NUMERICAL SCHEMES, WELL-POSEDNESS ANALYSIS AND ENGINEERING APPLICATIONS OF CONSERVATION LAWS AND RELATED EQUATIONS WITH DISCONTINUOUS FLUX

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Abstract

In recent years there has been an increased interest in the study of conservation laws and related equations, for example transport equations, strongly degenerate parabolic equations or systems of equations, with a flux that depends discontinuously on the space variable. In the simplest setting, the problem under study is

$$u_t + f(u,x)_x = 0, \quad x \in \mathbb{R}, \quad t > 0; \quad u(x,0) = u_0(x), \quad x \in \mathbb{R}; \quad f(u,x) = \begin{cases} g(u) & \text{for } x < 0, \\ h(u) & \text{for } x > 0, \end{cases} \tag{1}$$

where g and h are smooth functions. Solutions to conservation laws are in general discontinuous and need to be defined as weak solutions along with an entropy condition to select the unique physically relevant weak solution (the entropy solution). For (1), a new entropy solution concept is required, since standard results for conservation laws with a flux that depends smoothly on x [6] cannot be applied. In particular, it is not obvious a priori which discontinuities of the solution u across $x = 0$ are acceptable. Clearly, it is desirable to have a numerical scheme that provably converges to an entropy solution.

The first part of this contribution is devoted to a new mathematical model for so-called clarifier-thickener units [3, 4] used for the solid-liquid separation of suspensions in engineering applications. This model gives rise to a strongly degenerate parabolic equation with a discontinuous flux and a singular source term. Based on the physics of the problem, a definition of an entropy solution to this problem is introduced. A simple upwind-type finite difference scheme is introduced and its convergence to a solution that satisfies an entropy condition is proved. The uniqueness of an entropy solution can be shown by an analysis of the admissible jumps across the spatial points where the flux is discontinuous. Some numerical examples illustrate the predictions of the model.

In a second part, some recent extensions of the model and the underlying analysis are presented, including extensions to second order numerical schemes and the presence of sink terms [1].

Finally, other applications including two-phase flow in heterogeneous porous media, traffic flow with discontinuous road surface conditions [2] and population balance models [5] are reviewed.

References

[1] R. Bürger, A. García, K.H. Karlsen and J.D. Towers, On an extended clarifier-thickener model with singular source and sink terms, in preparation.

[2] R. Bürger and K.H. Karlsen, On a diffusively corrected kinematic-wave traffic model with changing road surface conditions, *Math. Models Meth. Appl. Sci.* **13** (2003), 1767–1799.

[3] R. Bürger, K.H. Karlsen, N.H. Risebro and J.D. Towers, Well-posedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units, *Numer. Math.* **97** (2004), 25–65.

[4] R. Bürger, K.H. Karlsen and J.D. Towers, A mathematical model of continuous sedimentation of flocculated suspensions in clarifier-thickener units, *SIAM J. Appl. Math.* **65** (2005), 882–940.

[5] R. Bürger, K.H. Karlsen and J.D. Towers, Closed-form and finite difference solutions to a population balance model of grinding mills, *J. Eng. Math.* **51** (2005), 165–195.

[6] S.N. Kružkov, First order quasi-linear equations in several independent variables, *Math. USSR Sb.* **10** (1970), 217–243.

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