Arch. Math. 00 (2005) 000–000
0003–889X/05/000000–00
DOI 10.1007/s00013-005-1531-7

Archiv der Mathematik

Stability and perturbations of the domain for the first eigenvalue of the 1-Laplacian

By

EMMANUEL HEBEY and NICOLAS SAINTIER

Abstract. We study the dependence of the first eigenvalue of the 1-Laplacian with respect to perturbations of the domain. We provide results ranging from general type of perturbations to regular perturbations by diffeomorphisms.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geqq 2$. The 1-Laplacian on Ω is the formal operator

$$\Delta_1 u = -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

we get by a formal derivation of $F(u) = \int_\Omega |\nabla u| dx$, or by letting $p \to 1$ in the definition of the p-Laplacian, $p > 1$. By analogy with the definition of the first eigenvalue $\lambda_{p,\Omega}$ of the p-Laplacian on Ω, we define the first eigenvalue $\lambda_{1,\Omega}$ of the 1-laplacian on Ω by the minimization problem

$$(0.1) \qquad \lambda_{1,\Omega} = \inf_{\begin{cases} u \in \dot{H}_1^1(\Omega) \\ \int_\Omega |u| dx = 1 \end{cases}} \int_\Omega |\nabla u| dx,$$

where $\dot{H}_1^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H_1^1(\Omega)$ of functions in $L^1(\Omega)$ with one derivative in L^1. By geometric measure theory, and the coarea formula, we also have that $\lambda_{1,\Omega} = h(\Omega)$, where $h(\Omega)$ is defined as the infimum of the ratio $|\partial D|/|D|$, D varies over all smooth subdomains $D \subset\subset \Omega$, and $|\partial D|$ and $|D|$ are the $(n-1)$-dimensional and n-dimensional measures of ∂D and D. The result is known as Cheeger's theorem [5], and $h(\Omega)$ is known as the Cheeger constant of Ω (see for instance Chavel [4]). Note the infimum in $h(\Omega)$ is not attained by a smooth subdomain $D \subset\subset \Omega$ since, if not, we may

Mathematics Subject Classification (2000): Primary 52A40; Secondary 49Q20.

blow it up by a factor larger than one. This would decrease h, contradicting the optimality of D. Minimizers for $h(\Omega)$ touch the boundary $\partial\Omega$.

The main purpose of this paper is the study of the dependence of $\lambda_{1,\Omega}$ under perturbations of Ω. The notion of perturbation is here quantified by mean of the 1-capacity. We provide results ranging from general type of perturbations to regular perturbations by diffeomorphisms. This type of problem has been widely studied in the case of the Laplacian, hardly in the case of the p-Laplacian and, as far as we know, has not been studied before in the case of the 1-laplacian. A natural space to study $\lambda_{1,\Omega}$ is $BV(\Omega)$, the space of functions of bounded variations (see, for instance, Evans and Gariepi [9], or Giusti [12]). By standard properties of the space $BV(\Omega)$, we can also define $\lambda_{1,\Omega}$ by the equation

$$\lambda_{1,\Omega} = \inf_{\begin{cases} u \in BV(\Omega) \\ \int_\Omega |u|dx = 1 \end{cases}} \left(\int_\Omega |\nabla u|dx + \int_{\partial\Omega} |u|d\sigma \right). \tag{0.2}$$

Note here that if $u \in BV(\Omega)$, and $\overline{u}$ is the extension of u by 0 in $\mathbb{R}^n \backslash \overline{\Omega}$, then $\overline{u} \in BV(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |\nabla \overline{u}|dx = \int_\Omega |\nabla u|dx + \int_{\partial\Omega} |u|d\sigma \tag{0.3}$$

By lower semicontinuity of the total variation, and compactness of the embedding $BV(\Omega) \subset L^1(\Omega)$, it easily follows from (0.3) that the infimum in (0.2) is attained by some nonnegative $u \in BV(\Omega)$. Then u is a solution of the equation $\Delta_1 u = \lambda_{1,\Omega}$ in the sense that there exists $\Lambda \in L^\infty(\Omega, \mathbb{R}^n)$, $\|\Lambda\|_\infty \leqq 1$, such that

$$\begin{cases} -\mathrm{div}\Lambda = \lambda_{1,\Omega}, u \geqq 0, \\ \Lambda \nabla u = |\nabla u| \text{ in } \Omega, \text{ and} \\ (\Lambda \nu)u = -|u| \text{ on} \partial\Omega, \end{cases} \tag{0.4}$$

where ν is the unit outer normal to $\partial\Omega$, and $\Lambda\nabla u$ is the distribution defined by integrating by parts $\int_\Omega (\Lambda\nabla u)v dx$ when $v \in C_0^\infty(\Omega)$ and $\mathrm{div}\Lambda$ makes sense. Moreover, see for instance Demengel [6], $u \in L^\infty(\Omega)$ and, since $\|u\|_1 = 1$, there exists $C = C(n, \Omega)$, $C > 0$, such that $\|u\|_\infty \leqq C$. We say u is an eigenfunction for $\lambda_{1,\Omega}$. Now define a Cacciopoli set in Ω as a set $D \subset \Omega$ such that $\chi_D \in BV(\Omega)$, where χ_D is the characteristic function of D. Since $\lambda_{1,\Omega} = h(\Omega)$, there are Caccioppoli sets $D \subset \Omega$, e.g., the level sets of eigenfunctions, such that $u = |D|^{-1}\chi_D$ is a minimizer for the right hand side in (0.2). Such sets are referred to as eigensets for $\lambda_{1,\Omega}$ (and sometimes also as Cheeger's sets). A general discussion about uniqueness and nonuniqueness of eigensets is in Fridman and Kawohl [10]. We refer also to Stredulinsky and Ziemer [18] (and to Belloni and Kawohl [3] for the p-Laplace case). Concerning regularity, possible references are Almgren [1], De Giorgi [8], Gonzales, Massari and Tamanini [13], and Stredulinsky and Ziemer [18]. By symmetrization, see Fridman and Kawohl [10] for details,

$$\lambda_{1,\Omega} \geqq n\omega_n^{1/n}|\Omega|^{-1/n},$$

where ω_n is the volume of the unit n-sphere.

Let K be a compact subset of $\mathbb{R}^n$. The 1-capacity of K, denoted by $\mathrm{cap}_1(K)$, is defined as the infimum of the L^1-norm of $|\nabla u|$, where the infimum is taken over all $u \in C_0^\infty(\mathbb{R}^n)$ such that $K \subset \mathrm{int}\{u \geqq 1\}$. Another possible definition (see, for instance, Maz'ja [16]) is that $\mathrm{cap}_1(K) = \inf |\partial\omega|$, where the infimum is taken over all smooth open bounded subset ω such that $K \subset \omega$. In particular, by the isoperimetric inequality, $|K|^{(n-1)/n} \leqq C\ \mathrm{cap}_1(K)$, where $C > 0$ does not depend on K (but only on the dimension). For A and B two subsets of $\mathbb{R}^n$, we denote by $A\Delta B$ the symmetric difference of A and B. Namely, $A\Delta B = (A\backslash B)\cup(B\backslash A)$. Our first result deals with general type of perturbations of a domain. It states as follows.

Theorem 0.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $(\Omega_\delta)_{\delta>0}$ be a sequence of smooth bounded domains in $\mathbb{R}^n$, and $K_\delta = adh(\Omega\Delta\Omega_\delta)$ be the closure of the symmetric difference $\Omega\Delta\Omega_\delta$. Let (A_δ) be a sequence of eigensets for the $\lambda_{1,\Omega_\delta}$'s. Assume* $\mathrm{cap}_1(K_\delta) \to 0$ *as $\delta \to 0$. Then, for any δ,*

$$\left|\lambda_{1,\Omega_\delta} - \lambda_{1,\Omega}\right| = \frac{\varepsilon_\delta}{|A_\delta|}\ \mathrm{cap}_1(K_\delta) + o(\mathrm{cap}_1(K_\delta)), \tag{0.5}$$

where $\varepsilon_\delta \in [0, 1]$ for all δ, and $|A_\delta| \geqq \Lambda$ for some $\Lambda > 0$ and all δ. In particular, $\lambda_{1,\Omega_\delta} \to \lambda_{1,\Omega}$ as $\delta \to 0$. Moreover, up to a subsequence, $\chi_{A_\delta} \to \chi_A$ in $L^1(\mathbb{R}^n)$ as $\delta \to 0$, where A is an eigenset for $\lambda_{1,\Omega}$.

We stated the second part of Theorem 0.1 for characteristic functions of eigensets. However, note the convergence holds also for eigenfunctions. In Section 1 we prove that if the u_δ's are eigenfunctions for $\lambda_{1,\Omega_\delta}$, then, up to a subsequence, with the notation in (0.3), $\overline{u}_\delta \to \overline{u}$ in $L^1(\mathbb{R}^n)$ as $\delta \to 0$, where u is an eigenfunction for $\lambda_{1,\Omega}$. We also get that the measures $\mu_\delta = |\nabla\overline{u}_\delta|$ and $\mu = |\nabla\overline{u}|$ satisfy $\mu_\delta \rightharpoonup \mu$ weakly as $\delta \to 0$.

A particular case of the general perturbations considered in Theorem 0.1 is when we consider domains with holes. Such domains (in the case of one ball) were considered by Sango [17] when discussing the first eigenvalue of the p-Laplace operator, $p > 1$. For A a Caccioppoli set, we let $\partial^\star A$ be its reduced boundary, namely (see [9] or [12]) the subset of the boundary which is C^1 in a measure theoretic sense. For $x \in \mathbb{R}^n$ and $r > 0$, we let also $B_x(r)$ be the n-dimensional ball of center x and radius r, and b_n be the volume of the unit n-dimensional ball.

Theorem 0.2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain in $\mathbb{R}^n$, A be an eigenset for $\lambda_{1,\Omega}$, $x_1, \dots, x_k$ be points in Ω, $(\varepsilon_{i,\delta})_{\delta>0}$, $i = 1, \dots, k$, be k sequences of positive real numbers converging to 0 as $\delta \to 0$, $K_\delta = \bigcup_{i=1}^{k} \overline{B}_{x_i}(\varepsilon_{i,\delta})$, $\delta > 0$ small, and $\Omega_\delta = \Omega\backslash K_\delta$. Assume there exists $i_0 = 1, \dots, k$ such that $\varepsilon_{i,\delta} = o(\varepsilon_{i_0,\delta})$ for all $i = 1, \dots, k$, $i \neq i_0$. Then,* $\mathrm{cap}_1(K_\delta) = \omega_{n-1}\varepsilon_{i_0,\delta}^{n-1} + o(\varepsilon_{i_0,\delta}^{n-1})$, *and*

$$\lambda_{1,\Omega} \leqq \lambda_{1,\Omega_\delta} \leqq \lambda_{1,\Omega} + \frac{\omega_{n-1}}{|A|}\varepsilon_{i_0,\delta}^{n-1} + o(\varepsilon_{i_0,\delta}^{n-1}) \tag{0.6}$$

for all δ*. Moreover,*

$$\lambda_{1,\Omega_\delta} \leqq \lambda_{1,\Omega} + \begin{cases} o(\varepsilon_{i_0,\delta}^{n-1}) \text{ if } x_{i_0} \in \mathrm{int}(\Omega\backslash A) \\ \frac{\omega_{n-1}-2b_{n-1}}{2|A|}\varepsilon_{i_0,\delta}^{n-1} + o(\varepsilon_{i_0,\delta}^{n-1}) \text{ if } x_{i_0} \in \partial^\star A, \end{cases} \tag{0.7}$$

where $\mathrm{int}(\Omega\backslash A)$ *is the interior of* $\Omega\backslash A$*, and* $\partial^\star A$ *is the reduced boundary of* A*.*

Needless to say, $\mathrm{cap}_1(K_\delta) \to 0$ as $\delta \to 0$ and the convergence of eigensets (resp. eigenfunctions) stated in Theorem 0.1 holds true. Characterizations in dimension 2 of convex Ω's for which $\Omega = A$ (as a by-product for which $\Omega\backslash A \neq \emptyset$, and thus for which equation (0.7) is not empty) are in Bellettini, Caselles and Novaga [2], and Kawohl and Lachand-Robert [14]. When Ω is convex and $n = 2$, the eigenset $A = A_\Omega$ is unique. The domain Ω is said to be a calibrable set when $A = \Omega$. The ovoid domain $(x^2 + y^2)^2 < x^3$ is a nice example in Kawohl and Lachand-Robert [14] of a noncalibrable set. With respect to the notation in Theorem 0.1, (0.7) gives that we can take $\varepsilon_\delta = 0$ if $x_{i_0} \in \mathrm{int}(\Omega\backslash A)$, and $\varepsilon_\delta < 1/2$ if $x_{i_0} \in \partial^\star A$. On the other hand, when $x_{i_0} \in \mathrm{int}(A)$, the upper bound in (0.6), where $\varepsilon_\delta = 1$, cannot (in general) be improved. If we let $\Omega = B_0(r)$ and $K_\varepsilon = \overline{B}_0(\varepsilon)$ be n-dimensional balls, $r > 0$, $0 < \varepsilon \ll 1$, then both Ω and the annulus $\Omega_\varepsilon = \Omega\backslash K_\varepsilon$ are calibrable sets (see, for instance, Kawohl and Lachand-Robert [14], Demengel, De Vuyst, and Motron [7]). In particular, the eigenvalues $\lambda_{1,\Omega}$ and $\lambda_{1,\Omega_\varepsilon}$ are given by $\lambda_{1,\Omega} = n/r$ and $\lambda_{1,\Omega_\varepsilon} = n(r^{n-1} + \varepsilon^{n-1})/(r^n - \varepsilon^n)$, and we get that

$$\lambda_{1,\Omega_\varepsilon} = \lambda_{1,\Omega} + \frac{\omega_{n-1}}{|A|}\varepsilon^{n-1} + o(\varepsilon^{n-1})$$

for all ε, where $A = \Omega$ is the eigenset of $\lambda_{1,\Omega}$. Without further assumptions, the upper bound in (0.6) is sharp.

We now consider the case of a regular perturbation of Ω by diffeomorphisms. We prove the differentiability at 0 of the map $\delta \to \lambda_{1,\Omega_\delta}$ and the convergence of the eigenfunctions without assumptions on the capacity of K_δ. In the case of the p-Laplacian, $p > 1$, such type of problems have been considered by Lamberti [15] and Garcia Melian and Sabina De Lis [11]. For Ω a smooth bounded open subset of $\mathbb{R}^n$, we let $(T_\delta)_\delta$ be a family of C^1-diffeomorphisms of the form

$$T_\delta(x) = (1 - \delta\Lambda)x + R(x, \delta), \tag{0.8}$$

where $x \in \overline{\Omega}$, $\Lambda \in \mathbb{R}$, $\delta \in (-\delta_0, \delta_0)$, $\delta_0 > 0$ is small, and $R(., \delta) \in C^1(\overline{\Omega}, \mathbb{R}^n)$ is a perturbative term such that $R(x, \delta) = o(\delta)$ and $D_x R(x, \delta) = o(\delta)$ as $\delta \to 0$, uniformly in x. In particular, $R(x, 0) = 0$, and if $\Omega_\delta = T_\delta(\Omega)$, then $\Omega_0 = \Omega$.

Theorem 0.3. *Let* Ω *be a smooth bounded open subset of* $\mathbb{R}^n$*, and* $\Omega_\delta = T_\delta(\Omega)$*, where the* T_δ*'s are* C^1*-diffeomorphisms like in* (0.8)*. The function* $\delta \to \lambda_{1,\Omega_\delta}$ *is continuous and differentiable at* $\delta = 0$*, and* $(\lambda_{1,\Omega_\delta})'(0) = \Lambda\lambda_{1,\Omega}$*, where* Λ *is as in* (0.8)*.*

In these examples of Theorem 0.3, the 1-capacity of $K_\delta = \mathrm{adh}(\Omega\Delta\Omega_\delta)$ can be large. For instance, if $\Omega = B_0(r)$ and $T_\delta = (1 - \delta\Lambda)x$, $\Lambda \neq 0$, then K_δ is an annulus with inner (or outer, depending on the sign of Λ) radius r. In particular, $\text{cap}_1(K_\delta) = \omega_{n-1}r^{n-1} + o(1)$, and $\text{cap}_1(K_\delta) \not\to 0$ as $\delta \to 0$. In other cases, the 1-capacity of K_δ may tend to zero, and we are back to the situation studied in Theorem 0.1. For instance, if $T_\delta(x) = (1 + \delta^\alpha R(\frac{1}{\delta}x))x$, $\alpha > 2$, $R \in C_0^1(\mathbb{R}^n)$, and $0 \in \partial\Omega$, then $K_\delta \subset B_0(r_0\delta)$ for some $r_0 > 0$, and $\text{cap}_1(K_\delta) \to 0$ as $\delta \to 0$. Combining Theorems 0.1 and 0.3, $\Lambda = 0$ if $\text{cap}_1(K_\delta) = o(\delta)$. The following sections are devoted to the proofs of the above theorems.

1. Proof of theorem 0.1. Let A_δ be an eigenset of $\lambda_{1,\Omega_\delta}$, $\delta > 0$ fixed. Let also ω_δ be a smooth bounded open subset such that $K_\delta \subset \omega_\delta$ and

$$\text{cap}_1(K_\delta) \leqq |\partial\omega_\delta| \leqq \text{cap}_1(K_\delta) + \varepsilon_\delta, \tag{1.1}$$

where $\varepsilon_\delta > 0$ is such that $\varepsilon_\delta = o(\text{cap}_1(K_\delta))$. We define v_δ by $v_\delta = \chi_{A_\delta}$ in $\mathbb{R}^n\backslash\overline{\omega}_\delta$, and $v_\delta = 0$ in ω_δ. Then $\text{supp}v_\delta \subset \overline{\Omega}$, where $\text{supp}v_\delta$ is the support of v_δ. Since $v_\delta \leqq 1$, we can write that

$$\int_{\mathbb{R}^n} |\nabla v_\delta| dx \ \leqq \int_{\mathbb{R}^n} |\nabla \chi_{A_\delta}| dx + |\partial\omega_\delta| = |A_\delta|\lambda_{1,\Omega_\delta} + (1 + o(1))\,\text{cap}_1(K_\delta), \tag{1.2}$$

where $o(1) \to 0$ as $\delta \to 0$. We can also write that

$$\int_\Omega v_\delta dx = |A_\delta| - \int_{\omega_\delta} \chi_{A_\delta} dx = |A_\delta| + O\left(|\omega_\delta|\right). \tag{1.3}$$

By the isoperimetric inequality, and by (1.1), $|\omega_\delta| \leqq C|\partial\omega_\delta|^{\frac{n}{n-1}} = o(\text{cap}_1(K_\delta))$, where $C > 0$ is a dimensional constant independent of δ. Coming back to (1.3), it follows that

$$\int_\Omega v_\delta dx = |A_\delta| + o(\text{cap}_1(K_\delta)) \tag{1.4}$$

and by the variational definition of $\lambda_{1,\Omega}$, (1.2), and (1.4), we get that

$$\lambda_{1,\Omega} \leqq \lambda_{1,\Omega_\delta} + \frac{1}{|A_\delta|}\text{cap}_1(K_\delta) + o(\text{cap}_1(K_\delta)) \tag{1.5}$$

for all $\delta > 0$. Similar arguments give that the converse inequality holds also. We let A be an eigenset for $\lambda_{1,\Omega}$, and let w_δ be the function given by $w_\delta = \chi_A$ in $\mathbb{R}^n\backslash\overline{\omega}_\delta$, and $w_\delta = 0$ in ω_δ, where ω_δ is as in (1.1). Then $\text{supp}w_\delta \subset \overline{\Omega}_\delta$, and, as above, we can write

$$\int_{\mathbb{R}^n} |\nabla w_\delta| dx \leqq |A|\lambda_{1,\Omega} + (1 + o(1))\,\text{cap}_1(K_\delta), \text{ and}$$

$$\int_{\Omega_\delta} w_\delta dx = |A| + o(\text{cap}_1(K_\delta)). \tag{1.6}$$

In particular, it follows from the variational definition of $\lambda_{1,\Omega_\delta}$ that

$$\lambda_{1,\Omega_\delta} \leqq \lambda_{1,\Omega} + \frac{1}{|A|}\mathrm{cap}_1(K_\delta) + o(\mathrm{cap}_1(K_\delta)) \tag{1.7}$$

for all $\delta > 0$. Without loss of generality, by the lower semicontinuity of the total variation, we may choose $A = A_0$ in (1.7) such that it is of maximum volume among the eigensets for $\lambda_{1,\Omega}$.

In what follows we let $u_\delta \in BV(\Omega_\delta)$ be an eigenfunction for $\lambda_{1,\Omega_\delta}$, like for instance $u_\delta = |A_\delta|^{-1}\chi_{A_\delta}$, and we assume that $\mathrm{cap}_1(K_\delta) \to 0$ as $\delta \to 0$. Then, by (1.5) and (1.7), $\lambda_{1,\Omega_\delta} \to \lambda_{1,\Omega}$ as $\delta \to 0$. Note here that $|A_\delta| \geqq C$ for some $C > 0$ (thanks for instance to the Sobolev inequality in $BV(\mathbb{R}^n)$ that we apply to the extensions by zero outside $\overline{\Omega}_\delta$ of the functions $|A_\delta|^{-1}\chi_{A_\delta}$). In what follows we let D be a smooth bounded open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset D$, and $\overline{\Omega}_\delta \subset D$ for all δ. We let $\overline{u}_\delta$ be the extension of u_δ by zero outside $\overline{\Omega}_\delta$. By (1.7), the sequence $(\overline{u}_\delta)$ is bounded in $BV(D)$. Then, by compactness of the embedding of $BV(D)$ into $L^1(D)$, we may assume that, up to a subsequence, $\overline{u}_\delta \to \overline{u}$ in $L^1(D)$ for some $\overline{u} \in BV(D)$. By the Sobolev inequality for BV-functions, we also have that the $\overline{u}_\delta$'s are bounded in the Lebesgue's space $L^{n/(n-1)}(D)$. On the one hand, we have that

$$\int_{D\backslash\Omega} |\overline{u}_\delta| dx \;\leqq\; \|\overline{u}_\delta\|_{L^{n/(n-1)}} |\Omega_\delta \backslash \Omega|^{1/n} \leqq C|\Omega_\delta\backslash\Omega|^{1/n}.$$

On the other hand, we can write with the isoperimetric inequality that

$$|\Omega_\delta\backslash\Omega| \leqq |K_\delta| \leqq C\mathrm{cap}_1(K_\delta)^{\frac{n}{n-1}} = o(\mathrm{cap}_1(K_\delta)),$$

where, as above, $C > 0$ is independent of δ. In particular, $\int_{D\backslash\Omega} |\overline{u}_\delta| dx = o(1)$. We regard $\overline{u}$ as a function in $\mathbb{R}^n$ (by letting $\overline{u} = 0$ outside D), and let $u = \overline{u}_{|\Omega}$ be the restriction of $\overline{u}$ to Ω. Then, according to what we just said, $\overline{u} = u$ in Ω, and $\overline{u} = 0$ in $\mathbb{R}^n\backslash\overline{\Omega}$. As is easily checked, $\int_\Omega |u| dx = 1$, while by lower semicontinuity of the total variation, and since $\lambda_{1,\Omega_\delta} \to \lambda_{1,\Omega}$, we can write that

$$\lambda_{1,\Omega} = \int_{\mathbb{R}^n} |\nabla\overline{u}_\delta| dx + o(1) \geqq \int_{\mathbb{R}^n} |\nabla\overline{u}| dx + o(1).$$

In particular, u is an eigenfunction for $\lambda_{1,\Omega}$, and

$$\lim_{\delta\to 0} \int_{\mathbb{R}^n} |\nabla\overline{u}_\delta| dx = \int_{\mathbb{R}^n} |\nabla\overline{u}| dx. \tag{1.8}$$

Letting $u_\delta = |A_\delta|^{-1}\chi_{A_\delta}$, we may assume $|A_\delta| \to \Lambda$ for some $\Lambda > 0$, and $\overline{u}_\delta \to \overline{u}$ a.e. Since $\int_\Omega |u| dx = 1$, we can write that $\overline{u} = |A|^{-1}\chi_A$ for some $A \subset \overline{\Omega}$. In particular, A is an eigenset for $\lambda_{1,\Omega}$, and $|A_\delta| \to |A|$ so that, by (1.5) and (1.7),

$$\lambda_{1,\Omega_\delta} = \lambda_{1,\Omega} + \frac{\varepsilon_\delta}{|A_\delta|}\mathrm{cap}_1(K_\delta) + o(\mathrm{cap}_1(K_\delta)),$$

where $\varepsilon_\delta \in [-1, 1]$ (since $|A| \geqq |A_0|$). This is equation (0.5) in Theorem 0.1, and the equation holds for all δ by contradiction. Noting that the convergence $\overline{u}_\delta \to \overline{u}$ in L^1 gives that $\chi_{A_\delta} \to \chi_A$ in L^1, Theorem 0.1 is proved.

For the remark following Theorem 0.1, there is still to prove that if $\mu_\delta = |\nabla \overline{u}_\delta|$ and $\mu = |\nabla \overline{u}|$, then $\mu_\delta \rightharpoonup \mu$ weakly as $\delta \to 0$. By lower semicontinuity of the total variation, $\mu(U) \leqq \liminf_{\delta \to 0} \mu_\delta(U)$ for all open subset U of $\mathbb{R}^n$. Conversely, let us assume that there exists a compact subset K of $\mathbb{R}^n$, and $\varepsilon > 0$ such that $\mu(K) + \varepsilon \leqq \mu_\delta(K)$ for a subsequence of the μ_δ's. Let Ω' be an open subset of $\mathbb{R}^n$ which contain $\overline{\Omega}$, K, and the $\overline{\Omega}_\delta$'s. Up to passing to another subsequence, by lower semicontinuity of the total variation, we can assume that $\mu(\Omega' \backslash K) \leqq \mu_\delta(\Omega' \backslash K) + \varepsilon'$ for all δ, where $\varepsilon' < \varepsilon$ is positive. Then, if $\hat{\varepsilon} = \varepsilon - \varepsilon'$, we can write that

$$\begin{aligned} \mu(\mathbb{R}^n) = \mu(\Omega') &= \mu(K) + \mu(\Omega' \backslash K) \\ &\leqq \mu_\delta(K) - \varepsilon + \mu_\delta(\Omega' \backslash K) + \varepsilon' \\ &= \mu_\delta(\Omega') - \hat{\varepsilon} = \mu_\delta(\mathbb{R}^n) - \hat{\varepsilon} \end{aligned}$$

for all δ, and we get a contradiction with (1.8) since $\hat{\varepsilon} > 0$. As a consequence, for any compact subset K of $\mathbb{R}^n$, $\mu(K) \geqq \limsup_{\delta \to 0} \mu_\delta(K)$ and, see for instance Evans-Gariepy [9], we actually proved that the measures μ_δ converge weakly to the measure μ.

2. Proof of Theorem 0.2. We now turn our attention to the Proof of Theorem 0.2. As is easily checked from the definition of the 1-capacity, the fact that the 1-capacity is an outer measure, and the isoperimetric inequality in Euclidean space,

$$\text{cap}_1(K_\delta) = \omega_{n-1} \varepsilon_{i_0,\delta}^{n-1} + o(\varepsilon_{i_0,\delta}^{n-1}). \tag{2.1}$$

From independent considerations, we clearly have that $\dot{H}_1^1(\Omega_\delta) \subset \dot{H}_1^1(\Omega)$. Hence, $\lambda_{1,\Omega} \leqq \lambda_{1,\Omega_\delta}$. On the other hand, by the proof of Theorem 0.1, see (1.7), and by (2.1), we also have that

$$\begin{aligned} \lambda_{1,\Omega_\delta} &\leqq \lambda_{1,\Omega} + \frac{1}{|A|} \text{cap}_1(K_\delta) + o(\text{cap}_1(K_\delta)) \\ &= \lambda_{1,\Omega} + \frac{\omega_{n-1}}{|A|} \varepsilon_{i_0,\delta}^{n-1} + o(\varepsilon_{i_0,\delta}^{n-1}). \end{aligned}$$

This proves (0.6). It remains to prove (0.7). For this, we need to be more careful than in the proof of Theorem 0.1. We let A be an eigenset for $\lambda_{1,\Omega}$, and let ω_δ be the union from $i = 1$ to k of balls $B_{x_i}(\tilde{\varepsilon}_{i,\delta})$, $\delta > 0$ small, where $\varepsilon_{i,\delta} < \tilde{\varepsilon}_{i,\delta}$, and $\tilde{\varepsilon}_{i,\delta} = (1 + o(1))\varepsilon_{i,\delta}$ for all i and δ. For $u = \chi_A$, we let also u_δ^+ be the trace of u when u is restricted to $\mathbb{R}^n \backslash \overline{\omega}_\delta$, and u_δ^- be the trace of u when u is restricted to ω_δ. Then, $|A|\lambda_{1,\Omega} = \int_{\mathbb{R}^n} |\nabla u| dx$, and

$$\int_{\mathbb{R}^n} |\nabla u| dx = \int_{\mathbb{R}^n \backslash \overline{\omega}_\delta} |\nabla u| dx + \int_{\omega_\delta} |\nabla u| dx + \int_{\partial \omega_\delta} |u_\delta^+ - u_\delta^-| d\sigma. \tag{2.2}$$

In particular, if we let w_δ be given by $w_\delta = u$ in $\mathbb{R}^n\backslash\overline{\omega}_\delta$, and $w_\delta = 0$ in ω_δ, we get with (2.2) that

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla w_\delta| dx &= \int_{\mathbb{R}^n\backslash\overline{\omega}_\delta} |\nabla u| dx + \int_{\partial\omega_\delta} u_\delta^+ d\sigma\\ &\leqq |A|\lambda_{1,\Omega} - \int_{\omega_\delta} |\nabla u| dx + \int_{\partial\omega_\delta} u_\delta^- d\sigma\\ &\leqq |A|\lambda_{1,\Omega} - \int_{B_\delta} |\nabla u| dx + \int_{\partial B_\delta} u_\delta^- d\sigma + o(\varepsilon_{i_0,\delta}^{n-1}),\end{aligned} \tag{2.3}$$

where $B_\delta = B_{x_{i_0}}(\tilde{\varepsilon}_{i_0,\delta})$. We also have that

$$\begin{aligned}&\int_{\partial B_\delta} u_\delta^- d\sigma = |\partial B_\delta| - \int_{\partial B_\delta} (1-u_\delta^-) d\sigma, \text{ and}\\ &\int_{B_\delta} |\nabla u| dx + \int_{\partial B_\delta} (1-u_\delta^-) d\sigma = \int_{\mathbb{R}^n} |\nabla v| dx,\end{aligned} \tag{2.4}$$

where v is the function $v = \chi_{A^c}$ in B_δ, $v = 0$ in $\mathbb{R}^n\backslash\overline{B}_\delta$, and $A^c = \mathbb{R}^n\backslash A$. If we assume that $x_{i_0} \in \mathrm{int}(\Omega\backslash A)$, then $u_\delta^- = 0$ on ∂B_δ, and it follows from the second equation in (1.6) of Section 1, from (2.1) and (2.3), and from the variational definition of $\lambda_{1,\Omega_\delta}$, that the first equation in (0.7) is true. Now we assume that $x_{i_0} \in \partial^\star A$. Then, the second equation in (1.6) of Section 1, (2.1), (2.3)–(2.4), and the variational definition of $\lambda_{1,\Omega_\delta}$ give that

$$\lambda_{1,\Omega_\delta} \leq \lambda_{1,\Omega} + \frac{1}{|A|}\left(\omega_{n-1}\varepsilon_{i_0,\delta}^{n-1} - \int_{\mathbb{R}^n} |\nabla v| dx\right) + o(\varepsilon_{i_0,\delta}^{n-1}) \tag{2.5}$$

for all δ. Let T_δ be the diffeomorphism given by $T_\delta(x) = x_{i_0} + \tilde{\varepsilon}_{i_0,\delta}^{-1}(x - x_{i_0})$. Then, by the change of variables formula for the total variation, see Giusti [12] or equation (3.1) below, we can write that

$$\int_{\mathbb{R}^n} |\nabla v| dx = \tilde{\varepsilon}_{i_0,\delta}^{n-1} \int_{\mathbb{R}^n} |\nabla (v \circ T_\delta^{-1})| dx. \tag{2.6}$$

Let A_δ be the set consisting of the x such that $T_\delta^{-1}(x) \in A^c$. Then $v \circ T_\delta^{-1} = \chi_{A_\delta}$ in B, and $v \circ T_\delta^{-1} = 0$ in $\mathbb{R}^n\backslash\overline{B}$, where $B = B_{x_{i_0}}(1)$. Since $\partial^\star A^c = \partial^\star A$, $x_{i_0} \in \partial^\star A^c$. By the blow-up property of the reduced boundary (see, for instance, Evans-Gariepy [9]), we can write that

$$\chi_{A_\delta} \to \chi_{H^-(x_{i_0})} \ \text{in} L^1_{\mathrm{loc}}(\mathbb{R}^n) \tag{2.7}$$

as $\delta \to 0$, where $H^-(x_{i_0})$ consists of the $y \in \mathbb{R}^n$ such that $\nu_{A^c}(x_{i_0}).(y - x_{i_0}) \leqq 0$, and where $\nu_{A^c}(x_{i_0})$ is the generalized exterior normal to A^c at x_{i_0}. By (2.7), $v \circ T_\delta^{-1} \to \hat{v}$ in

$L^1(\mathbb{R}^n)$, where $\hat{v} = \chi_{H^-(x_{i_0})}$ in B, and $\hat{v} = 0$ in $\mathbb{R}^n \backslash \overline{B}$. By lower semicontinuity of the total variation, it follows that

$$\int_{\mathbb{R}^n} |\nabla(v \circ T_\delta^{-1})| dx \geqq \int_{\mathbb{R}^n} |\nabla \hat{v}| dx + o(1), \tag{2.8}$$

while we easily check that

$$\int_{\mathbb{R}^n} |\nabla \hat{v}| dx = \frac{1}{2}\omega_{n-1} + b_{n-1}, \tag{2.9}$$

where b_n is the volume of the unit ball in $\mathbb{R}^n$. Combining (2.5)–(2.9), we get that the second equation in (0.7) is also true. This ends the proof of Theorem 0.2.

The proof of Theorem 0.2, and hence the theorem itself, easily extend to other, more general, types of holes. For instance, when we do not assume anymore that only one of the $\varepsilon_{i,\delta}$ is leading, or when we substract $K_{i,\delta} \subset B_{x_i}(\varepsilon_{i,\delta})$ instead of the whole ball. Only slight modifications in the proof, that we leave to the reader, are required to get such extensions.

3. proof of theorem 0.3. By the change of variables formula for the total variation (see Giusti [12]), if T is a C^1-diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, Ω is a smooth open subset of $\mathbb{R}^n$, and $u \in BV(\Omega)$, then

$$\int_{\Omega^\star} |\nabla u^\star| dx = \int_{\Omega} |(DT)^{-1}\nu_u||DT||\nabla u| dx, \tag{3.1}$$

where $\Omega^\star = T(\Omega)$, $u^\star = u \circ T^{-1}$, ν_u is the Radon-Nikodym derivative of ∇u with respect to $|\nabla u|$, and $|DT|$ is the absolute value of the determinant of DT. By (3.1) with $T = T_\delta$, noting that $|\nu_u| = 1$ for $|\nabla u|$-almost all x, by the variational definition of $\lambda_{1,\Omega_\delta}$, and by (0.8), we easily get that $\limsup_{\delta\to 0} \lambda_{1,\Omega_\delta} \leqq \lambda_{1,\Omega}$. Conversely, we let u_δ be a nonnegative eigenfunction for $\lambda_{1,\Omega_\delta}$, and we define the function v_δ by $v_\delta = u_\delta \circ T_\delta$. By (0.8), (3.1), and what we just said, the sequence $(\overline{v}_\delta)$ is bounded in $BV(\mathbb{R}^n)$, with the additional properties that

$$\int_{\mathbb{R}^n} |\nabla \overline{v}_\delta| dx \leqq (1 + o(1))\, \lambda_{1,\Omega_\delta}, \text{ and}$$

$$\int_{\Omega} v_\delta dx = (1 + o(1)) \int_{\Omega_\delta} u_\delta dx = 1 + o(1), \tag{3.2}$$

where $o(1) \to 0$ as $\delta \to 0$. As above, we adopt the notation $\overline{v}_\delta$ for the extension of v_δ by zero outside Ω. Let D be a bounded domain in $\mathbb{R}^n$ which contain both Ω and the Ω_δ's. By compactness of the embedding of $BV(D)$ into $L^1(D)$, we may assume that, up to a subsequence, $\overline{v}_\delta \to v$ in $L^1(D)$ and almost everywhere as $\delta \to 0$. Let u be the restriction of v to Ω. Then $u \geqq 0$ and, by the second equation in (3.2), $\int_\Omega u dx = 1$. Moreover, by the first equation in (3.2), and by lower semicontinuity of the total variation,

$$\lambda_{1,\Omega} \leqq \int_{\mathbb{R}^n} |\nabla \overline{u}| dx \leqq \int_D |\nabla v| dx \leqq \liminf_{\delta \to 0} \lambda_{1,\Omega_\delta},$$

where $\overline{u}$ stands for the extension of u by zero outside Ω. In particular, $\lambda_{1,\Omega_\delta} \to \lambda_{1,\Omega}$ as $\delta \to 0$, and the function $\delta \to \lambda_{1,\Omega_\delta}$ is continuous at $\delta = 0$. This proves the first assertion in Theorem 0.3.

As a consequence of the above developments,

$$\int_{\mathbb{R}^n} |\nabla \overline{v}_\delta| dx \to \int_{\mathbb{R}^n} |\nabla \overline{u}| dx$$

as $\delta \to 0$, and u is an eigenfunction for $\lambda_{1,\Omega}$. In particular, like in Section 1, $\mu_\delta \rightharpoonup \mu$ weakly, where $\mu_\delta = |\nabla \overline{v}_\delta|$ and $\mu = |\nabla \overline{u}|$. In what follows we let A_δ be an eigenset for $\lambda_{1,\Omega_\delta}$ and let $u_\delta = |A_\delta|^{-1}\chi_{A_\delta}$. Then (we refer again to Section 1 for the simple argument involved here), $\overline{u} = |A|^{-1}\chi_A$, where A is an eigenset for $\lambda_{1,\Omega}$. In particular, $|\nabla \overline{v}_\delta| \rightharpoonup \mu$ weakly, and $\mu = |A|^{-1}\mathcal{H}^{n-1}\lfloor \partial_\star A$, where $\partial_\star A$ is the measure theoretic boundary of A (see for instance Evans-Gariepy [9]), and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Now we prove the second assertion in Theorem 0.3, namely the differentiability of the eigenvalue $\lambda_{1,\Omega_\delta}$ at $\delta = 0$ and the equation $(\lambda_{1,\Omega_\delta})'(0) = \Lambda\lambda_{1,\Omega}$. As is easily checked from (0.8), for any $x \in \overline{\Omega}$, and any $X \in \mathbb{R}^n$ such that $|X| = 1$,

$$\begin{aligned} &|DT_\delta(x)| = 1 - n\Lambda\delta + o(\delta), \text{ and} \\ &|(DT_\delta)(x)^{-1}.X| = 1 + \Lambda|X|^2\delta + o(\delta), \end{aligned} \tag{3.3}$$

where the $o(\delta)$'s are uniform in x and X. By (0.2) and (3.1) with $T = T_\delta$, we can write that

$$\lambda_{1,\Omega_\delta} \leqq \frac{\int_{\mathbb{R}^n} |(DT_\delta)^{-1}.\nu_u|\,|DT_\delta|\,|\nabla \overline{u}| dx}{\int_\Omega |DT_\delta|\, u dx}, \tag{3.4}$$

where u and $\overline{u}$ are as above. By (3.3),

$$\int_\Omega |DT_\delta|\, u dx = 1 - n\Lambda\delta + o(\delta). \tag{3.5}$$

Since $|\nu_u| = 1$ a.e w.r.t μ, and $\mathcal{H}^{n-1}(\partial_\star A \backslash \partial^\star A) = 0$, we also get with (3.3) that

$$\frac{1}{\lambda_{1,\Omega}} \int_{\mathbb{R}^n} |(DT_\delta)^{-1}.\nu_u|\,|DT_\delta|\,|\nabla \overline{u}| dx = 1 - (n-1)\Lambda\delta + o(\delta). \tag{3.6}$$

Plugging (3.5) and (3.6) into (3.4), it follows that

$$\lambda_{1,\Omega_\delta} - \lambda_{1,\Omega} \leqq \Lambda\delta\lambda_{1,\Omega} + o(\delta). \tag{3.7}$$

In order to get the converse inequality, we write, still using (0.2) and (3.1), that

$$\lambda_{1,\Omega_\delta} - \lambda_{1,\Omega} \geqq \frac{\int_{\mathbb{R}^n} |(DT_\delta)^{-1}.\nu_\delta|\,|DT_\delta|\,|\nabla\overline{v}_\delta|dx}{\int_\Omega |DT_\delta|\,v_\delta dx} - \frac{\int_{\mathbb{R}^n} |\nabla\overline{v}_\delta|dx}{\int_\Omega v_\delta dx}, \tag{3.8}$$

where ν_δ is the Radon-Nikodym derivative of $\nabla\overline{v}_\delta$ with respect to $|\nabla\overline{v}_\delta|$. By (3.3), since $|\nu_\delta| = 1$ for μ_δ-almost all points,

$$\int_{\mathbb{R}^n} |(DT_\delta)^{-1}.\nu_\delta||DT_\delta||\nabla\overline{v}_\delta|dx = \int_{\mathbb{R}^n} |\nabla\overline{v}_\delta|dx - (n-1)\Lambda\delta \int_{\mathbb{R}^n} |\nabla\overline{v}_\delta|dx + o(\delta) \tag{3.9}$$

and

$$\int_\Omega |DT_\delta|\,v_\delta dx = \int_\Omega v_\delta dx - n\Lambda\delta \int_\Omega v_\delta dx + o(\delta). \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), it follows that

$$\lambda_{1,\Omega_\delta} - \lambda_{1,\Omega} \geqq \frac{\int_{\mathbb{R}^n} |\nabla\overline{v}_\delta|dx}{\int_\Omega v_\delta dx}\Lambda\delta + o(\delta). \tag{3.11}$$

Since $\int_\Omega v_\delta dx \to 1$ and $\int_{\mathbb{R}^n} |\nabla\overline{v}_\delta|dx \to \lambda_{1,\Omega}$ as $\delta \to 0$, it follows from (3.7) and (3.11) that $\lambda_{1,\Omega_\delta} - \lambda_{1,\Omega} = \Lambda\delta\lambda_{1,\Omega} + o(\delta)$. The equation holds for a subsequence, but since the right hand side in the equation does not depend on the subsequence, it holds true for all δ. In particular, $(\lambda_{1,\Omega_\delta})'(0) = \Lambda\lambda_{1,\Omega}$ and this ends the proof of Theorem 0.3.

The proof of the first assertion in Theorem 0.3, and hence the continuity of $\lambda_{1,\Omega_\delta}$ at $\delta = 0$, extend to very general T_δ's. We basically only need that $T_\delta \to Id$ in the C^1-topology as $\delta \to 0$.

References

[1] F. J. ALMGREN, Existence and regularity almost everywhere of solutions to elliptic variational problems. Mem. Am. Math. Soc. **165**(4), 1976.

[2] G. BELLETTINI, V. CASELLES and M. NOVAGA, The total variation flow in $\mathbb{R}^n$. J. Differential Equations. **184**, 475–525 (2002).

[3] M. BELLONI and B. KAWOHL, A direct uniqueness proof for equations involving the p-Laplace operator. Manuscripta Math. **109**, 229–231 (2002).

[4] I. CHAVEL, Riemannian geometry – a modern introduction. Cambridge Tracts in Mathematics **108**, Cambridge 1993.

[5] J. CHEEGER, A lower bound for the smallest eigenvalue of the Laplacian. In Problems in Analysis, A Symposium in honor of S. Bochner. Princeton 195–199 (1970).

[6] F. DEMENGEL, Functions locally almost 1-harmonic. Appl. Anal. **83**, 865–896 (2004).

[7] F. DEMENGEL, F. DE VUYST and M. MOTRON, A numerical approach of the first eigenvalue for the 1-Laplacian on the square and other particular sets. Preprint 2002.

[8] E. DE GIORGI, Frontiere orientate di misura minima, Seminario di Matematica della Scuola Normale Superiore di Pisa. Pisa 1961.

[9] L. C. EVANS and R. F. GARIEPY, Measure theory and fine properties of functions. Studies in Advanced Math., CRC Press, Ann Arbor 1992.

[10] V. FRIDMAN and B. KAWOHL, Isoperimetric estimates for the first eignevalue of the p-Laplace operator and the Cheeger constant. Comment. Math. Univ. Carolinae **44**, 659–667 (2003).

[11] J. GARCIA MELIAN and J. SABINA DE LIS, On the perurbation of eigenvalues for the p-laplacian. C.R. Acad. Sci. Paris, Série **332**(1), 893–898 (2001).

[12] E. GIUSTI, Minimal surfaces and functions of bounded variation. Monographs in Mathematics 1984.

[13] E. GONZALEZ, U. MASSARI and I. TAMANINI, On the regularity of boundaries of sets minimizing perimeter with a volume constraint. Indiana Univ. Math. J. **32**, 25–37 (1983).

[14] B. KAWOHL and T. LACHAND-ROBERT, Characterization of Cheeger sets for convex subsets of the plane will appear in Pacific J. Math. 2004.

[15] P. D. LAMBERTI, A differentiability result for the first eigenvalue of the p-Laplacian upon domain perturbation. Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday **1**(2), Dordrecht 741–754 (2003).

[16] V. G. MAZ'JA, Sobolev Spaces. Springer Series in Soviet Mathematics. Berlin-New York 1985.

[17] M. SANGO, Behaviour of the first eigenvalue of the p-Laplacian in a domain with a hole. Colloq. Math. **87**, 103–111 (2001).

[18] E. STREDULINSKY and W. P. ZIEMER, Area minimizing sets subject to a volume constraint in a convex set. J. Geom. Anal. **7**, 653–677 (1997).

Received: 31 March 2005

E. Hebey
Université de Cergy-Pontoise
Département de Mathématiques
Site de Saint-Martin
2 avenue Adolphe Chauvin
F-95302 Cergy-Pontoise cedex
France
emmanuel.hebey@math.u-cergy.fr

N. Saintier
Université Pierre et Marie Curie
Département de Mathématiques
4 place Jussieu
F-75252 Paris cedex 05
France
saintier@math.jussieu.fr