A PACKAGE FOR COMPUTING IMPLICIT EQUATIONS OF PARAMETRIZATIONS FROM TORIC SURFACES

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Abstract. In this paper we present an algorithm for computing a *matrix representation* for a surface in $\mathbb{P}^3$ parametrized over a 2-dimensional toric variety $\mathscr{T}$ embedded into a projective space. This algorithm follows the ideas of [BDD09] and [Bot11] and it is implemented in *Macaulay2* [GS]. The authors show in [BDD09] that such a surface can be represented by a matrix of linear syzygies if the base points are finite in number and form locally a complete intersection, and in [Bot11] that this can be generalized to the case where the base locus is not necessarily a locally complete intersection, but locally an almost complete intersection. The key point consists in exploiting the sparse structure of the parametrization, which allows us to avoid non almost complete intersection points and to obtain significantly smaller matrices than in the homogeneous case. This implementation contributes to computing implicit equations for rational surfaces in $\mathbb{P}^3$, avoiding costly Gröbner Bases methods and further, it permits to obtain matrix representations of such surfaces, which is better adapt to practical problems.

1. Introduction

The aim of this article is to provide and describe a package for *Macaulay2* [GS] for computing a matrix representation and the implicit equation of parametrized toric surfaces in $\mathbb{P}^3$ as well as routines for computing the toric coordinate ring of a projective toric variety $\mathscr{T}_P$ associated to a polytope P. Precisely, let $f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^3$ be an affine generically finite rational map, let P be a 2-dimensional lattice polytope, and $\mathscr{T} := \mathscr{T}_P$ be the two-dimensional projective toric variety associated to P. A dashed arrow $\dashrightarrow$ or the map being rational means that the map f might not be defined everywhere. The set of points where it is not defined is called the base locus of f. Let $g : \mathscr{T} \dashrightarrow \mathbb{P}^3$ be the rational map induced by f on $\mathscr{T}$, and $\mathcal{S} := \overline{\mathrm{im}(g)} \subset \mathbb{P}^3$ the parametrized image of g (and of f), which is a hypersurface in $\mathbb{P}^3$.

Given a hypersurface $\mathcal{S} \subset \mathbb{P}^3$, a matrix M with entries in the polynomial ring $\mathbb{K}[X_0, \ldots, X_3]$ is called a *representation matrix* of $\mathcal{S}$ if it is generically of full rank and if the rank of M evaluated in a point p of $\mathbb{P}^3$ drops if and only if the point p lies on $\mathcal{S}$. The `MatrixRepToric` package provides methods for computing a representation matrix M for a surfaces $\mathcal{S}$.

It follows immediately that such a matrix M represents $\mathcal{S}$ if and only if the greatest common divisor D of all its minors of maximal size is a power of a homogeneous implicit equation $F \in \mathbb{K}[X_0, \ldots, X_3]$ of $\mathcal{S}$. When the base locus is locally almost a complete intersection, there is a matrix M such that D factors as $D = F^\delta G$ where $\delta \in \mathbb{N}$ and $G \in \mathbb{K}[X_0, \ldots, X_3]$ (see [Bot11, Sec. 3.2] for a description of the surface $(D = 0)$).

To sum up, we assume the parametrization f is given but the matrix M representing $\mathcal{S}$ is not known. That is, the input is a list of 4 polynomials in 2 variables defining a surface $\mathcal{S}$, and the output is a matrix M of linear forms in $\mathbb{K}[X_0, \ldots, X_3]$ representing $\mathcal{S}$. Observe that M as well as the implicit equation of $\mathcal{S}$ is defined up to multiplicative constant of $\mathbb{K}$.

In [BDD09] and in [Bot11] the authors show how to compute a representation matrix M and an implicit equation for $\mathcal{S}$, assuming that the base locus $\mathscr{P}$ of f is 0-dimensional (or empty) and locally almost a complete intersection. The results in [BDD09] and [Bot11] are a further generalization of the foundational work by Jouanolou, Busé and Chardin, in [BJ03, BC05, Cha06, BD07] on implicitization of rational hypersurfaces via approximation complexes.

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Given a generically finite rational map $f : \mathbb{A}^2 \dashrightarrow \mathbb{A}^3$, where $f(s,t) = (f_1/f_0, f_2/f_0, f_3/f_0)(s,t)$, the method we propose in this article can be sketched as follows:

(1) Choose a lattice polytope Q in $\mathbb{R}^2$ such that $\mathcal{N}(f) \subseteq d \cdot Q$ (cf. method `teHomothPolytope`).
(2) Take $\mathscr{T} := \mathscr{T}_Q \subset \mathbb{P}^m$ the projective toric variety associated to Q, with $m = \#(Q \cap \mathbb{Z}^2) - 1$. Let $A = \mathbb{K}[T_0, \ldots, T_m]/I(\mathscr{T})$ be its coordinate ring (cf. method `polynomialsToPolytope`, `ToricEmbedding`, `newToricEmbedding` and `teToricRing`).
(3) Interpret f as $g : \mathscr{T} \dashrightarrow \mathbb{P}^3$, with $g = (g_0 : g_1 : g_2 : g_3)$ (cf. method `teToricRationalMap`).
(4) Set $\nu_0 := 2d - \alpha$, where $d = \deg(g_i)\ \forall\ i$, $\alpha := \max\{i : i \cdot Q$ has no interior lattice points$\}$ (cf. method `isGoodDegree`).
(5) Compute the matrix $M_\nu : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$, for any $\nu \geq \nu_0$ (cf. method `representationMatrix`).
(6) If M_ν is full-rank, compute the equation of $\overline{\mathrm{im}(g)}$ as a factor of gcd(maximal minors ofM_ν) (cf. method `implicitEq`).

From a practical point of view this method constitutes a major improvement with respect to previous theoretical implicitization methods, as it makes it applicable for a wider range of parametrizations (for example, by avoiding unnecessary base points with bad properties) and leads to significantly smaller representation matrices making it computationally better.

There are several advantages of this perspective: The method works in a very general setting and makes only minimal assumptions on the parametrization. In particular, as we have mentioned, it works well in the presence of "nice" base points. Unlike the method of toric resultants (cf. for example [KD06]), there is no need to extract a maximal minor of unknown size, since the matrices are generically of full rank. In this terms, in our algorithm we fully exploit the structure of $\mathcal{N}(f)$ (cf. Definition 2.1), so one obtains much better results for sparse parametrizations, both in terms of computation time and in terms of the size of the representation matrix. One important point is that representation matrices can be efficiently constructed by solving a linear system of relatively small size. This means that the computation of representation matrices is much faster than the computation of the implicit equation and they are thus an interesting alternative as an implicit representation of the surface.

The package `MatrixRepToric` contains the methods that are briefly described below. With the notation before: `isGoodDegree`: verifies if the Euler Chrasteristric of the $\mathcal{Z}$-complex associated to the f_i is zero in a given degree ν. `degreeImplicitEq`: Computes the degree of the polynomial $det((\mathcal{Z})_\nu)$. `ToricEmbedding`: Computes the toric embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^m$, associated to a given polytope Q. `newToricEmbedding`: is the constructor for the object ToricEmbedding. `teToricRing`: Returns the coordinate ring A of the toric variety $\mathscr{T}$. `teGetPolytope`: returns the Polytope Q associated to a given ToricEmbedding. `teToricRationalMap`: Computes the rational map $g : \mathscr{T} \dashrightarrow \mathbb{P}^3$ defined over the toric coordinate ring A from the given polynomials f_i. `representationMatrix`: Computes the right-most map of the $\mathcal{Z}$-complex in degree ν. `implicitEq`: computes the gcd of the right-most map of the $\mathcal{Z}$-complex in degree ν. `polynomialsToPolytope`: Return the convexHull $\mathcal{N}(f)$of the union of the Newton polytope of a given list of polynomials f. `teLatticePotintsFromHomothethicPolytope`: Given two polytopes N and P, it returns the list of coordinates of N based on the smallest homothety of P containing N.

The code can be downloaded from:
"`http://mate.dm.uba.ar/∼nbotbol/Macaulay2/MatrixRepToric.m2`".

The complete documentation can be seen inside the code or in html format in
"`http://mate.dm.uba.ar/∼nbotbol/Macaulay2/doc/MatrixRepToric/html/`".

2. The MatrixRepToric package

The starting point of our approach is the generalization of matrix representations of curves to rational surfaces defined as the image of a map

$$\mathbb{A}^2 \overset{f}{\dashrightarrow} \mathbb{A}^3 : (s,t) \mapsto \left(\frac{f_1(s,t)}{f_0(s,t)}, \frac{f_2(s,t)}{f_0(s,t)}, \frac{f_3(s,t)}{f_0(s,t)}\right) \tag{1}$$

where $f_i \in \mathbb{K}[s,t]$ are coprime polynomials. In order to put the problem in the context of graded modules, one first has to consider an associated projective map

$$\mathscr{T} \overset{g}{\dashrightarrow} \mathbb{P}^3 : x \mapsto (g_0(x) : g_1(x) : g_2(x) : g_3(x)),$$

where $\mathscr{T}$ is a 2-dimensional projective toric variety (for example $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$) embedded in $\mathbb{P}^m$ with coordinate ring $A = \mathbb{K}[T_0, \dots, T_m]/I(\mathscr{T})$ and the $g_i \in A$ are homogenized versions of their affine counterparts f_i. In other words, $\mathscr{T}$ is a suitable compactification of the affine space $(\mathbb{A}^*)^2$ [CLS11, Ful93]. In this case, a linear syzygy (or moving plane) of the parametrization g is a linear relation on the polynomials $g_0, \dots, g_3$, i.e. a linear form $L = h_0X_0 + h_1X_1 + h_2X_2 + h_3X_3$ in the variables X_0, X_1, X_2, X_3 with $h_i \in \mathbb{K}[s,t]$ such that $\sum_{i=0,\dots,3} h_i g_i = 0$.

In [BDD09] and [Bot11] the authors show that this generalization to toric varieties allows to choose a "good" (toric) compactification of $(\mathbb{A}^*)^2$ depending on the polynomials $f_0, \dots, f_3$, which makes the method applicable in cases where it failed over $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. This is also significantly more efficient and leads to much smaller representation matrices.

The main idea of the method is to use a toric embedding to consider our domain as a 2-dimensional toric variety contained in a higher-dimensional projective space $\mathbb{P}^m$, which we recall here. In [BDD09] and [Bot11] they show that the coordinate ring A is in general not Gorenstein which makes the algebra behind this method much more complicated than in the projective setting.

2.1. **The toric embedding.**

We briefly recall some basic notions from toric geometry. These constructions are investigated in more detail in [KD06, Sect. 2], [GKZ94, Ch. 5 & 6], [Ful93] and [CLS11].

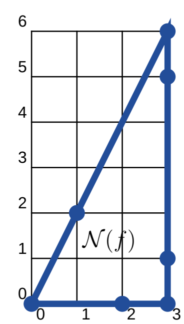
All the varieties considered hereafter are understood to be taken over a field $\mathbb{K}$ ($\mathbb{Q}$ in our computations) and assume f is a rational map as in (1) such that

- f is a generically finite map onto its image and hence parametrizes an irreducible surface $\mathcal{S} \subset \mathbb{P}^3$
- $\gcd(f_0, \dots, f_3) = 1$, which means that there are only finitely many base points.

Definition 2.1. Let $p = \sum_{(\alpha,\beta)\in\mathbb{Z}^2} p_{\alpha,\beta} s^\alpha t^\beta \in \mathbb{K}[s,t]$. Define the support $\mathrm{Supp}(p)$ to be the set of all the exponents which appear in p, i.e. $\mathrm{Supp}(p) = \{(\alpha,\beta) \in \mathbb{Z}^2 \mid p_{\alpha,\beta} \neq 0\} \subset \mathbb{Z}^2$. The Newton polytope $\mathcal{N}(f) \subset \mathbb{R}^2$, where $f = (f_0, f_1, , f_2, f_3)$, is defined as the convex hull of the union $\bigcup_i \mathrm{Supp}(f_i)$ in $\mathbb{R}^2$ of the supports of the f_i. In other words, $\mathcal{N}(f)$ is the smallest convex lattice polygon in $\mathbb{R}^2$ containing all the exponents appearing in one of the f_i. Note that our hypothesis that f is generically finite implies that $\mathcal{N}(f)$ is two-dimensional.

Example 2.2. We first treat an example for which the usual projective compactification, $\mathscr{T} = \mathbb{P}^2$, fails. This could not be solved in a satisfactory manner neither in [BD07], but can be solved with the method in [BDD09] and [Bot11], implemented in `MatrixRepToric` package. Consider the surface parametrized by

```
i1 :  S = QQ[s,t];
i2 : f0 = s^2+s^3*t; f1 = s^3*t^6+1; f2 = s*t^2+2*s^3*t^5; f3 = s^2+s^3*t^6;
i3 : L={f0,f1,f2,f3};
i4 : latticePoints polynomialsToPolytope L
o4 = {0, | 1 |, | 1 |, | 1 |, | 2 |, | 2 |, | 2 |, | 2 |,
         | 0 |  | 1 |  | 2 |  | 0 |  | 1 |  | 2 |  | 3 |
     ------------------------------------------------------
     | 2 |, | 3 |, | 3 |, | 3 |, | 3 |, | 3 |, | 3 |, | 3 |}
     | 4 |  | 0 |  | 1 |  | 2 |  | 3 |  | 4 |  | 5 |  | 6 |
```



Assume $\mathcal{N}(f)$ has $m+1$ lattice points, this is, $\#(\mathcal{N}(f) \cap \mathbb{Z}^2) = m+1$. For example, in Example 2.2, $m = 15$. Then $\mathcal{N}(f)$ defines a two-dimensional projective toric variety $\mathscr{T} \subseteq \mathbb{P}^m$, as explained in [CLS11], where $m+1$ is the cardinality of $\mathcal{N}(f) \cap \mathbb{Z}^2$. It is defined as the closed image of the embedding

$$(\mathbb{A}^*)^2 \overset{\rho}{\hookrightarrow} \mathbb{P}^m : (s,t) \mapsto (\dots : s^i t^j : \dots)$$

where $(i,j) \in \mathcal{N}(f) \cap \mathbb{Z}^2$. For example, the triangle between the points $(0,1)$, $(1,0)$, and $(0,0)$ corresponds to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ has a rectangle as polygon. The rational map f factorizes through $\mathscr{T}$ via g, where g is given by four polynomials $g_0, \dots, g_3$ of degree d in m variables, making the following diagram commute

(2)
$$\begin{tikzcd}
(\mathbb{A}^*)^2 \arrow[r, dashed, "f"] \arrow[d, hook, "\rho"] & \mathbb{A}^3 \arrow[d, hook, "\iota"] \\
\mathscr{T} \arrow[r, dashed, "g"'] & \mathbb{P}^3
\end{tikzcd}$$

The map g induces a morphism between the homogeneous coordinate rings of $\mathbb{P}^3$ and $\mathscr{T}$

$$\mathbb{K}[X_0, X_1, X_2, X_3] \xrightarrow{h} A = \mathbb{K}[T_0, \ldots, T_m]/I(\mathscr{T}) : X_i \mapsto g_i(T_0, \ldots, T_m).$$

Note that the variables T_k correspond to monomials $s^i t^j$ and the ideal $I(\mathscr{T})$ is the ideal of relations between these monomials. The implicit equation of $\mathcal{S}$ is a generator of the principal ideal $\ker(h)$. We should remark that the toric ideals $I(\mathscr{T})$ are very well understood and there exist highly efficient software systems to compute their Gröbner bases.

Following Example 2.2, the homogeneous coordinate ring $A = \mathbb{K}[T_0, \ldots, T_m]/I(\mathscr{T})$ of $\mathscr{T}$ can be computed in the following way:

```
i5 : A = teToricRing newToricEmbedding L;
i6 : describe A
o6 =             QQ[T , T , T , T , T , T , T , T , T , T , T  , T  , T  , T  , T  , T  ]
                     0   1   2   3   4   5   6   7   8   9   10   11   12   13   14   15
 -------------------------------------------------------------------------------------
       2
      (T   - T  T  , T  T   - T  T  , T  T   - T  T  , T  T   - T  T  , T  T   - T T  ,...
        14    13 15   13 14    12 15   12 14     11 15   11 14     10 15   10 14    9 15
```

We compute the toric map g as:

```
i7 : g = teToricRationalMap L
o7 = | T_4+T_10 T_0+T_15 T_3+2T_14 T_9+T_15 |
      1       4
o7 : A  <--- A
```

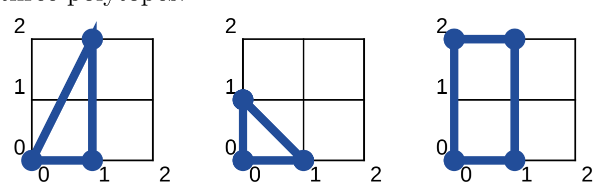
As usual for any quotient ring in *Macaulay2*, we can obtain the ideal $I(\mathscr{T})$ by doing `ideal A`.

The same we have dome with $\mathcal{N}(f)$ can be done with any lattice polytope Q for which an integer multiple of it contains $\mathcal{N}(f)$. We have $\mathscr{T}_Q$ the toric variety associated to Q, and a map $g_Q : \mathscr{T}_Q \dashrightarrow \mathbb{P}^3$ defined by g'_0, g'_1, g'_2 and g'_3 that satisfies $\iota \circ f = g_Q \circ \rho$. This map induces a morphism between the homogeneous coordinate rings

$$\mathbb{K}[X_0, X_1, X_2, X_3] \xrightarrow{h_Q} A_Q = \mathbb{K}[T_0, \ldots, T_m]/I(\mathscr{T}_Q) : X_i \mapsto g'_i(T_0, \ldots, T_m).$$

Here, the variables T_k correspond to monomials $s^i t^j$ in $\mathcal{N}(f)$. Assume that $k = \min\{i : i \cdot Q \supset \mathcal{N}(f)\}$. Since every point in $\mathcal{N}(f)$ is the sum of k points in Q, then deg $g'_i = k$ for all i. Thus, there is an interplay between number of variables $T's$ and degree of g. Also Q needs to be "good enough" in order to avoid base points in $\mathscr{T}_Q$, e.g. an integer contraction of $\mathcal{N}(f)$ if possible.

Consider the following three polytopes:



Polytope Q_1 Polytope Q_2 Polytope Q_3

Following Example 2.2, the homogeneous coordinate ring A_Q of $\mathscr{T}_Q$ can be computed in the following way:

```
i8 : Q1=convexHull matrix{{0,1,1},{0,0,2}};
i9 : latticePoints Q1
o9 = {0, | 1 |, | 1 |, | 1 |}
         | 0 |  | 1 |  | 2 |
i10 : gQ1 = teToricRationalMap (L,Q1)
o10 = | T_0T_1^2+T_1^2T_2 T_0^3+T_3^3 T_0^2T_3+2T_2T_3^2 T_1^3+T_3^3 |
```

2.2. **The representation matrix.** Next we give the main ideas behind the construction of a representation matrix M of the closed image of f. First, notice that we aim to provide a map of graded A-modules $M : (\mathcal{Z}_1) \to (\mathcal{Z}_0)$, that for any degree $\nu \in \mathbb{Z}$ in the grading of A gives a map $M_\nu : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$. Second, observe that it is necessary to compute the "good" degree ν as small as possible, in order to keep matrices small.

For the first point, take $\mathcal{Z}_0 := A[X_0, X_1, X_2, X_3]$ and $\mathcal{Z}_1 := Z_1(d) \otimes_A A[X_0, X_1, X_2, X_3]$, where Z_1 is the first syzygy module of (g_0, g_1, g_2, g_3). In *Macaulay2*, Z_1 can be computed by doing

```
i11 : Z1= kernel koszul(1,g)
```

It is important to note that this implementation in *Macaulay2* interprets the module Z_1 as a submodule of $A^4(-d)$. The shift of d considered in $Z_1(d)$ is just to recover the original grading of A. Now, take $M : (\mathcal{Z}_1) \to (\mathcal{Z}_0)$ define by multiplication by (X_0, X_1, X_2, X_3). Precisely, for given $a = (a_0, a_1, a_2, a_3) \in Z_1(d) \subset A^4$, $M(a) := \sum a_i X_i$.

Now, we move to the second point. The key point here is that if ν is a degree such that $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu$ vanishes, then ν is a good degree. Precisely, denote

$$\alpha := \max\{i : i \cdot \mathcal{N}(f) \text{ has no interior lattice points}\}$$

and let $\nu \in \mathbb{Z}$. If $(g_0, \ldots, g_3)$ has only finitely many zeroes (or empty) and is locally almost a complete intersection, then ν can be taken $\geq \nu_0 = 2d - \alpha$. Moreover, if there is at least one common zero of $g_0, \ldots, g_3$, then the bound can be improved. For more details see [BDD09, Thm. 11] and [Bot11, Lemma 11] where there are explicit computations for ν.

Following [BDD09, Ch. 3] and [Bot11, Ch. 3] it follows that the determinant of the $\mathcal{Z}_\bullet$-complex is a power of the implicit equation of $\mathcal{S}$:

Theorem 2.3. *Assume that $\mathscr{P} = \mathrm{Proj}(A/(g_0, \ldots, g_3))$ is finite in number or empty. Let α be as before and $\nu_0 = 2d - \alpha$.*

If $\mathscr{P}$ is locally a complete intersection, then, for any integer $\nu \geq \nu_0$ the determinant D of the complex $(\mathcal{Z}_\bullet)_\nu$ of $\mathbb{K}[X_0, X_1, X_2, X_3]$-modules defines (up to multiplication with a constant) the same non-zero element in $\mathbb{K}[X_0, X_1, X_2, X_3]$ and $D = F^{\deg(g)}$ where F is the implicit equation of $\mathcal{S}$, and the right-most map of the complex $(\mathcal{Z}_\bullet)_\nu$ is a representation matrix M_ν for $\mathcal{S}$.

If $\mathscr{P}$ is locally almost a complete intersection, then $D = F^{\deg(g)}G$, where F is the implicit equation of $\mathcal{S}$, and G may be written as a product of linear forms in a splitting field. The matrix M_ν represents the surface $\mathcal{S}$ together with finitely many planes, one for each base point almost complete intersection that is not complete intersection.

By [GKZ94, Appendix A], the determinant D can be computed either as an alternating product of sub-determinants of the differentials in $(\mathcal{Z}_\bullet)_\nu$ or as the greatest common divisor of the maximal-size minors of the matrix M associated to the right-most map $(\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$. Note that this matrix is nothing else than the matrix M_ν of linear syzygies as described in the introduction; it can be computed with the same algorithm as in [BD07].

In Example 2.2, we have first computed the new parametrization over the toric variety $\mathscr{T}_{\mathcal{N}(f)}$, which is given by linear forms $g_0, \ldots, g_3$, of $\mathcal{N}(f)$ and the coordinate ring is $A = \mathbb{K}[T_0, \ldots, T_{15}]/J$. Recall that the 16 variables correspond to the 16 points in the Newton polytope. Since $d = 1$ and $\mathcal{N}(f)$ has interior points, thus $\alpha = 0$, then the optimal degree is $\nu_0 = 2$. In degree 2 we have that the surface $\mathcal{S}$ is represented by a 46×90-matrix.

```
i6 : rM = representationMatrix (g,2);
                                     46                        90
o6 : Matrix (QQ[X , X , X , X ])   <--- (QQ[X , X , X , X ])
                 0   1   2   3               0   1   2   3
```

Taking as polytope Q_1 we obtain a rather smaller matrix than the one before.

```
i7 : rMQ1 = representationMatrix (gQ1,5);
                                     36                        63
o7 : Matrix (QQ[X , X , X , X ])   <--- (QQ[X , X , X , X ])
                 0   1   2   3               0   1   2   3
```

The implicit equation can be computed as:

```
i8 : EqQ1 = implicitEq (L,5,Q1);
i9 : (factor EqQ1)#4
          13 4         12 5         11 6        10 7        9 8       8 9        7 10       6 11
o8 = 36000X  X  - 21600X  X  - 17280X  X  - 2304X  X  + 1728X X  - 576X X  - 1896X X   - 720X X   ...
          0  1         0  1         0  1        0  1        0 1       0 1        0 1        0 1
```

When we consider Q_2 we get a base point $p \in \mathscr{T}_{Q_2} = \mathbb{P}^2$ which is not an almost complete intersection. This leads to a matrix which is never of full rank, hence not a representation matrix for $\mathcal{S}$. Thus, the implicit equation would be identical to zero.

```
i9 : Q2 = convexHull matrix{{0,0,1},{0,1,0}};
i10 : gQ2 = teToricRationalMap (L,Q2)
o10 = |T_0^7T_2^2+T_0^5T_1T_2^3 T_0^9+T_1^6T_2^3 T_0^6T_1^2T_2+2T_0T_1^5T_2^3 T_0^6T_2^3+T_1^6T_2^3|
                                        1                    4
o10 : Matrix (QQ[T , T , T ])  <--- (QQ[T , T , T ])
                  0   1   2             0   1   2
i11 : rMQ2 = representationMatrix (gQ2,16);
                                           153                        309
o11 : Matrix (QQ[X , X , X , X ])    <--- (QQ[X , X , X , X ])
                  0   1   2   3                0   1   2   3
i12 : rank rMQ2
o12 = 152
```

We observe that in this case there are two base points $(0 : 0 : 1)$ and $(0 : 1 : 0)$. We see that $(0 : 0 : 1)$ is not even an a.c.i. base point:

```
i16 : localize (ideal gQ2, ideal (T_0,T_1))
             6     5   5     7
o16 = ideal (T , T T , T T , T )
             1   0 1   0 1   0
o16 : Ideal of QQ[T , T , T ]
                   0   1   2
```

Observe that this procedure can be used in general to check if the base locus is locally a complete intersection or locally almost a complete intersection In practice, checking if a base point is locally almost a complete intersection should not be necessary, since in that case the matrix will not be of full rank.

Considering Q_3, we get $\mathscr{T}_{Q_3} = \mathbb{P}^1 \times \mathbb{P}^1$. By computing the radical of the ideal associated to gQ3 one can observe that this also leads to base points. We show that we still get 66×125 matrix of full rank. However, the equation computed from this matrix contains extraneous factors.

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