Modified Taylor formulas and hp Clouds

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Summary. We study the numerical properties of a cloud based *hp* finite element method. Contrary to the proposed methods of Duarte *et al.*, enriched linearly dependent basis shape functions are used. Numerical examples are analyzed to show the approximative power of the method.

Spectral meshless or mesh free methods for numerical analysis of partial differential equations have recently emerged as important numerical tools for their ability to solve some complex problems more effectively than the finite element method. Recent developments have demonstrated the simplicity of adding hierarchical refinements to a low order reproducing set of shape functions which satisfy the *partition of unity* requirement. In particular, in the *hp* cloud method of C. A. Duarte and J. T. Oden [1, 2, 3], the basic idea is to multiply functions in a partition of unity $\{\mathcal{W}_i\}_{i=1}^N$ by Taylor's polynomials at nodes x_i. The resulting functions, called *hp* cloud shape functions, have good properties, such as high regularity and compactness; and linear combinations of these functions can represent polynomials of any degree. This property allows the implementation of p and *hp* adaptivity leading in many situation to spectral convergence.

The *partition of unity* method also impacted on mesh-based finite elements and led to the development of *hp* clouds FEM [7, 8]. The introduction of spectral degrees of freedom over a conventional finite element *partition of unity* can also lead to fast and accurate methods for solving numerical problems. On the other hand, a meshless finite element method has been presented by S. Idelsohn *et al.* in [9], which has the advantages of the good meshless methods and the shape functions depend only on the node positions. The proposed method shares several of the advantages of the finite element method such as the simplicity of the shape functions in a large part of the domain and the C^0 continuity between elements. Furthermore, the resulting *partition of unity* has algebraic precision equal to one. It appears as a difficult issue, however, to obtain higher order algebraic precision with this kind of methodologies and we believe MFEM could also be greatly benefited by the use of a *hp* cloud scheme.

Characterizing the algebraic precision of the shape functions in these methodologies is an important ingredient in rate of convergence and error estimates. In [4], we consider a *partition of unity* which has algebraic precision equal to $m \geq 1$, and we study quasi-interpolation operators of the form

$$\mathcal{IS}(x) := \sum_{i=1}^{N} \mathcal{T}_i[x_i, x]\, \mathcal{W}_i(x), \tag{1}$$

where $\mathcal{T}_i[x_i, x]$ are modified Taylor polynomials of degree r expanded at x_i. In the univariate case, Xuli proved in [5] that an appropriate combination of Taylor polynomials yields algebraic precision equal to $m + r$. This result was generalized by Guessab *et al.* [6] to the multivariate case when the domain is convex. Xuli's work however was preceded by Duarte. In fact, several years before Xuli, Duarte [1, 2, 3] noted that the use of Taylor polynomials of the same degree of those that are reproduced by $\{\mathcal{W}_i\}$ yields singular or near singular stiffness matrices in Galerkin schemes. Therefore, he proposed to use only polynomials that are missing in the linear combinations of $\{\mathcal{W}_i\}$ and a reproduction formula which he proved only in the univariate case [1].

The first contribution in [4] deals with reproduction formulas:

- Xuli's reproduction formula: we have shown that the convexity assumption in [6] can be relaxed. In fact, it is only needed that the support of function $\mathcal{W}_i$ be star shaped w.r.t. node x_i, $\forall i$, $i = 1, ..., N$.
- Duarte's reproduction formula: we have proved it in the multivariate case.

These different approaches suggest different *hp* cloud function spaces: $\mathcal{F}_{\mathcal{X}}^{m,r}$ and $\mathcal{F}_{\mathcal{D}}^{m,r}$. In the first one, Taylor polynomials of degree $\leq r$ are added at each node, while Taylor polynomials of degree between $m+1$ and r are added in the second case. there is an algebraic redundancy in $\mathcal{F}_{\mathcal{X}}^{m,r}$, but the Galerkin method is stable in the following sense: even with this redundancy, it yields the right solution. Furthermore, and perhaps amazingly, under a Galerkin scheme the first approach produces better numerical results and more accurate solutions. In fact, $\mathcal{F}_{\mathcal{X}}^{m,r}$ built over a FEM *partition of unity* of algebraic precision m produces comparable results than a standard FEM of precision equal to $m + r$ (even slightly better if degrees of freedom are considered). An univariate example was analyzed in [4] showing this fact. Furthermore, it should be remarked that the Generalized Finite Element Method can also yield sparse positive semi-definite linear system [7]. However, in [10], the use of direct solvers as subroutines MA27 and MA47 of *Harwell Subroutine Library* was successful even when the nullity of the stiffness matrix was large. It was also shown in [10] that round-off errors did not play a significant role in solving the linear system, *i.e.*, the round-off error was also the same as when finite element linear system is solved. An iterative algorithm was also given in [10]. Therefore, there exist nowadays efficient solvers to deal with singular or near singular linear systems.

In this work we study the multivariate case with several numerical examples and we illustrate the main ideas of the method.

References

[1] C. A. M. Duarte, *hp Clouds-an hp Meshless Method*, Ph. D Thesis, TICAM, The University of Texas at Austin, 1995.

[2] C. A. M. Duarte and J. T. Oden, *Hp clouds - an hp meshless method to solve boundary value problems*, Technical Report 59-05, TICAM, The University of Texas at Austin, 1995.

[3] C. A. M. Duarte and J. T. Oden, *hp Clouds-an hp Meshless Method,* Numer. Methods for Partial Diff. Eq. (1996), pp. 1-34.

[4] C. Zuppa, *Modified Taylor formulas and hp Clouds* (preprint).

[5] H. Xuli, *Multi-node higher order expansions of a functions,* J. Approx. Theory **124** (2003), pp. 242-253.

[6] A. Guessab, O. Nouisser and G. Schmeisser, *Multivariate approximation by a combination of modified Taylor polynomials,* to appear in Comput. Math. Appl. (2005).

[7] I. Babuśka and J. M. Melenk, *The partition of unity finite element method*, Technical Report BN-1185, Inst. for Phys. Sci. and Tech., University of Maryland, 1995.

[8] J. T. Oden, C. A. M. Duarte and O. C. Zienkiewicz, *A new cloud-based hp finite element method,* Comput. Methods Appl. Mech. Engrg. **153** (1998), pp. 117-126.

[9] S. R. Idelsohn, E. Oñate, N. Calvo and F. Del Pin, *The meshless finite element method*, Int. J. Num. Meth. Engrg. **58** (2003), pp. 893-912.

[10] T. Strouboulis, I. Babuška and K. Copps, *The design and analysis of the generalized finite element method*, Comput. Methods Appl. Mech. Engrg. **181** (2001), pp. 43-69.