v_\alpha|^{p^*-2}v_\alpha - |u^0|^{p^*-2}u^0$$

We let $C > 0$, given by (11), be such that for any α

$$\begin{aligned} & ||v_\alpha + u^0|^{p^*-2}(v_\alpha + u^0) - |v_\alpha|^{p^*-2}v_\alpha - |u^0|^{p^*-2}u^0| \\ & \leq C(|v_\alpha|^{p^*-1-\theta}|u^0|^\theta + |v_\alpha|^\theta |u^0|^{p^*-1-\theta}) \end{aligned}$$

Then, using Hölder's inequality and convexity, we get that

$$\left|\int_M \Phi_\alpha \phi dv_g\right| \leq C\|\phi\|_{p^*}\left(\||v_\alpha|^{p^*-1-\theta}|u^0|^\theta\|_{\frac{p^*}{p^*-1}} + \||u^0|^{p^*-1-\theta}|v_\alpha|^\theta\|_{\frac{p^*}{p^*-1}}\right)$$

By standard integration theory,

$$\begin{aligned} \int_M \Phi_\alpha \phi dv_g &= o(1)\|\phi\|_{p^*} \\ &= o(1)\|\phi\|_{H_1^p(M)} \end{aligned}$$

Since u^0 is a weak solution of the limit equation, it remains to prove that

$$\begin{aligned} o(1)\|\phi\|_{H_1^p} = & \int_M |\nabla u_\alpha|^{p-2}\nabla u_\alpha \nabla\phi dv_g - \int_M |\nabla v_\alpha|^{p-2}\nabla v_\alpha\nabla\phi dv_g \\ & - \int_M |\nabla u^0|^{p-2}\nabla u^0\nabla\phi dv_g \\ & + \int_M h_\alpha|u_\alpha|^{p-2}u_\alpha\phi dv_g - \int_M h_\infty|u^0|^{p-2}u^0\phi dv_g \end{aligned}$$

and thus, by Hölder's inequality, that

$$\int_M ||\nabla u_\alpha|^{p-2}\nabla u_\alpha - |\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u^0|^{p-2}\nabla u^0|^{\frac{p}{p-1}} dv_g = o(1) \tag{15}$$

and

$$\int_M |h_\alpha|u_\alpha|^{p-2}u_\alpha - h_\infty|u^0|^{p-2}u^0|^{\frac{p}{p-1}} dv_g = o(1) \tag{16}$$

The proof of (15) uses (11) as above, whereas (16) is a consequence of [3] (since $u_\alpha \to u^0$ a.e. and $(|u_\alpha|^{p-2}u_\alpha)_\alpha$ is bounded in $L^{\frac{p}{p-1}}(M)$). This proves Step 1.3.

In what follows, we let $K(n, p)$ be the sharp constant K in the Euclidean Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{1/p^*} \leq K\left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{1/p} .$$

The value of $K(n, p)$ is well known and can be found, for instance, in Hebey [17]. We let also β^* be given by

$$\beta^* = \frac{1}{n}K(n, p)^{-n} \tag{17}$$

Step 1.4 in the proof of (7) and (8) is as follows.

Step 1.4 *Let $(v_\alpha)_\alpha$ be a P-S sequence for I_g such that $v_\alpha \rightharpoonup 0$ in $H_1^p(M)$ and $I_g(v_\alpha) \to \beta$ as $\alpha \to +\infty$. If $\beta < \beta^*$ then $\beta = 0$ and $v_\alpha \to 0$ in $H_1^p(M)$ as $\alpha \to +\infty$.*

Proof of step 1.4. We write that

$$\begin{aligned} o(1) &= DI_g(v_\alpha).v_\alpha \\ &= \int_M |\nabla v_\alpha|^p dv_g - \int_M |v_\alpha|^{p^*} dv_g \end{aligned}$$

and $I_g(v_\alpha) = \beta + o(1)$. Combining these two equations with the definition of I_g we get that

$$\int_M |\nabla v_\alpha|^p dv_g = n\beta + o(1)$$

and

$$\int_M |v_\alpha|^{p^*} dv_g = n\beta + o(1)$$

In particular $\beta \geq 0$. Since the embedding $H_1^p(M) \hookrightarrow L^p(M)$ is compact, we can assume that $v_\alpha \to 0$ in $L^p(M)$. For $\epsilon > 0$, there exists a constant B_ϵ such that for any α,

$$\|v_\alpha\|_{p^*}^p \leq (K(n,p)^p + \epsilon)\|\nabla v_\alpha\|_p^p + B_\epsilon \|v_\alpha\|_p^p$$

(see, for instance, Hebey [17]). Passing to the limit in this equation, we get that

$$(n\beta)^{\frac{p}{p^*}} \leq (K(n,p)^p + \epsilon) n\beta$$

and since $\epsilon > 0$ is arbitrary, we obtain that

$$(n\beta)^{\frac{p}{p^*}} \leq K(n,p)^p n\beta$$

If we assume that β is positive, then

$$(n\beta)^{\frac{p}{p^*}-1} = (n\beta)^{-\frac{p}{n}} \leq K(n,p)^p$$

and we get that

$$K(n,p)^p = (n\beta^*)^{-\frac{p}{n}} < (n\beta)^{-\frac{p}{n}} \leq K(n,p)^p$$

which is absurd. Hence, $\beta = 0$ and

$$\int_M |\nabla v_\alpha|^p dv_g = o(1)$$

Since $v_\alpha \to 0$ in $L^p(M)$, this proves that $v_\alpha \to 0$ in $H_1^p(M)$. Step 1.4 is proved.

Another step we need in the proof of (7) and (8) is as follows. We let $D_1^p(\mathbb{R}^n)$ be the Beppo-Levi space defined above. Namely the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\| = \|\nabla u\|_p$.

Step 1.5 *Let $u \in D_1^p(\mathbb{R}^n)$ be a nontrivial solution of the critical Euclidean equation $\Delta_p u = |u|^{p^*-2}u$. Then $E(u) \geq \beta^*$.*

Proof of step 1.5. We let $(u_n)_n$ be a sequence of smooth functions with compact support such that $\|u_n - u\| \to 0$ as $n \to +\infty$. Then

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \nabla u_n dx = \int_{\mathbb{R}^n} |u|^{p^*-2} u u_n dx \tag{18}$$

and

$$\left| \int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \nabla u_n dx - \int_{\mathbb{R}^n} |\nabla u|^p dx \right| \leq \int_{\mathbb{R}^n} |\nabla u|^{p-1} |\nabla(u_n - u)| dx$$
$$\leq \|u_n - u\|.\|u\|^{p-1}$$

which goes to 0 as $n \to +\infty$. Similarly, by the Sobolev theorem for the embedding $D_1^p(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} |u|^{p^*-2} u u_n dx - \int_{\mathbb{R}^n} |u|^{p^*} dx \right| \le \int_{\mathbb{R}^n} |u|^{p^*-1} |u - u_n| dx$$
$$\le \|u - u_n\|_{p^*} \|u\|_{p^*}^{p^*-1}$$

which also goes to 0 as $n \to +\infty$. Passing to the limit in (18), we then obtain that

$$\int_{\mathbb{R}^n} |\nabla u|^p dx = \int_{\mathbb{R}^n} |u|^{p^*} dx$$

The sharp Euclidean Sobolev inequality gives that

$$\int_{\mathbb{R}^n} |\nabla u|^p dx = \int_{\mathbb{R}^n} |u|^{p^*} dx$$
$$\le K(n, p)^{p^*} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx \right)^{\frac{p^*}{p}}$$

and thus that

$$\int_{\mathbb{R}^n} |\nabla u|^p dx \ge K(n, p)^{-n}$$

It follows that

$$E(u) = \left(\frac{1}{p} - \frac{1}{p^*} \right) \int_{\mathbb{R}^n} |\nabla u|^p dx$$
$$= \frac{1}{n} \int_{\mathbb{R}^n} |\nabla u|^p dx$$
$$\ge \frac{1}{n} K(n, p)^{-n} = \beta^*$$

and this proves Step 1.5.

In addition to steps 1.1–1.5, we need the following lemma in the proof of (7) and (8). Given a converging sequence $(x_\alpha)_\alpha$ of points in M, and a sequence $(R_\alpha)_\alpha$ of positive real numbers, with the property that $R_\alpha \to +\infty$ as $\alpha \to +\infty$, we define a generalized bubble as a sequence $(\hat{B}_\alpha)_\alpha$ of functions in M defined by the equations

$$\hat{B}_\alpha(x) = \eta_{\delta, x_\alpha}(x) R_\alpha^{\frac{n-p}{p}} v\big(R_\alpha \exp_{x_\alpha}^{-1}(x)\big) \qquad (19)$$

where v is a solution of the critical Euclidean equation $\Delta_p u = |u|^{p^*-2} u$.

Lemma 1.1 *Let $(v_\alpha)_\alpha$ be a P-S sequence for I_g such that $v_\alpha \rightharpoonup 0$ in $H_1^p(M)$ but not strongly. Then there exists a generalized bubble $(\hat{B}_\alpha)_\alpha$ such that, up to a subsequence, $(w_\alpha)_\alpha$ where $w_\alpha = v_\alpha - \hat{B}_\alpha$ is a P-S sequence for I_g, $w_\alpha \rightharpoonup 0$ in $H_1^p(M)$, and $I_g(w_\alpha) = I_g(v_\alpha) - E(v) + o(1)$ where $v \in D_1^p(\mathbb{R}^n)$ is the function from which the $\hat{B}_\alpha$'s are defined as in (19).*

Lemma 1.1 is the main step in the proof of (7) and (8). We postpone its proof to the next section, and show now how we get (7) and (8) from Lemma 1.1 and Steps 1.1–1.5.

Proof of (7) and (8). We let $(u_\alpha)_\alpha$ be a P-S sequence for I_g^α. By Step 1.1, $(u_\alpha)_\alpha$ is bounded in $H_1^p(M)$. We may then assume that there exists $u^0 \in H_1^p(M)$ and $c \in \mathbb{R}$ such that, up to a subsequence, $u_\alpha \to u^0$ weakly in $H_1^p(M)$, strongly in $L^p(M)$, and a.e, and such that $I_g^\alpha(u_\alpha) \to c$. By Step 1.2, u^0 is a solution of the limit equation and by Step 1.3, $v_\alpha = u_\alpha - u^0$ is a P-S sequence for I_g satisfying that

$$\begin{aligned} I_g(v_\alpha) &= I_g^\alpha(u_\alpha) - I_g^\infty(u^0) + o(1) \\ &= c - I_g^\infty(u^0) + o(1) \end{aligned}$$

If $c - I_g^\infty(u^0) < \beta^*$ then, according to Step 1.4, $(v_\alpha)_\alpha$ converges strongly to 0 in $H_1^p(M)$ and we get that (7) and (8) hold with $k = 0$. If not, applying Lemma 1.1, we get a new P-S sequence $(v_\alpha^1)_\alpha$ for I_g converging weakly to 0 in $H_1^p(M)$ and such that

$$I_g\big(v_\alpha^1\big) = I_g(v_\alpha) - E(v) + o(1)$$

where $v \in D_1^p(\mathbb{R}^n)$ is a solution of the critical Euclidean equation $\Delta_p u = |u|^{p^*-2}u$. In view of Step 1.5, $E(v) \geq \beta^*$ and thus

$$I_g\big(v_\alpha^1\big) \leq I_g(v_\alpha) - \beta^* + o(1)$$

If $c - I_g^\infty(u^0) < 2\beta^*$, we may again apply Step 1.4 to get that $(v_\alpha^1)_\alpha$ converges strongly to 0 in $H_1^p(M)$. In particular, (7) and (8) hold with $k = 1$. If not, namely if $c - I_g^\infty(u^0) \geq 2\beta^*$, we apply once again Lemma 1.1 and get a new P-S sequence $(v_\alpha^2)_\alpha$ for I_g. Then either $c - I_g^\infty(u^0) < 3\beta^*$, or $c - I_g^\infty(u^0) \geq 3\beta^*$. Going on with such a process, we clearly get by finite induction that (7) and (8) hold with some $k \geq 1$.

2 Proof of Lemma 1.1

We prove Lemma 1.1 in this section. Up to a subsequence, we may assume that $I_g(v_\alpha) \to \beta$ as $\alpha \to +\infty$. We may also assume that the v_α's are smooth since, if not, using the density of $C^\infty(M)$ in $H_1^p(M)$, there always exists $\overline{v}_\alpha$ smooth and such that $\|v_\alpha - \overline{v}_\alpha\|_{H_1^p} \to 0$ as $\alpha \to +\infty$. Then $I_g(v_\alpha) = I_g(\overline{v}_\alpha) + o(1)$ and $DI_g(v_\alpha)\phi = DI_g(\overline{v}_\alpha)\phi + o(1)\|\phi\|_{H_1^p}$ for any $\phi \in H_1^p(M)$ anf thus $(\overline{v}_\alpha)_\alpha$ is a P-S sequence for I_g. Moreover $\overline{v}_\alpha \rightharpoonup 0$ in $H_1^p(M)$ but not strongly and if the conclusion of Lemma 1.1 holds for $(\overline{v}_\alpha)_\alpha$, i.e. if there exists a generalized bubble $(\hat{B}_\alpha)$ built from $v \in D_1^p(\mathbb{R}^n)$ as in (19) such that, up to a subsequence, $\bar{w}_\alpha = \bar{v}_\alpha - \hat{B}_\alpha$ is a P-S sequence for I_g with $I_g(\bar{w}_\alpha) = I_g(\bar{v}_\alpha) - E(v) + o(1)$, then it holds also for $(v_\alpha)_\alpha$ since then $w_\alpha := v_\alpha - \hat{B}_\alpha$ satisfies $I_g(w_\alpha) = I_g(\bar{w}_\alpha) + o(1)$ and $DI_g(w_\alpha)\phi = DI_g(\bar{w}_\alpha)\phi + o(1)\|\phi\|_{H_1^p}$ for any $\phi \in H_1^p(M)$. Since $DI_g(v_\alpha) \to 0$,

$$\int_M |\nabla v_\alpha|^p dv_g = n\beta + o(1) \tag{20}$$

while, by Step 1.4 of Sect. 1, $\beta \geq \beta^*$. For $t > 0$, we let

$$\mu_\alpha(t) = \max_{x\in M} \int_{B_x(t)} |\nabla v_\alpha|^p dv_g$$

where $B_x(t)$ is the geodesic ball of center x and radius t. Given $t_0 > 0$ small, it follows from (20) that there exist $x_0 \in M$ and $\lambda_0 > 0$ such that, up to a subsequence,

$$\int_{B_{x_0}(t_0)} |\nabla v_\alpha|^p dv_g \geq \lambda_0$$

for all α. Then, since $t \to \mu_\alpha(t)$ is continuous, we get that for any $\lambda \in (0, \lambda_0)$, there exists $t_\alpha \in (0, t_0)$ such that $\mu_\alpha(t_\alpha) = \lambda$. Clearly, there also exists $x_\alpha \in M$ such that

$$\mu_\alpha(t_\alpha) = \int_{B_{x_\alpha}(t_\alpha)} |\nabla v_\alpha|^p dv_g \ .$$

Up to a subsequence, $(x_\alpha)_\alpha$ converges. We let $r_0 \in (0, i_g/2)$ be such that for all $x \in M$ and all $y, z \in \mathbb{R}^n$, if $|y| \leq r_0$ and $|z| \leq r_0$, then

$$d_g\left(\exp_x(y), \exp_x(z)\right) \leq C_0|z - y|$$

for some $C_0 \in [1, 2]$ independent of x, y, and z. Given $R_\alpha \geq 1$ and $x \in \mathbb{R}^n$ such that $|x| < i_g R_\alpha$, we let

$$\tilde{v}_\alpha(x) = R_\alpha^{-\frac{n-p}{p}} v_\alpha\left(\exp_{x_\alpha}\left(R_\alpha^{-1}x\right)\right)$$
$$\tilde{g}_\alpha(x) = \left(\exp_{x_\alpha}^\star g\right)\left(R_\alpha^{-1}x\right) \ .$$

Then, if $|z| + r < i_g R_\alpha$, we get that

$$\int_{B_z(r)} |\nabla \tilde{v}_\alpha|^p dv_{\tilde{g}_\alpha} = \int_{\exp_{x_\alpha}\left(R_\alpha^{-1}B_z(r)\right)} |\nabla v_\alpha|^p dv_g \ . \tag{21}$$

When $|z| + r < r_0 R_\alpha$,

$$\exp_{x_\alpha}\left(R_\alpha^{-1}B_z(r)\right) \subset B_{\exp_{x_\alpha}\left(R_\alpha^{-1}z\right)}\left(C_0 r R_\alpha^{-1}\right) \tag{22}$$

while

$$\exp_{x_\alpha}\left(R_\alpha^{-1}B_0(C_0 r)\right) = B_{x_\alpha}\left(C_0 r R_\alpha^{-1}\right) \ . \tag{23}$$

Given $r \in (0, r_0)$, we fix t_0 such that $C_0 r t_0^{-1} \geq 1$. Then, for any $\lambda \in (0, \lambda_0)$, to be fixed later on, we let $R_\alpha \geq 1$ be such that $C_0 r R_\alpha^{-1} = t_\alpha$. By (21) to (23), for any $z \in \mathbb{R}^n$ such that $|z| < r_0 R_\alpha - r$,

$$\begin{aligned} &\int_{B_z(r)} |\nabla \tilde{v}_\alpha|^p dv_{\tilde{g}_\alpha} \leq \lambda \text{ , and} \\ &\int_{B_0(C_0 r)} |\nabla \tilde{v}_\alpha|^p dv_{\tilde{g}_\alpha} = \lambda \text{ .} \end{aligned} \tag{24}$$

We let $\delta \in (0, i_g)$ and $C_1 > 1$ be such that for any $x \in M$, and any $R \geq 1$, if $\tilde{g}_{x,R}(y) = \exp_x^\star g(R^{-1}y)$, then

$$\frac{1}{C_1} \int_{\mathbb{R}^n} |\nabla u|^p dx \leq \int_{\mathbb{R}^n} |\nabla u|^p dv_{\tilde{g}_{x,R}} \leq C_1 \int_{\mathbb{R}^n} |\nabla u|^p dx \tag{25}$$

for all $u \in D_1^p(\mathbb{R}^n)$ such that supp $u \subset B_0(\delta R)$. Without loss of generality, we also assume that

$$\frac{1}{C_1} \int_{\mathbb{R}^n} |u| dx \leq \int_{\mathbb{R}^n} |u| dv_{\tilde{g}_{x,R}} \leq C_1 \int_{\mathbb{R}^n} |u| dx \tag{26}$$

for all $u \in L^1(\mathbb{R}^n)$ such that supp $u \subset B_0(\delta R)$. We let $\tilde{\eta} \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $0 \leq \tilde{\eta} \leq 1$, $\tilde{\eta} = 1$ in $B_0(1/4)$, and $\tilde{\eta} = 0$ in $\mathbb{R}^n \backslash B_0(3/4)$. We set $\tilde{\eta}_\alpha(x) = \tilde{\eta}(\delta^{-1} R_\alpha^{-1} x)$, where δ is as above. Then,

$$\int_{\mathbb{R}^n} |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|^p dv_{\tilde{g}_\alpha} = O(1)$$

and it follows from (25) that the sequence $(\tilde{\eta}_\alpha \tilde{v}_\alpha)_\alpha$ is bounded in $D_1^p(\mathbb{R}^n)$. In particular, up to a subsequence, there exists $v \in D_1^p(\mathbb{R}^n)$ such that $\tilde{\eta}_\alpha \tilde{v}_\alpha \rightharpoonup v$ weakly in $D_1^p(\mathbb{R}^n)$. Now we divide the proof of Lemma 1.1 into several steps. As a first step, we claim that the following holds.

Step 2.1 *For r and λ sufficiently small,*

$$\tilde{\eta}_\alpha \tilde{v}_\alpha \to v \;\; in \; H_1^p(B_0(C_0 r)) \tag{27}$$

as $\alpha \to +\infty$.

Proof of Step 2.1. We fix $x_0 \in \mathbb{R}^n$. Then by Fatou's lemma and Fubini's theorem

$$\begin{aligned} &\int_r^{2r} \left(\liminf_{\alpha \to \infty} \int_{S_{x_0}(r)} N_\xi(\tilde{\eta}_\alpha \tilde{v}_\alpha) dv_{h_\rho} \right) d\rho \\ &\leq \liminf_{\alpha \to \infty} \int_{B_{x_0}(2r)} N_\xi(\tilde{\eta}_\alpha \tilde{v}_\alpha) dx \leq C \end{aligned}$$

where h_ρ and ξ stand respectively for the standard metric on the sphere $S_{x_0}(\rho)$ and the Euclidean metric, and where $N_\xi(u) = |\nabla u|_\xi^p + |u|^p$. Then there exists $\rho \in [r, 2r]$ such that up to a subsequence, and for any α,

$$\int_{S_{x_0}(\rho)} N_\xi(\tilde{\eta}_\alpha \tilde{v}_\alpha) dv_{h_\rho} \le C$$

Let $C = C(\rho) > 0$ be such that for any $\phi \in C^\infty(\mathbb{R})$,

$$N_{h_\rho}\left(\phi_{|S_{x_0}(\rho)}\right) \le C N_\xi(\phi)$$

on $S_{x_0}(\rho)$. Then $((\tilde{\eta}_\alpha \tilde{v}_\alpha)_{|S_{x_0}(\rho)})_\alpha$ is bounded in $H_1^p(S_{x_0}(\rho))$ and, by compactness of the embedding $H_1^p(S_{x_0}(\rho)) \hookrightarrow H^p_{\frac{p-1}{p}}(S_{x_0}(\rho))$, we get that a subsequence $\tilde{\eta}_\alpha \tilde{v}_\alpha$ converges to v in $H^p_{\frac{p-1}{p}}(S_{x_0}(\rho))$. Let $A = B_{x_0}(3r) - B_{x_0}(\rho)$ and

$$\psi_\alpha = \begin{cases} \tilde{\eta}_\alpha \tilde{v}_\alpha - v & \text{in } \bar{B}_{x_0}(\rho) \\ z_\alpha & \text{in } \bar{B}_{x_0}(3r) - B_{x_0}(\rho) \\ 0 & \text{otherwise} \end{cases}$$

where z_α is the solution of the Dirichlet problem

$$\begin{aligned} &\Delta_p u = 0 \ \text{ in } A\ , \\ &u = \tilde{\eta}_\alpha \tilde{v}_\alpha - v \ \text{ on } S_{x_0}(\rho)\ , \text{ and} \\ &u = 0 \ \text{ on } S_{x_0}(3r) \end{aligned} \tag{28}$$

The existence of z_α follows from Step 2.2 below. Moreover, still by Step 2.2, $\psi_\alpha \to 0$ in $H_1^p(A)$ and we also have that $\psi_\alpha \rightharpoonup 0$ in $D_1^p(\mathbb{R}^n)$. We fix $r < \frac{\delta}{24}$ and denote by $\tilde{\psi}_\alpha \in H_1^p(M)$ the function in M obtained by rescaling ψ_α. Namely,

$$\tilde{\psi}_\alpha(x) = \begin{cases} R_\alpha^{\frac{n-p}{p}} \psi_\alpha\left(R_\alpha \exp_{x_\alpha}^{-1}(x)\right) & \text{if } d_g(x_\alpha, x) < 6r \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{\eta}(\delta^{-1} \exp_{x_\alpha}^{-1}(x)) = 1$ if $d_g(x_\alpha, x) < 6r$ and, if in addition $|x_0| < 3r$, we have that

$$\begin{aligned} DI_g(v_\alpha).\tilde{\psi}_\alpha &= DI_g(\tilde{\eta}_\alpha v_\alpha).\tilde{\psi}_\alpha \\ &= \int_{B_{x_0}(3r)} |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|_{\tilde{g}_\alpha}^{p-2} < \nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha); \nabla\psi_\alpha >_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} \\ &\quad - \int_{B_{x_0}(3r)} |\tilde{\eta}_\alpha \tilde{v}_\alpha|^{p^*-2} \tilde{\eta}_\alpha \tilde{v}_\alpha \psi_\alpha dv_{\tilde{g}_\alpha} \end{aligned}$$

The sequence $(\psi_\alpha)_\alpha$ is bounded in $D_1^p(\mathbb{R}^n)$. The Sobolev inequality then gives that $(\psi_\alpha)_\alpha$ is also bounded in $H_1^p(\mathbb{R}^n)$. By the definition of $\tilde{\psi}_\alpha$ and $\tilde{g}_\alpha$ we then get that $(\tilde{\psi}_\alpha)_\alpha$ is bounded in $H_1^p(M)$. Since $(v_\alpha)_\alpha$ is a P-S sequence for I_g, we thus obtain that

$$DI_g(v_\alpha).\tilde{\psi}_\alpha = o(1) \tag{29}$$

We remark that if

$$I = \int_A |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|_{\tilde{g}_\alpha}^{p-2} (\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha).\nabla\psi_\alpha)_{\tilde{g}_\alpha}\, dv_{\tilde{g}_\alpha}$$

then $I = o(1)$. Indeed, by Hölder's inequality,

$$|I| \le \left(\int_A |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|_{\tilde{g}_\alpha}^p dv_{\tilde{g}_\alpha}\right)^{\frac{p-1}{p}} \left(\int_A |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^p dv_{\tilde{g}_\alpha}\right)^{\frac{1}{p}}$$

Up to reduce r, we may assume that $supp(\psi_\alpha) \subset B_{x_0}(3r) \subset B_0(\delta) \subset B_0(\delta R_\alpha)$. Then

$$\begin{aligned}\int_A |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^p dv_{\tilde{g}_\alpha} &\le C\int_A |\nabla\psi_\alpha|^p dx \\ &= o(1)\end{aligned}$$

Moreover, by the definition of $\tilde{\eta}_\alpha$, $supp(\tilde{\eta}_\alpha \tilde{v}_\alpha) \subset B_0(\frac{3\delta R_\alpha}{4}) \subset B_0(\delta R_\alpha)$, and then

$$\begin{aligned}\int_A |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|_{\tilde{g}_\alpha}^p dv_{\tilde{g}_\alpha} &\le C\int_A |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|^p dx \\ &= O(1)\end{aligned}$$

We eventually get that $I = o(1)$. Now we prove that

$$\begin{aligned}&\int_{B_{x_0}(3r)} |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|_{\tilde{g}_\alpha}^{p-2} (\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha).\nabla\psi_\alpha)_{\tilde{g}_\alpha}\, dv_{\tilde{g}_\alpha} \\ &= \int_{\mathbb{R}^n} |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^p dv_{\tilde{g}_\alpha} + o(1)\end{aligned} \tag{30}$$

In order to prove (30) we first note that

$$\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha) \to \nabla v \ a.e \tag{31}$$

We proceed as in the proof of Step 1.2 to get that (31) holds true. If we let $\Sigma_\alpha = |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|_\xi^{p-2}\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)$, it suffices to prove that, for $\epsilon > 0$, there exists $C > 0$ such that $\int_\Omega \Sigma_\alpha \nabla\beta_{\epsilon,\alpha} dx \le C\epsilon$ where $\beta_{\epsilon,\alpha} = \beta_\epsilon \circ (\tilde{\eta}_\alpha \tilde{v}_\alpha - v)$. Now, for $\theta > 0$ small and $C > 0$ given by (11) we can write that

$$\begin{aligned}&\int_{B_{x_0}(\rho)} \left| |\nabla(\psi_\alpha + v)|_{\tilde{g}_\alpha}^{p-2}\nabla(\psi_\alpha + v) - |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^{p-2}\nabla\psi_\alpha - |\nabla v|_{\tilde{g}_\alpha}^{p-2}\nabla v\right|^{\frac{p}{p-1}} dv_{\tilde{g}_\alpha} \\ &\le C\int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^{\frac{p(p-1-\theta)}{p-1}} |\nabla v|_{\tilde{g}_\alpha}^{\frac{p\theta}{p-1}} dv_{\tilde{g}_\alpha} + C\int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^{\frac{p\theta}{p-1}} |\nabla v|_{\tilde{g}_\alpha}^{\frac{p(p-1-\theta)}{p-1}} dv_{\tilde{g}_\alpha} \\ &\le C\int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|_\xi^{\frac{p(p-1-\theta)}{p-1}} |\nabla v|_\xi^{\frac{p\theta}{p-1}} dx + C\int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|_\xi^{\frac{p\theta}{p-1}} |\nabla v|_\xi^{\frac{p(p-1-\theta)}{p-1}} dx\end{aligned}$$

Since $|\nabla\psi_\alpha| \to 0$ a.e by (31), and $(\psi_\alpha)_\alpha$ is bounded in $D_1^p(\mathbb{R}^n)$, it follows from standard integration theory that

$$\int_{B_{x_0}(\rho)} \left| |\nabla(\psi_\alpha+v)|_{\tilde{g}_\alpha}^{p-2}\nabla(\psi_\alpha+v) - |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^{p-2}\nabla\psi_\alpha - |\nabla v|_{\tilde{g}_\alpha}^{p-2}\nabla v \right|^{\frac{p}{p-1}} dv_{\tilde{g}_\alpha} = o(1)$$

By Hölder's inequality and since $(\psi_\alpha)_\alpha$ is bounded in $D_1^p(\mathbb{R}^n)$, we then get that

$$\begin{aligned} &\int_{B_{x_0}(\rho)} |\nabla(\psi_\alpha + v)|_{\tilde{g}_\alpha}^{p-2} \left(\nabla(\psi_\alpha + v).\nabla\psi_\alpha\right)_{\tilde{g}_\alpha} \\ &= \int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^{p} dv_{\tilde{g}_\alpha} + \int_{B_{x_0}(\rho)} |\nabla v|_{\tilde{g}_\alpha}^{p-2} \left(\nabla v.\nabla\psi_\alpha\right)_{\tilde{g}_\alpha} |dv_{\tilde{g}_\alpha} + o(1) \end{aligned} \tag{32}$$

Using (29), (32), the fact that $\psi_\alpha \rightharpoonup 0$ in $D_1^p(\mathbb{R}^n)$ and the fact that $\psi_\alpha \to 0$ in $D_1^p(A)$, we eventually obtain that

$$\begin{aligned} &\int_{B_{x_0}(3r)} |\nabla(\tilde{\eta}_\alpha\tilde{v}_\alpha)|_{\tilde{g}_\alpha}^{p-2} \left(\nabla(\tilde{\eta}_\alpha\tilde{v}_\alpha).\nabla\psi_\alpha\right)_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} \\ &= \int_{B_{x_0}(\rho)} |\nabla(\tilde{\eta}_\alpha\tilde{v}_\alpha)|_{\tilde{g}_\alpha}^{p-2} \left(\nabla(\tilde{\eta}_\alpha\tilde{v}_\alpha).\nabla\psi_\alpha\right)_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + o(1) \\ &= \int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^{p} dv_{\tilde{g}_\alpha} + \int_{B_{x_0}(\rho)} |\nabla v|_{\tilde{g}_\alpha}^{p-2} \left(\nabla v.\nabla\psi_\alpha\right)_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + o(1) \\ &= \int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^{p} dv_{\tilde{g}_\alpha} + o(1) \\ &= \int_{\mathbb{R}^n} |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^{p} dv_{\tilde{g}_\alpha} + o(1) \end{aligned}$$

and this proves (30). In a similar way we can prove that

$$\begin{aligned} &\int_{B_{x_0}(3r)} |\tilde{\eta}_\alpha\tilde{v}_\alpha|^{p^*-2}\tilde{\eta}_\alpha\tilde{v}_\alpha\psi_\alpha dv_{\tilde{g}_\alpha} \\ &= \int_{B_{x_0}(\rho)} |\psi_\alpha + v|^{p^*-2}(\psi_\alpha + v)\psi_\alpha dv_{\tilde{g}_\alpha} + o(1) \\ &= \int_{B_{x_0}(\rho)} |\psi_\alpha|^{p^*} dv_{\tilde{g}_\alpha} + \int_{B_{x_0}(\rho)} |v|^{p^*-2} v\psi_\alpha dv_{\tilde{g}_\alpha} + o(1) \\ &= \int_{\mathbb{R}^n} |\psi_\alpha|^{p^*} dv_{\tilde{g}_\alpha} + o(1) \end{aligned}$$

Finally, by (29), and according to what we just proved, we get that

$$\int_{\mathbb{R}^n} |\nabla\psi_\alpha|_{\tilde{g}_\alpha}^{p} dv_{\tilde{g}_\alpha} - \int_{\mathbb{R}^n} |\psi_\alpha|^{p^*} dv_{\tilde{g}_\alpha} = o(1) \tag{33}$$

Now we prove that

$$\int_{\mathbb{R}^n} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} = \int_{B_{x_0}(\rho)} |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} - \int_{B_{x_0}(\rho)} |\nabla v|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + o(1) \quad (34)$$

We fix $\theta > 0$ small and $C > 0$ such that

$$\begin{aligned}
&\int_{B_{x_0}(\rho)} \left| |\nabla(\psi_\alpha + v)|^p_{\tilde{g}_\alpha} - |\nabla v|^p_{\tilde{g}_\alpha} - |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} \right| dv_{\tilde{g}_\alpha} \\
&\le C \int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|^{p-\theta}_{\tilde{g}_\alpha} |\nabla v|^\theta_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + C \int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|^\theta_{\tilde{g}_\alpha} |\nabla v|^{p-\theta}_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} \\
&\le C' \int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|^{p-\theta} |\nabla v|^\theta dx + C' \int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|^\theta |\nabla v|^{p-\theta} dx
\end{aligned}$$

By standard integration theory, we then get that

$$\begin{aligned}
&\int_{B_{x_0}(\rho)} |\nabla(\psi_\alpha + v)|^p_{\tilde{g}_\alpha} \\
&= \int_{B_{x_0}(\rho)} |\nabla v|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + \int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + o(1)
\end{aligned}$$

Writing that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} &= \int_{B_{x_0}(\rho)} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + o(1) \\
&= \int_{B_{x_0}(\rho)} |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} - \int_{B_{x_0}(\rho)} |\nabla v|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + o(1)
\end{aligned}$$

this proves (34). In particular, it follows from (34) that

$$\int_{\mathbb{R}^n} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} \le \int_{B_{x_0}(\rho)} |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + o(1) \quad (35)$$

From now on, we let $N \in \mathbb{N}$ be such that $B_0(2)$ is covered by N balls of radius 1 centered in $B_0(2)$. Then there exist N points $x_1, \ldots, x_N$ in $B_{x_0}(2r)$ such that

$$B_{x_0}(\rho) \subset B_{x_0}(2r) \subset \bigcup_{i=1}^N B_{x_i}(r)$$

and we get with (24) and (35) that for x_0 and r such that $|x_0| + 3r < r_0$,

$$\int_{\mathbb{R}^n} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} \le N\lambda + o(1) \quad (36)$$

For x_0 and r such that $|x_0| + 3r < \delta$, with (33) and the Sobolev inequality, there exists $C > 0$ such that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha}\right)^{p/p^*} &= \left(\int_{\mathbb{R}^n} |\psi_\alpha|^{p^*} dv_{\tilde{g}_\alpha}\right)^{p/p^*} + o(1)\\ &\le C \int_{\mathbb{R}^n} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + o(1)\end{aligned}$$

and with (36), we get the existence of a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} \le \left(C\lambda^{\frac{p^*}{p}-1} + o(1)\right) \int_{\mathbb{R}^n} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} + o(1)$$

Choosing $\lambda > 0$ sufficiently small such that $C\lambda^{\frac{p^*}{p}-1} < 1$, we get that

$$\int_{\mathbb{R}^n} |\nabla\psi_\alpha|^p_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha} = o(1)$$

and thus that $\psi_\alpha \to 0$ in $\mathcal{D}_1^p(\mathbb{R}^n)$. Since $r \le \rho$, it follows that

$$\tilde{\eta}_\alpha \tilde{v}_\alpha \to v \text{ in } H_1^p(B_{x_0}(r)) \tag{37}$$

and the convergence holds as soon as $C\lambda^{\frac{p^*}{p}-1} < 1$, $|x_0| < 3r$, $|x_0| + 3r < \min\{r_0, \delta\}$ and r is sufficiently small. We fix $r > 0$ and λ such that the above are satisfied. Then (37) holds for any $x_0 \in B_0(2r)$. Since $C_0 \le 2$, $B_0(C_0 r)$ is covered by N balls of radius r centered in $B_0(2r)$. It follows that $\tilde{\eta}_\alpha \tilde{v}_\alpha \to v$ in $H_1^p(B_0(C_0 r))$ and this proves Step 2.1.

Step 2.2 below was used in the proof of Step 2.1.

Step 2.2 *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $p \in (1, n)$, and $\hat{p}$ be given by $\hat{p} = \frac{p-1}{p}$. Let also $h \in H^p_{\hat{p}}(\partial\Omega)$. Then there exists a solution $u \in H_1^p(\Omega)$ of the equation*

$$\begin{aligned}\Delta_p u = 0 \ \ &in\ \Omega\ ,\\ u = h \ \ &on\ \partial\Omega\ .\end{aligned}$$

Moreover, $\|u\|_{H_1^p(\Omega)} \le C\|h\|_{H^p_{\hat{p}}(\partial\Omega)}$ where $C > 0$ is independent of u and h.

Proof of Step 2.2. Following Struwe [23], see Appendix A in [23], we easily get that there exist $C_1, C_2 > 0$ such that for any $u \in H_1^p(\Omega)$,

$$\int_\Omega |u|^p dx \le C_1 \int_\Omega |\nabla u|^p dx + C_2 \int_{\partial\Omega} |u_{|\partial\Omega}|^p dx \tag{38}$$

Let $h \in H^p_{\hat{p}}(\partial\Omega)$ and $\mathcal{H}$ be the set consisting of the functions $v \in H_1^p(\Omega)$ which are such that $v - h \in H_0^{1,p}(\Omega)$, where $H_0^{1,p}(\Omega)$ is the closure in $H_1^p(\Omega)$ of the space of smooth functions with compact support in Ω. We let

$$\lambda = \inf_{v\in\mathcal{H}} \int_\Omega |\nabla v|^p dx$$

and let $(v_m)_m$ be a minimizing sequence for λ. Thanks to (38), $(v_m)_m$ is then bounded in $H_1^p(\Omega)$. We may therefore assume that $v_m \rightharpoonup u$ in $H_1^p(\Omega)$, that $v_m \to u$ in $L^p(\Omega)$, and that $v_m \to u$ in $L^p(\partial\Omega)$. In particular, $u \in \mathcal{H}$ and $\int_\Omega |\nabla u|^p dx = \lambda$. It follows that u is a weak solution of $\Delta_p u = 0$ in Ω and $u = h$ on $\partial\Omega$. Let H be given by the extension operator from $H_{\hat{p}}^p(\partial\Omega)$ into $H_1^p(\Omega)$. Then $H \in H_1^p(\Omega)$ and $H_{|\partial\Omega} = h$. It easily follows from the equation satisfied by u that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = 0$$

where $v = u - H \in H_0^{1,p}(\Omega)$, and hence, by Hölder's inequality, that

$$\int_\Omega |\nabla u|^p dx \le \int_\Omega |\nabla H|^p dx$$

In particular, by the continuity of the extension operator,

$$\|\nabla u\|_p \le C\|h\|_{H_{\hat{p}}^p(\partial\Omega)}$$

where $C > 0$ is independent of u and h. Coming back to (38), it follows that $\|u\|_{H_1^p(\Omega)} \le C\|h\|_{H_{\hat{p}}^p(\partial\Omega)}$ where $C > 0$ is independent of u and h. This ends the proof of Step 2.2.

It easily follows from Step 2.1 that $v \not\equiv 0$. To see this we return to (24) and write that

$$\begin{aligned}\lambda &= \int_{B_0(C_0 r)} |\nabla(\tilde{\eta}_\alpha \tilde{v}_\alpha)|^p dv_{\tilde{g}_\alpha} \\ &\le C_1 \int_{B_0(C_0 r)} |\nabla v|^p dx + o(1)\end{aligned}$$

Hence, $v \not\equiv 0$. Similarly, it also follows from Step 2.1 that $R_\alpha \to +\infty$ as $\alpha \to +\infty$. Indeed, if $R_\alpha \to R$ as $\alpha \to +\infty$, $R \ge 1$, then $\tilde{v}_\alpha \rightharpoonup 0$ in $H_1^p(B_0(C_0 r))$ since $v_\alpha \rightharpoonup 0$ in $H_1^p(M)$. A contradiction with Step 2.1 and the above claim that $v \not\equiv 0$. Hence,

$$R_\alpha \to +\infty \tag{39}$$

as $\alpha \to +\infty$. Thanks to (39) we can then prove that for any $R > 0$,

$$\tilde{v}_\alpha \to v \text{ in } H_1^p(B_0(R)) \tag{40}$$

as $\alpha \to +\infty$. To see this, we let $R \ge 1$ be given. Since $R_\alpha \to \infty$, we have that $R_\alpha > R$ for α large. Then (24) holds for $z \in \mathbb{R}^n$ such that $|z| < r_0 R - r$, and it follows from the proof of Step 2.1 that (37) holds if $|x_0| < 3r(2R-1)$, $|x_0| + 3r < r_0 R$, and $|x_0| + 3r < \delta R$. In particular, (37) holds if $|x_0| < 2rR$ and thus $\tilde{\eta}_\alpha \tilde{v}_\alpha \to v$ strongly in $H_1^p(B_0(2rR))$. Since $R \ge 1$ is arbitrary and $\tilde{\eta}_\alpha(x) = 1$ for α large if $|x| \le R$, we obtain (40). An easy claim then is the following.

Step 2.3 *The limit v of Step 2.1 is a solution of $\Delta_p u = |u|^{p^*-2}u$.*

Proof of Step 2.3. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ and $R_0 > 0$ such that $supp\ \phi \subset B_0(R_0)$, and let $\hat{\phi}_\alpha \in C_c^\infty(\mathbb{R}^n)$ be given by

$$\hat{\phi}_\alpha(x) = R_\alpha^{\frac{n-p}{p}} \phi(R_\alpha x)$$

Then $supp\ \hat{\phi}_\alpha \subset B_0(R_0 R_\alpha^{-1})$. For α large, we define $\phi_\alpha \in C^\infty(M)$ by the equation $\hat{\phi}_\alpha = \phi_\alpha \circ \exp_{x_\alpha}$. Then, for α large, we get that

$$\begin{aligned}\int_M |\nabla v_\alpha|_g^{p-2} (\nabla v_\alpha.\nabla\phi_\alpha)_g\, dv_g &= \int_{\mathbb{R}^n} |\nabla \tilde{v}_\alpha|_{\tilde{g}_\alpha}^{p-2} (\nabla\tilde{v}_\alpha.\nabla\phi)_{\tilde{g}_\alpha}\, dv_{\tilde{g}_\alpha} \\ &= \int_{\mathbb{R}^n} |\nabla(\tilde{\eta}_\alpha\tilde{v}_\alpha)|_{\tilde{g}_\alpha}^{p-2} (\nabla(\tilde{\eta}_\alpha\tilde{v}_\alpha).\nabla\phi)_{\tilde{g}_\alpha}\, dv_{\tilde{g}_\alpha}\end{aligned}$$

and that

$$\begin{aligned}\int_M |v_\alpha|^{p^*-2} v_\alpha\phi_\alpha dv_g &= \int_{\mathbb{R}^n} |\tilde{v}_\alpha|^{p^*-2}\tilde{v}_\alpha\phi dv_{\tilde{g}_\alpha} \\ &= \int_{\mathbb{R}^n} |\tilde{\eta}_\alpha\tilde{v}_\alpha|^{p^*-2}\tilde{\eta}_\alpha\tilde{v}_\alpha\phi dv_{\tilde{g}_\alpha}\end{aligned}$$

Moreover $(\phi_\alpha)_\alpha$ is bounded in $H_1^p(M)$. Thus

$$\begin{aligned}o(1) &= DI_g(v_\alpha).\phi_\alpha \\ &= \int_{\mathbb{R}^n} |\nabla(\tilde{\eta}_\alpha\tilde{v}_\alpha)|_{\tilde{g}_\alpha}^{p-2} (\nabla(\tilde{\eta}_\alpha\tilde{v}_\alpha).\nabla\phi)_{\tilde{g}_\alpha}\, dv_{\tilde{g}_\alpha} \\ &\quad - \int_{\mathbb{R}^n} |\tilde{\eta}_\alpha\tilde{v}_\alpha|^{p^*-2}\tilde{\eta}_\alpha\tilde{v}_\alpha\phi\, dv_{\tilde{g}_\alpha}\end{aligned}$$

Since $R_\alpha \to \infty$, we can write that $\tilde{g}_\alpha \to \xi$ in $C^1(B_0(R))$ and thus that $dv_{\tilde{g}_\alpha} = \epsilon_\alpha dx$ where $\epsilon_\alpha \to 1$ uniformly in $B_0(R)$. Since in addition $\tilde{\eta}_\alpha\tilde{v}_\alpha \to v$ in $D_1^p(\mathbb{R}^n)$, we get, by passing to the limit in the above equation that, for any $\phi \in D_1^p(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla v|^{p-2}\nabla v\nabla\phi dx = \int_{\mathbb{R}^n} |v|^{p^*-2} v\phi dx$$

In other words, v is a weak solution of $\Delta_p u = |u|^{p^*-2}u$. This ends the proof of Step 2.3.

For $x \in M$ and $\hat{\delta} \in (0, \frac{\delta}{8})$, we let

$$V_\alpha(x) = \eta_\alpha(x) R_\alpha^{\frac{n-p}{p}} v\big(R_\alpha \exp_{x_\alpha}^{-1}(x)\big)$$

where $\eta_\alpha = \eta_{\hat{\delta},x_\alpha}$, and set $w_\alpha = v_\alpha - V_\alpha$. A last step in the proof of Lemma 1.1 is as follows.

Step 2.4 *On the one hand,*

$$w_\alpha \rightharpoonup 0 \ \ in\ H_1^p(M) \tag{41}$$

as $\alpha \to +\infty$. *On the other hand,*

$$DI_g(V_\alpha) \to 0 \ \ \textit{and} \ \ DI_g(w_\alpha) \to 0 \tag{42}$$

in $H_1^p(M)'$ *as* $\alpha \to +\infty$. *At last,*

$$I_g(w_\alpha) = I_g(v_\alpha) - E(v) + o(1) \tag{43}$$

where $o(1) \to 0$ *as* $\alpha \to +\infty$.

Proof of Step 2.4. First we prove that $w_\alpha \rightharpoonup 0$ in $H_1^p(M)$. There it clearly suffices to prove that $V_\alpha \rightharpoonup 0$ in $H_1^p(M)$. We note that $(V_\alpha)_\alpha$ is bounded in $H_1^p(M)$ and, since $H_1^p(M) \hookrightarrow L^p(M)$ compactly, it suffices in turn to prove that $V_\alpha \rightharpoonup 0$ in $L^p(M)$. In what follows we let $f \in L^q(M)$ where $q = \frac{p}{p-1}$. Given $R > 0$ arbitrary, we let

$$B_\alpha = B_{x_\alpha}(R_\alpha^{-1}R) \ , \ \ B_\alpha^c = B_{x_\alpha}(2\hat{\delta})\backslash B_{x_\alpha}(R_\alpha^{-1}R) \ , \ \text{ and } g_\alpha = \exp_{x_\alpha}^* g$$

Then, we can write that

$$\begin{aligned}
&\left|\int_{B_\alpha} f V_\alpha dv_g\right| \\
&\le R_\alpha^{\frac{n-p}{p}} \int_{B_0(R_\alpha^{-1}R)} \left|f(\exp_{x_\alpha}(x))\right| |v(R_\alpha x)| \, dv_{g_\alpha} \\
&\le C.R_\alpha^{\frac{n-p}{p}} \left(\int_{B_0(R_\alpha^{-1}R)} |v(R_\alpha x)|^p \, dx\right)^{\frac{1}{p}} \left(\int_{B_0(R_\alpha^{-1}R)} \left|f(\exp_{x_\alpha}(x))\right|^q \, dv_{g_\alpha}\right)^{\frac{1}{q}} \\
&\le C\|f\|_q R_\alpha^{-1} \left(\int_{B_0(R)} |v|^p dx\right)^{\frac{1}{p}}
\end{aligned}$$

and, in a similar way, that

$$\begin{aligned}
\left|\int_{B_\alpha^c} f V_\alpha dv_g\right| &\le C\|f\|_q R_\alpha^{-1} \left(\int_{B_0(2\hat{\delta}R_\alpha)\backslash B_0(R)} |v|^p dx\right)^{\frac{1}{p}} \\
&\le C\|f\|_q \left(\int_{B_0(2\hat{\delta}R_\alpha)\backslash B_0(R)} |v|^{p^*} dx\right)^{\frac{1}{p^*}}
\end{aligned}$$

Since $R_\alpha \to \infty$, and $R > 0$ is arbitrary, we get that

$$\int_M f V_\alpha dv_g = o(1)$$

Noting that $f \in L^q(M)$ is arbitrary, this proves that $V_\alpha \to 0$ weakly in $H_1^p(M)$, and thus that (41) holds. The proof of (42) is an easy adaptation of [18]. We skip it here and restrict ourself to prove (43). We write that

$$I_g(w_\alpha) = \frac{1}{p}\int_M |\nabla w_\alpha|_g^p dv_g - \frac{1}{p^*}\int_M |w_\alpha|^{p^*} dv_g$$

and that

$$\int_M |\nabla w_\alpha|_g^p dv_g = \int_{B_\alpha} |\nabla w_\alpha|_g^p dv_g + \int_{B_\alpha^c} |\nabla w_\alpha|_g^p dv_g \\ + \int_{M\backslash B_{x_\alpha}(2\hat{\delta})} |\nabla w_\alpha|_g^p dv_g \tag{44}$$

where B_α and B_α^c are as above. On the one hand,

$$\int_{B_\alpha} |\nabla w_\alpha|_g^p dv_g = \int_{B_0(R)} |\nabla \tilde{v}_\alpha - \nabla v|_{\tilde{g}_\alpha}^p \, dv_{\tilde{g}_\alpha} \\ \leq C \int_{B_0(R)} |\nabla \tilde{v}_\alpha - \nabla v|^p \, dx$$

and since $\tilde{v}_\alpha \to v$ in $H_1^p(B_0(R))$, we get that

$$\int_{B_\alpha} |\nabla w_\alpha|_g^p dv_g = o(1) \tag{45}$$

On the other hand, mimicking what was done in [18], we also have that

$$\int_{B_\alpha^c} |\nabla w_\alpha|_g^p dv_g = \int_{B_\alpha^c} |\nabla v_\alpha|_g^p dv_g + B_r(\alpha) + o(1) \tag{46}$$

where here, and in what follows, $B_R(\alpha)$ stands for any expression such that

$$\lim_{R\to+\infty} \limsup_{\alpha\to+\infty} B_R(\alpha) = 0$$

At last, by definition of $\eta_{\hat{\delta},x_\alpha}$, we can write that

$$\int_{M\backslash B_{x_\alpha}(2\hat{\delta})} |\nabla w_\alpha|_g^p dv_g \\ = \int_M |\nabla v_\alpha|_g^p dv_g - \int_{B_\alpha} |\nabla v_\alpha|_g^p dv_g - \int_{B_\alpha^c} |\nabla v_\alpha|_g^p dv_g \tag{47}$$

Plugging (45), (46), and (47) into (44), we get that

$$\int_M |\nabla w_\alpha|_g^p dv_g = \int_M |\nabla v_\alpha|_g^p dv_g - \int_{B_\alpha} |\nabla v_\alpha|_g^p dv_g + B_R(\alpha) + o(1)$$

Since $\tilde{g}_\alpha \to \xi$ in $C^0(B_0(R))$ and $\tilde{v}_\alpha \to v$ in $H_1^p(B_0(R))$,

$$\int_{B_\alpha} |\nabla v_\alpha|_g^p dv_g = \int_{B_0(R)} |\nabla \tilde{v}_\alpha|_{\tilde{g}_\alpha}^p dv_{\tilde{g}_\alpha} \\ = \int_{\mathbb{R}^n} |\nabla v|^p dx + B_R(\alpha) + o(1)$$

and we eventually get that

$$\int_M |\nabla w_\alpha|_g^p dv_g = \int_M |\nabla v_\alpha|_g^p dv_g - \int_{\mathbb{R}^n} |\nabla v|^p dx + B_R(\alpha) + o(1)$$

In a asimilar way we can also write that

$$\int_M |w_\alpha|^{p^*} dv_g = \int_M |v_\alpha|^{p^*} dv_g - \int_{\mathbb{R}^n} |v|^{p^*} dx + B_R(\alpha) + o(1)$$

and since $R > 0$ is arbitrary, it follows that (43) holds true. This ends the proof of Step 2.4.

According to what we said up to now, and thanks to Steps 2.1–2.4, Lemma 1.1 holds for some $\delta \in (0, \frac{i_g}{2})$. Given $\delta_1 < \delta_2$ in $(0, \frac{i_g}{2})$, we can check, using the definition of $\eta_{\delta_i, x_\alpha}$, that

$$\left\| \hat{B}_\alpha^1 - \hat{B}_\alpha^2 \right\|_{H_1^p} = o(1)$$

where $\hat{B}_\alpha^i(x) = \eta_{\delta_i, x_\alpha}(x) R_\alpha^{\frac{n-p}{p}} v(R_\alpha \exp_{x_\alpha}^{-1}(x))$, $i = 1, 2$. For instance,

$$\int_M \left| \hat{B}_\alpha^1 - \hat{B}_\alpha^2 \right|^{p^*} dv_g \le 2 \int_{B_0(2\delta_2 R_\alpha) - B_0(\delta_1 R_\alpha)} |v|^{p^*} dv_{\tilde{g}_\alpha} = o(1)$$

and we proceed in a similar way for the gradient term. It follows that Lemma 1.1 holds for any $\delta \in (0, \frac{i_g}{2})$. This ends the proof of Lemma 1.1.

3 Positivity of the bubbles

We prove in this section that if the u_α's of Sect. 1 are nonnegative, then u^0 and the u^i's of Sect. 1 are also nonnegative. That u^0 is nonnegative is straightforward. On what concerns the u^i's, we proceed as follows. We fix an integer N in $\{1, .., k\}$ and prove that $u^N \ge 0$ by showing that $\tilde{u}_\alpha^N \to u^N$ a.e in $\mathbb{R}^n$ where

$$\tilde{u}_\alpha^N(x) = \left(\mu_\alpha^N\right)^{\frac{n-p}{p}} u_\alpha\left(\exp_{x_\alpha^N}\left(\mu_\alpha^N x\right)\right) \tag{48}$$

We let $\mu_\alpha^i = 1/R_\alpha^i$, $v_\alpha = u_\alpha - u^0$ and $\tilde{v}_\alpha^N$ and $\tilde{u}_\alpha^{0,N}$ be given by

$$\tilde{v}_\alpha^N(x) = \left(\mu_\alpha^N\right)^{\frac{n-p}{p}} v_\alpha\left(\exp_{x_\alpha^N}\left(\mu_\alpha^N x\right)\right)$$
$$\tilde{u}_\alpha^{0,N}(x) = \left(\mu_\alpha^N\right)^{\frac{n-p}{p}} u^0\left(\exp_{x_\alpha^N}\left(\mu_\alpha^N x\right)\right)$$

We assume for the moment that there exist an integer p and p sequences $(y_\alpha^j)_\alpha$ in M, and (λ_α^j) of positive real numbers, $j = 1 \dots p$, such that $\lambda_\alpha^j / \mu_\alpha^N \to 0$ as $\alpha \to +\infty$, the sequence consisting of the $d_g(x_\alpha^N, y_\alpha^j)/\mu_\alpha^N$ is bounded, and such that for any $R, R' > 0$,

$$\int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right) \setminus \bigcup_{j=1}^p B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)} \left| v_\alpha - u_\alpha^N \right|^{p^*} dv_g = o(1) + \epsilon(R') \tag{49}$$

where $\epsilon(R^{'}) \to 0$ as $R^{'} \to +\infty$.
We then get with (49) that

$$\int_{B_0(R)-\bigcup_{j=1}^{p} B_{\tilde{y}_\alpha^j}\left(R'C\frac{\lambda_\alpha^j}{\mu_\alpha^N}\right)} \left|\tilde{v}_\alpha^N - u^N\right|^{p^*} dx = o(1) + \epsilon(R') \tag{50}$$

where $y_\alpha^j = \exp_{x_\alpha^N}(\mu_\alpha^N \tilde{y}_\alpha^j)$. Noting that the $\tilde{y}_\alpha^j$'s are bounded, we may assume that $\tilde{y}_\alpha^j \to \tilde{y}^j$ as $\alpha \to \infty$. Then (50) gives that

$$\tilde{v}_\alpha^N \to u^N \text{ in } L_{loc}^{p^*}(B_0(R)\backslash\{\tilde{y}^j, j = 1, .., p\})$$

for any $R > 0$, and we may thus assume that

$$\tilde{u}_\alpha^N - \tilde{u}_\alpha^{0,N} \to u^N \text{ a.e in } \mathbb{R}^n \tag{51}$$

Moreover, if $\tilde{g}_\alpha(x) = ((\exp_{x_\alpha^N})^* g)(\mu_\alpha^N x)$, we can write that for any $R > 0$,

$$\int_{B_0(R)} \left|\tilde{u}_\alpha^{0,N}\right|^{p^*} dx \leq C \int_{B_0(R)} \left|\tilde{u}_\alpha^{0,N}\right|^{p^*} dv_{\tilde{g}_\alpha}$$
$$= \int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)} \left|u^0\right|^{p^*} dv_g$$

which goes to 0 as $\alpha \to +\infty$. Hence, $\tilde{u}_\alpha^{0,N} \to 0$ in $L^{p^*}(B_0(R))$ for all $R > 0$, and we may thus also assume that $\tilde{u}_\alpha^{0,N} \to 0$ a.e in $\mathbb{R}^n$. By (51) we then get that $\tilde{u}_\alpha^N \to u^N$ a.e in $\mathbb{R}^n$. In particular, the fact that u^0 and the u^i's of Sect. 1 are nonnegative if the u_α's are nonnegative follows from (49). Now we prove (49) as a particular case of the following statement. Namely we claim that for any integer $N \in \{1, .., k\}$ and for any integer $s \in \{0, .., N-1\}$, there exist an integer p and p sequences $(y_\alpha^j)_\alpha$ in M, and (λ_α^j) of positive real numbers, $j = 1 \ldots p$, such that $\lambda_\alpha^j/\mu_\alpha^N \to 0$ as $\alpha \to +\infty$, the sequence consisting of the $d_g(x_\alpha^N, y_\alpha^j)/\mu_\alpha^N$ is bounded, and such that for any $R, R^{'} > 0$,

$$\int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)-\bigcup_{j=1}^{p} B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)} \left|v_\alpha - \sum_{i=1}^{s} u_\alpha^i - u_\alpha^N\right|^{p^*} dv_g = o(1) + \epsilon(R') \tag{52}$$

where $\epsilon(R') \to 0$ as $R' \to +\infty$ and the $(u_\alpha^i)'s$ and $(x_\alpha^i)'s$ are the ordered sequences in i that we got in the preceding section. Clearly, (49) follows from (52) when $s = 0$.
In order to prove (49), we fix an integer N in $\{1, .., k\}$ and prove that (52) holds for all s by inverse induction on s. If $s = N - 1$, then, by (40), for any $R > 0$,

$$\int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)} \left|v_\alpha - \sum_{i=1}^{N} u_\alpha^i\right|^{p^*} dv_g = o(1)$$

and it follows that (52) holds with $p = 0$. Now we suppose that (52) holds for some $s \leq N - 1$ and we fix $R, R^{'} > 0$. If $d_g(x_\alpha^s, x_\alpha^N) \not\to 0$ as $\alpha \to \infty$, then, for $\tilde{R} > 0$ given, and up to a subsequence, $B_{x_\alpha^N}(R\mu_\alpha^N) \bigcap B_{x_\alpha^s}(\tilde{R}\mu_\alpha^s) = \emptyset$. As a consequence,

$$\int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)\setminus\bigcup_{j=1}^p B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)} |u_\alpha^s|^{p^*} dv_g \le \int_{M\setminus B_{x_\alpha^s}\left(\tilde{R}\mu_\alpha^s\right)} |u_\alpha^s|^{p^*} dv_g \le C \int_{\mathbb{R}^n\setminus B_0(\tilde{R})} |u^s|^{p^*} dx \tag{53}$$

Since $u^s \in L^{p^*}(\mathbb{R}^n)$ and $\tilde{R} > 0$ is arbitrary, we get that

$$\int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)\setminus\bigcup_{j=1}^p B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)} |u_\alpha^s|^{p^*} dv_g = o(1)$$

and thus that

$$\begin{aligned}
&\int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)\setminus\bigcup_{j=1}^p B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)} \left|v_\alpha - \sum_{i=1}^{s-1} u_\alpha^i - u_\alpha^N\right|^{p^*} dv_g \\
&\le C \int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)\setminus\bigcup_{j=1}^p B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)} \left|v_\alpha - \sum_{i=1}^{s} u_\alpha^i - u_\alpha^N\right|^{p^*} dv_g \\
&\quad + C \int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)\setminus\bigcup_{j=1}^p B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)} |u_\alpha^s|^{p^*} dv_g \\
&= o(1) + \epsilon(R')
\end{aligned}$$

This proves (52) for $s-1$. We may thus assume in what follows that $d_g(x_\alpha^s, x_\alpha^N) \to 0$ as $\alpha \to \infty$. Then we need to compare carefully the respective growth of the distances $d_g(x_\alpha^s, x_\alpha^N)$ and the radii μ_α^s and μ_α^N. We let $r_0 > 0$ and $C \ge 1$ be such that for all $x \in M$ and all $y, z \in \mathbb{R}^n$, if $|y|, |z| \le r_0$ then

$$\frac{1}{C}|z - y| \le d_g(\exp_x(y), \exp_x(z)) \le C|z - y|$$

If $\tilde{x}_\alpha^s$ and $\tilde{y}_\alpha^j$ are such that $x_\alpha^s = \exp_{x_\alpha^N}(\mu_\alpha^N \tilde{x}_\alpha^s)$ and $y_\alpha^j = \exp_{x_\alpha^N}(\mu_\alpha^N \tilde{y}_\alpha^j)$, then

$$B_{\tilde{y}_\alpha^j}\left(\frac{R'}{C}\frac{\lambda_\alpha^j}{\mu_\alpha^N}\right) \subset \frac{1}{\mu_\alpha^N}\exp_{x_\alpha^N}^{-1}\left(B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)\right) \subset B_{\tilde{y}_\alpha^j}\left(R'C\frac{\lambda_\alpha^j}{\mu_\alpha^N}\right) \tag{54}$$

and

$$B_{\tilde{x}_\alpha^s}\left(\frac{R'}{C}\frac{\mu_\alpha^s}{\mu_\alpha^N}\right) \subset \frac{1}{\mu_\alpha^N}\exp_{x_\alpha^N}^{-1}\left(B_{x_\alpha^s}\left(R'\mu_\alpha^s\right)\right) \subset B_{\tilde{x}_\alpha^s}\left(R'C\frac{\mu_\alpha^s}{\mu_\alpha^N}\right) \tag{55}$$

Given $\tilde{R} > 0$, we have by (40) that

$$\int_{B_{x_\alpha^s}\left(\tilde{R}\mu_\alpha^s\right)} \left|v_\alpha - \sum_{i=1}^{s} u_\alpha^i\right|^{p^*} dv_g = o(1)$$

Hence, by (52),

$$\int_{\left(B_{x_\alpha^N}\left(R\mu_\alpha^N\right)\setminus\bigcup_{j=1}^{p}B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)\right)\bigcap B_{x_\alpha^s}\left(\tilde{R}\mu_\alpha^s\right)}|u_\alpha^N|^{p^*}dv_g=o(1)+\epsilon(R')$$

and it follows from (54) and (55) that

$$\int_{\left(B_0(R)\setminus\bigcup_{j=1}^{p}B_{\tilde{y}_\alpha^j}\left(R'C\frac{\lambda_\alpha^j}{\mu_\alpha^N}\right)\right)\bigcap B_{\tilde{x}_\alpha^s}\left(\frac{\tilde{R}}{C}\frac{\mu_\alpha^s}{\mu_\alpha^N}\right)}|u^N|^{p^*}dx=o(1)+\epsilon(R') \tag{56}$$

Now we ditinguish two cases. In the first case, we assume that $d_g(x_\alpha^s,x_\alpha^N)/\mu_\alpha^N$ is such that $d_g(x_\alpha^s,x_\alpha^N)/\mu_\alpha^N\to+\infty$ as $\alpha\to\infty$. Then $d_g(x_\alpha^s,x_\alpha^N)/\mu_\alpha^s\to+\infty$ since, if not, we get by (56) with $\tilde{R}$ large enough that $\mu_\alpha^s/\mu_\alpha^N\to 0$, while

$$\frac{d_g\left(x_\alpha^s,x_\alpha^N\right)}{\mu_\alpha^s}=\frac{d_g\left(x_\alpha^s,x_\alpha^N\right)}{\mu_\alpha^N}\frac{\mu_\alpha^N}{\mu_\alpha^s}$$

It follows that $B_{x_\alpha^N}(R\mu_\alpha^N)\bigcap B_{x_\alpha^s}(\tilde{R}\mu_\alpha^s)=\emptyset$ for $\tilde{R}>0$ and we may proceed as in the case where $d_g(x_\alpha^s,x_\alpha^N)$ does not converge to 0 to get that (52) holds for s-1. In the second case, we assume that the $d_g(x_\alpha^s,x_\alpha^N)/\mu_\alpha^N$'s converge as $\alpha\to+\infty$. By (56), we must have that $\mu_\alpha^s/\mu_\alpha^N\to 0$. We set $y_\alpha^{p+1}=x_\alpha^s$ and $\lambda_\alpha^{p+1}=\mu_\alpha^s$. Clearly, by (52),

$$\int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)\setminus\bigcup_{j=1}^{p+1}B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)}\left|v_\alpha-\sum_{i=1}^{s}u_\alpha^i-u_\alpha^N\right|^{p^*}dv_g=o(1)+\epsilon(R')$$

while

$$\begin{aligned}\int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)\setminus\bigcup_{j=1}^{p+1}B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)}|u_\alpha^s|^{p^*}dv_g&\le\int_{M\setminus B_{x_\alpha^s}\left(R'\mu_\alpha^s\right)}|u_\alpha^s|^{p^*}dv_g\\&=\epsilon(R')\end{aligned}$$

It follows that

$$\int_{B_{x_\alpha^N}\left(R\mu_\alpha^N\right)\setminus\bigcup_{j=1}^{p+1}B_{y_\alpha^j}\left(R'\lambda_\alpha^j\right)}\left|v_\alpha-\sum_{i=1}^{s-1}u_\alpha^i-u_\alpha^N\right|^{p^*}dv_g=o(1)+\epsilon(R')$$

and (52) holds for s-1. As already mentioned, this implies that (49) is true, and thus that u^0 and the u^i's of Sect. 1 are nonnegative if the u_αs are nonnegative.

4 The C^0-estimate

By the maximum principle and standard regularity results as in Druet [8], Guedda-Véron [16], and Tolksdorf [24], $u^0 \in C^{1,\theta}(M)$ and either $u^0 > 0$ everywhere, or $u^0 \equiv 0$. In order to prove the C^0-estimate of Theorem 0.1 it suffices thus to prove that there exists $C > 0$ such that for any α, and any x,

$$\left(\min_{i=1,\ldots,k} d_g\left(x_\alpha^i, x\right)\right)^{\frac{n-p}{p}} u_\alpha(x) \le C \tag{57}$$

where d_g is the distance with respect to the metric g, and the x_α^i's are the centers of the bubbles $(B_\alpha^i)_\alpha$ in Theorem 0.1. We define the function Φ_α by

$$\Phi_\alpha(x) = \min_{i=1,\ldots,k} d_g\left(x_\alpha^i, x\right)$$

and the function v_α by

$$v_\alpha(x) = \Phi_\alpha^{\frac{n-p}{p}}(x) u_\alpha(x)$$

We let $y_\alpha \in M$ be such that

$$v_\alpha(y_\alpha) = \max_{x \in M} v_\alpha(x)$$

and assume by contradiction that $v_\alpha(y_\alpha) \to +\infty$ as $\alpha \to +\infty$. We let $\mu_\alpha > 0$ be given by $\mu_\alpha = u_\alpha(y_\alpha)^{-p/(n-p)}$. Then $\mu_\alpha \to 0$ as $\alpha \to +\infty$. For $\delta \in (0, i_g)$, where i_g is the injectivity radius of (M, g), we define the function w_α in the Euclidean ball $B_0(\delta\mu_\alpha^{-1})$ of center 0 and radius $\delta\mu_\alpha^{-1}$ by

$$w_\alpha(x) = \mu_\alpha^{\frac{n-p}{p}} u_\alpha\left(\exp_{y_\alpha}(\mu_\alpha x)\right)$$

By the definition of y_α,

$$\lim_{\alpha \to +\infty} \frac{d_g\left(x_\alpha^i, y_\alpha\right)}{\mu_\alpha} = +\infty \tag{58}$$

for all i. For $R > 0$, $x \in B_0(R)$, and $i = 1, \ldots, k$, we write that

$$\begin{aligned} d_g\left(x_\alpha^i, \exp_{y_\alpha}(\mu_\alpha x)\right) &\ge d_g\left(x_\alpha^i, y_\alpha\right) - d_g\left(y_\alpha, \exp_{y_\alpha}(\mu_\alpha x)\right) \\ &\ge \Phi_\alpha(y_\alpha) - \mu_\alpha |x| \\ &\ge \left(1 - \frac{R\mu_\alpha}{\Phi_\alpha(y_\alpha)}\right) \Phi_\alpha(y_\alpha) \end{aligned}$$

Thanks to (58), the right hand side in the above equation is positive. Then, we can write that

$$\begin{aligned} w_\alpha(x) &= \frac{\mu_\alpha^{\frac{n-p}{p}} v_\alpha\left(\exp_{y_\alpha}(\mu_\alpha x)\right)}{\Phi_\alpha\left(\exp_{y_\alpha}(\mu_\alpha x)\right)^{\frac{n-p}{p}}} \\ &\le \left(1 - \frac{R\mu_\alpha}{\Phi_\alpha(y_\alpha)}\right)^{-\frac{n-p}{p}} \frac{u_\alpha(y_\alpha)^{-1} v_\alpha(y_\alpha)}{\Phi_\alpha(y_\alpha)^{\frac{n-p}{p}}} \\ &= \left(1 - \frac{R\mu_\alpha}{\Phi_\alpha(y_\alpha)}\right)^{-\frac{n-p}{p}} \end{aligned}$$

and we get that the w_α's are uniformly bounded in any compact subset of $\mathbb{R}^n$. Let g_α be the Riemannian metric on $\mathbb{R}^n$ given by

$$g_\alpha(x) = \left(\exp^\star_{y_\alpha} g\right)(\mu_\alpha x)$$

Then, Eq. (2) becomes

$$(\Delta_p)_{g_\alpha} w_\alpha + \mu_\alpha^p \tilde{h}_\alpha w_\alpha^{p-1} = w_\alpha^{p^\star-1} \tag{59}$$

where $\tilde{h}_\alpha(x) = h_\alpha(\exp_{y_\alpha}(\mu_\alpha x))$. For any compact subset K of $\mathbb{R}^n$, $g_\alpha \to \xi$, the Euclidean metric, in $C^2(K)$ as $\alpha \to +\infty$. By (59) and the De Giorgi-Nash-Moser iterative scheme we then get the existence of $C > 0$, independent of α, such that for any α,

$$\sup_{x\in B_0(1)} w_\alpha(x) \le C\left(\int_{B_0(2)} w_\alpha^{p^\star} dv_{g_\alpha}\right)^{1/p^\star} \tag{60}$$

Independently,

$$\int_{B_{y_\alpha}(2\mu_\alpha)} u_\alpha^{p^\star} dv_g = \int_{B_0(2)} w_\alpha^{p^\star} dv_{g_\alpha} \tag{61}$$

while, thanks to the H_1^p-decomposition of the first part of Theorem 0.1,

$$\int_{B_{y_\alpha}(2\mu_\alpha)} u_\alpha^{p^\star} dv_g = \int_{B_{y_\alpha}(2\mu_\alpha)} \left(u^0 + \sum_{i=1}^k B_\alpha^i + R_\alpha\right)^{p^\star} dv_g$$

where u^0, the B_α^i's and the R_α's are as in Theorem 0.1. In particular, since u^0 is a continuous function,

$$\int_{B_{y_\alpha}(2\mu_\alpha)} u_\alpha^{p^\star} dv_g \le C \sum_{i=1}^k \int_{B_{y_\alpha}(2\mu_\alpha)} \left(B_\alpha^i\right)^{p^\star} dv_g + o(1) \tag{62}$$

where $C > 0$ is independent of α, and $o(1) \to 0$ as $\alpha \to +\infty$. We fix $i = 1, \ldots, k$, and let the x_α^i's and $\mu_\alpha^i = (R_\alpha^i)^{-1}$'s be the centers and weights of $(B_\alpha^i)_\alpha$ as given by (4). We distinguish two cases: Case 1: For any $R > 0$, and any α, $B_{y_\alpha}(2\mu_\alpha) \cap B_{x_\alpha^i}(R\mu_\alpha^i) = \emptyset$, and Case 2: There exists $R > 0$ such that for any α, $B_{y_\alpha}(2\mu_\alpha) \cap B_{x_\alpha^i}(R\mu_\alpha^i) \ne \emptyset$. Up to a subsequence, we are either in case 1 or in case 2. In case 1 we write that

$$\int_{B_{y_\alpha}(2\mu_\alpha)} \left(B_\alpha^i\right)^{p^\star} dv_g \le \int_{M\backslash B_{x_\alpha^i}\left(R\mu_\alpha^i\right)} \left(B_\alpha^i\right)^{p^\star} dv_g$$

and noting that

$$\lim_{R\to+\infty} \lim_{\alpha\to+\infty} \int_{M\backslash B_{x_\alpha^i}\left(R\mu_\alpha^i\right)} \left(B_\alpha^i\right)^{p^\star} dv_g = 0$$

we get that

$$\int_{B_{y_\alpha}(2\mu_\alpha)} \left(B_\alpha^i\right)^{p^\star} dv_g = o(1) \tag{63}$$

In case 2 we have that $d_g(x_\alpha^i, y_\alpha) \le 2\mu_\alpha + R\mu_\alpha^i$, and it follows from (58) that $\mu_\alpha = o(\mu_\alpha^i)$ and that $d_g(x_\alpha^i, y_\alpha) = O(\mu_\alpha^i)$. Writing that

$$B_{y_\alpha}(2\mu_\alpha) \subset \exp_{x_\alpha^i}\left(\mu_\alpha^i B_{z_\alpha}\left(C\frac{2\mu_\alpha}{\mu_\alpha^i}\right)\right)$$

where

$$z_\alpha = \frac{1}{\mu_\alpha^i}\exp_{x_\alpha^i}^{-1}(y_\alpha)$$

converges in $\mathbb{R}^n$ (up to a subsequence) and $C > 1$ is independent of α, we then get that

$$\int_{B_{y_\alpha}(2\mu_\alpha)} (B_\alpha^i)^{p^\star} dv_g \le \int_{B_{z_\alpha}\left(C\frac{2\mu_\alpha}{\mu_\alpha^i}\right)} u^{p^\star} dv_{g_\alpha}$$

where u is given by (4). Since $\mu_\alpha = o(\mu_\alpha^i)$,

$$\int_{B_{z_\alpha}\left(C\frac{R\mu_\alpha}{\mu_\alpha^i}\right)} u^{p^\star} dv_{g_\alpha} = o(1)$$

and we get here again that

$$\int_{B_{y_\alpha}(2\mu_\alpha)} (B_\alpha^i)^{p^\star} dv_g = o(1) \tag{64}$$

In particular, thanks to (63) and (64), we get that, up to a subsequence,

$$\int_{B_{y_\alpha}(2\mu_\alpha)} (B_\alpha^i)^{p^\star} dv_g = o(1) \tag{65}$$

for all $i = 1, \ldots, k$. Combining (61), (62) and (65), it follows that

$$\lim_{\alpha\to+\infty}\int_{B_0(2)} w_\alpha^{p^\star} dv_{g_\alpha} = 0 \tag{66}$$

Noting that $w_\alpha(0) = 1$, we then get a contradiction by combining (66) and (60). This proves the C^0-estimate of Theorem 0.1.

As a consequence of the C^0-estimate, we see that the u_α's are uniformly bounded on any compact subset of $M - \mathcal{S}$ where

$$\mathcal{S} = \left\{\lim_{\alpha\to\infty} x_\alpha^i, i = 1, \ldots, k\right\}$$

is the set of the geometrical blow-up points of the u_α's ($\mathcal{S} = \emptyset$ if the u_α's don't blow-up).
Since $u_\alpha \to 0$ in $H_{1,loc}^p(M - \mathcal{S})$, the Moser iterative scheme implies that, up to a subsequence,

$$u_\alpha \to 0 \text{ in } C_{loc}^0(M - \mathcal{S}) \tag{67}$$

In order to conclude, we need to prove the remark after the theorem. Namely that

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\sup_{x\in M\backslash\Omega_\alpha(R)} R_\alpha^k(x)^{\frac{n-p}{p}}\left|u_\alpha(x)-u^0(x)\right|=0 \tag{68}$$

where, $R_\alpha^k(x)=\min_{i=1,\dots,k} d_g(x_\alpha^i,x)$, and, for $R>0$

$$\Omega_\alpha(R)=\bigcup_{i=1}^k B_{x_\alpha^i}\left(R\mu_\alpha^i\right)$$

We prove (68) by contradiction and assume that there exists a sequence $(y_\alpha)_\alpha$ of points in M, and that there exists $\delta_0>0$ such that for any $i=1,\dots,k$,

$$\frac{d_g\left(x_\alpha^i,y_\alpha\right)}{\mu_\alpha^i}\to+\infty \tag{69}$$

as $\alpha\to+\infty$, and such that for any α,

$$R_\alpha^k(y_\alpha)^{\frac{n-p}{p}}\left|u_\alpha(y_\alpha)-u^0(y_\alpha)\right|\geq\delta_0\ . \tag{70}$$

Clearly, $R_\alpha^k(y_\alpha)\to 0$ as $\alpha\to+\infty$ since if not, by (67), $u_\alpha(y_\alpha)-u^0(y_\alpha)\to 0$ as $\alpha\to+\infty$ which contradicts (70). We let $\mu_\alpha=u_\alpha(y_\alpha)^{-p/(n-p)}$. Then we can rewrite (70) as

$$\frac{R_\alpha^k(y_\alpha)}{\mu_\alpha}\geq\delta_1\ , \tag{71}$$

where $\delta_1^{(n-p)/p}=\delta_0/2$. In particular, $\mu_\alpha\to 0$ as $\alpha\to+\infty$. Given $\delta>0$ less than the injectivity radius of (M,g), we define the function w_α in the Euclidean ball $B_0(\delta\mu_\alpha^{-1})$ by

$$w_\alpha(x)=\mu_\alpha^{\frac{n-p}{p}}u_\alpha\left(\exp_{y_\alpha}(\mu_\alpha x)\right)$$

and let g_α be the metric given by $g_\alpha(x)=(\exp_{y_\alpha}^\star g)(\mu_\alpha x)$. For any compact subset K of $\mathbb{R}^n$, and if ξ stands for the Euclidean metric, we have that $g_\alpha\to\xi$ in $C^2(K)$ as $\alpha\to+\infty$. By (71) we can write that if (x_α) is a sequence in $B_0(\delta_1/2)$, then

$$\begin{aligned}d_g\left(x_\alpha^i,\exp_{y_\alpha}(\mu_\alpha x_\alpha)\right)&\geq d_g\left(y_\alpha,x_\alpha^i\right)-d_g\left(y_\alpha,\exp_{y_\alpha}(\mu_\alpha x_\alpha)\right)\\&\geq\delta_1\mu_\alpha-|x_\alpha|\mu_\alpha\end{aligned}$$

for all i and all α. In particular, $d_g(x_\alpha^i,\exp_{y_\alpha}(\mu_\alpha x_\alpha))\geq C\mu_\alpha$ for some $C>0$ independent of α, and up to a subsequence, we get with the C^0-estimate that

$$w_\alpha(x)\leq C \tag{72}$$

for all $x\in B_0(\delta_1/2)$ and all α, where $C>0$ is independent of α and x. Now we may follow the arguments of the proof of the C^0-estimate. On the one hand, the w_α's are solutions of an equation like

$$(\Delta_p)_{g_\alpha}w_\alpha+\mu_\alpha^p\tilde{h}_\alpha w_\alpha^{p-1}=w_\alpha^{p^*-1} \tag{73}$$

in $B_0(\delta_1/2)$, where $\tilde{h}_\alpha(x) = h_\alpha(exp_{y_\alpha}(\mu_\alpha x))$. On the other hand, they are bounded in $B_0(\delta_1/2)$ by (72). We then can assume (see [19]) that, up to a subsequence, $w_\alpha \to w$ in $C^0(B_0(\delta_1/8))$ as $\alpha \to +\infty$ where w satisfies

$$\Delta_p w = w^{p^*-1}$$

Moreover

$$w(0) = 1$$

since $w_\alpha(0) = 1$ for all α.
Let $\delta_2 = \delta_1/8$. We have that

$$\begin{aligned}\int_{B_{y_\alpha}(\delta_2\mu_\alpha)} u_\alpha^{p^*} dv_g &= \int_{B_0(\delta_2)} w_\alpha^{p^*} dv_{g_\alpha} \\ &= \int_{B_0(\delta_2)} w^{p^*} dx + o(1) \ ,\end{aligned} \tag{74}$$

where $o(1) \to 0$ as $\alpha \to +\infty$, while, by the H_1^p-decomposition of the first part of theorem 0.1,

$$\int_{B_{y_\alpha}(\delta_2\mu_\alpha)} u_\alpha^{p^*} dv_g \le C \sum_{i=1}^{k} \int_{B_{y_\alpha}(\delta_2\mu_\alpha)} \left(B_\alpha^i\right)^{p^*} dv_g + o(1) \ , \tag{75}$$

where $C > 0$ is independent of α. Independently, as in the proof of C^0-estimate, see equation (65), we have that

$$\int_{B_{y_\alpha}(\delta_2\mu_\alpha)} \left(B_\alpha^i\right)^{p^*} dv_g = o(1) \tag{76}$$

for all i. We prove (76) as we prove (65) by considering the two cases where $B_{y_\alpha}(\delta_2\mu_\alpha) \cap B_{x_\alpha^i}(R\mu_\alpha^i) = \emptyset$ for all $R > 0$, and $B_{y_\alpha}(\delta_2\mu_\alpha) \cap B_{x_\alpha^i}(R\mu_\alpha^i) \neq \emptyset$ for some $R > 0$. In the second case we recover (58) thanks to (69) by noting that (71) and the nonempty intersection give that $\delta_1\mu_\alpha \le \delta_2\mu_\alpha + R\mu_\alpha^i$ so that $\mu_\alpha \le C\mu_\alpha^i$. Then, combining (74)-(76), we get that w satisfies

$$\int_{B_0(\delta_2)} w^{p^*} dx = 0$$

and this is impossible since w is continuous, nonnegative, and such that $w(0) = 1$. This proves Lemma 76).

References

1. Alves, C.O.: Existence of positive solutions for a problem with lack of compactness involving the p-Laplacian. Nonlinear Anal. **51**(7), 1187–1206 (2002)
2. Brezis, H., Coron, J.M.: Convergence of solutions of H-systems or how to blow bubbles. Arch. Rational Mech. Anal. **89**(1), 21–56 (1985)
3. Brezis, H., Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc. **88**(3), 486–490 (1983)

4. Brezis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math. **36**(4), 437–477 (1983)
5. Damascelli, L., Pacella, F.: Monotonicity and symmetry results for p-Laplace equations and applications. Adv. Differential Equations **5**, 1179–1200 (2000)
6. Damascelli, L., Pacella, F., Ramaswamy, M.: Symmetry of ground states of p-Laplace equations via the moving plane method. Arch. Rational Mech. Anal. **148**, 291–308 (1999)
7. Demengel, F., Hebey, E.: On some nonlinear equations involving the p-Laplacian with critical Sobolev growth. Adv. Differential Equations **3**(4), 533–574 (1998)
8. Druet, O.: Generalized scalar curvature type equations on compact Riemannian manifolds. Proc. Roy. Soc. Edinburgh Sect. A **130**(4), 767–788 (2000)
9. Druet, O.: Isoperimetric inequalities on compact manifolds. Geom. Dedicata **90**, 217–236 (2002)
10. Druet, O.: Sharp local isoperimetric inequalities involving the scalar curvature. Proc. Amer. Math. Soc. **130**(8), 2351–2361 (2002)
11. Druet, O.: The best constants problem in Sobolev inequalities. Math. Ann. **314**(2), 327–346 (1999)
12. Druet, O., Hebey, E.: The AB program in geometric analysis: sharp Sobolev inequalities and related problems. Mem. Amer. Math. Soc. **160**(761), (2002)
13. Druet, O., Hebey, E., Robert, F.: A C^0-theory for the blow-up of second order elliptic equations of critical Sobolev growth. Electron. Res. Announc. Amer. Math. Soc. **9**, 19–25 (2003)
14. Evans, L.C.: Weak convergence methods for nonlinear partial differential equations. Conference Board of the Mathematical Sciences **74** (1990)
15. Ghoussoub, N., Yuan, C.: Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. Trans. Amer. Math. Soc. **352**(12), 5703–5743 (2000)
16. Guedda, M., Veron, L.: Quasilinear elliptic equations involving critical Sobolev exponents. Nonlinear Anal. **13**(8), 879–902 (1989)
17. Hebey, E.: Nonlinear analysis on manifolds: Sobolev spaces and inequalities. Courant Lecture Notes in Mathematics **5** (1999)
18. Hebey, E., Robert, F.: Coercivity and Struwe's compactness for Paneitz type operators with constant coefficients. Calc. Var. Partial Differential Equations **13**(4), 491–517 (2001)
19. Ladyzhenskaya, O., Ural'tseva, N.: Linear and Quasilinear Elliptic Equations. Academic Press (1968)
20. Lions, P.L: The concentration-compactness principle in the calculus of variations, the limit case, parts 1 and 2. Rev. Mat. Iberoamericana **1**(1/2), 145–201, 45–121 (1985)
21. Sacks, J., Uhlenbeck, K.: The existence of minimal immersions of two-spheres. Ann. of Math. (2), **113**(1), 1–24 (1981)
22. Schoen, R., Zhang, D.: Prescribed scalar curvature on the n-sphere. Calc. Var. Partial Differential Equations **4**(1), 1–25 (1996)
23. Struwe, M.: Variational methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Third edition. Springer-Verlag (2000)
24. Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. J. Differential Equations **51**(1), 126–150 (1984)