Exact bounded PML for the Helmholtz equation

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Abstract

We consider the following Helmholtz problem which models the propagation of a wave of frequency $\omega > 0$ and velocity of propagation $c > 0$ in an unbounded homogeneous medium:

$$\begin{cases} \Delta u + k^2 u = 0 \qquad \text{in } \Omega_{\mathrm{E}}, \\ u = g \qquad \text{on } \Gamma, \\ \displaystyle\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \end{cases}$$

where $k := \omega/c$ is the wave number, $\Omega_{\mathrm{E}} := \mathbb{R}^2 \setminus \overline{\Omega}_{\mathrm{I}}$, with $\Omega_{\mathrm{I}} \subset \mathbb{R}^2$ being a simply connected bounded domain with regular boundary Γ, and $g \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$ is a given source function. The third equation is a typical *Sommerfeld* condition modeling the radiation of the wave at infinity. This is is a classical scattering problem, whose existence and uniqueness of solution is well known (see for instance [4]).

The typical first step for the numerical solution by finite elements or finite differences of such a scattering problem is to truncate the unbounded computational domain, which entails an inherent difficulty: *how to choose boundary conditions to replace the Sommerfeld radiation condition at infinity* (see for instance [5]). There are several techniques to deal with this: boundary element methods, infinite element methods, Dirichlet-to-Neumann methods based on Fourier expansions, or the use of absorbing boundary conditions.

An alternative approach to deal with the truncation of unbounded domains is the so called *PML* (Perfectly Matched Layer) technique, introduced by Berenger [1] for Maxwell's equations in electromagnetism. It is based on simulating an absorbing layer of damping material surrounding the domain of interest, like a thin sponge which absorbs the scattered field radiated to the exterior of the domain. The absorbing material is characterized by a damping function varying through the thickness of the layer. Typical examples are linear and quadratic damping functions ([1, 3]).

In a recent paper [2], we have introduced an 'exact' bounded PML, based on using a singular damping function. 'Exactness' must be understood in the sense that this technique allows exact recovering of the solution to time-harmonic scattering problems in unbounded domains. In spite of the singularity of the damping function, the procedure is shown to lead to a well posed conforming finite element discretization.

The efficiency of this approach is shown first in a simplified one-dimensional framework, which allows us to choose a convenient singular damping function, as well. Subsequently, the high accuracy of this technique is numerically demonstrated by means of two-dimensional tests, as well as its advantages as compared with other classical PML techniques.

References

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