Análisis II - Matemática 3
Análisis Matemático II

Marco Farinati

FCEN UBA mfarinat@dm.uba.ar

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El rotor y el teorema de Stokes

El rotor y test de derivadas cruzadas

$$\vec{F}(x,y,z) = \big(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\big)$$

Será $\vec{F}$ un gradiente? $\exists \phi : \mathbb{R}^3 \to \mathbb{R}$ tal que $\vec{F} = \nabla \phi$?

En ese caso, $\displaystyle\int_a^b \langle \vec{F}(\sigma(t)), \sigma'(t) \rangle dt = \phi(\sigma(b)) - \phi(\sigma(a))$.

Si $\vec{F} = \nabla \phi$:

$$(F_1, F_2, F_3) = (\partial_x \phi, \partial_y \phi, \partial_z \phi)$$

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i} \qquad \Rightarrow \qquad \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

El rotor y test de derivadas cruzadas

Definición

$\vec{F}: \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ un campo C^1. Se define su *rotor* como

$$\Big(\partial_y F_3 - \partial_z F_2 \ , \ \partial_z F_1 - \partial_x F_3 \ , \ \partial_x F_2 - \partial_y F_1\Big)$$

$$rot(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Observamos: $\vec{F} = \nabla\phi = (\partial_x\phi, \partial_y\phi, \partial_z\phi)$ para alguna $\phi \in C^2$ entonces $rot(\vec{F}) = 0$.

Ejemplo: $\vec{F} = (P(x,y), Q(x,y), 0) \Rightarrow rot(\vec{F}) = (0 \ , \ 0 \ , \partial_x Q - \partial_y P)$

Teorema de Stokes

$\mathcal{S}$ una superficie regular orientada y parametrizada $T : D \to \mathcal{S}$. $\mathcal{C} := T(\partial D) = \partial\mathcal{S}$ el borde de la superficie. Si D es un dominio donde vale el teorema de Green

Teorema de Stokes, o del rotor

$\forall$ campo $\vec{F}$ de clase $\mathcal{C}^1$ definido en un abierto que contiene a $\mathcal{S}$

$$\int_{\partial\mathcal{S}^+} \langle \vec{F}, d\ell \rangle = \iint_{\mathcal{S}} \langle \nabla \times \vec{F}, N \rangle dA$$

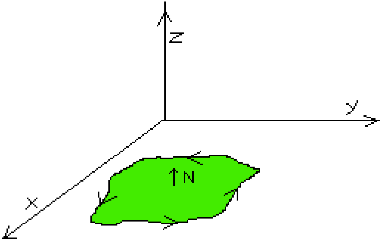
donde $\partial\mathcal{S}^+ = T(\partial(D^+))$

Ejemplo: $D \subset \mathbb{R}^2$ dominio donde vale Green.
$\mathcal{S} = \{(x,y,0) : (x,y) \in D\}$, $F = (P(x,y), Q(x,y), 0)$.
$T(x,y) = (x,y,0)$.
Orientamos con N hacia arriba.

$$\langle rot(\vec{F}), N\rangle = \langle rot(\vec{F}), (0,0,1)\rangle = \partial_x Q - \partial_y P$$

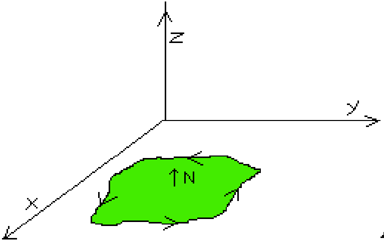
$$\int_{\partial\mathcal{S}^+} \langle \vec{F}, d\ell\rangle = \iint_{\mathcal{S}} \langle rot(\vec{F}), N\rangle dA$$

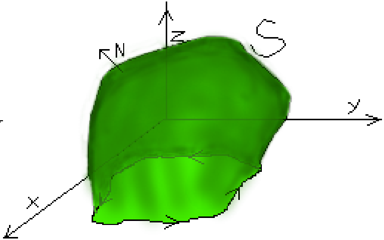
Stokes dice lo mismo que Green:



$$\int_{\partial D^+} Pdx + Qdy = \iint_D (\partial_x Q - \partial_y P)dxdy$$

pero... ahora también sabemos

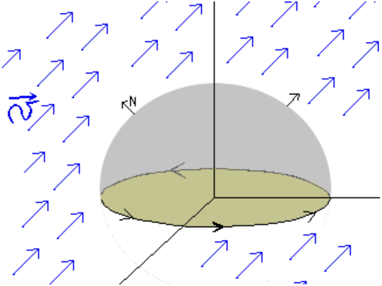




$$\int_{\partial D^+} \langle \vec{F}, d\vec{\ell} \rangle = \iint_{\mathbf{S}} \langle \nabla \times \vec{F}, N \rangle dA$$

Para cualquier superficie orientada que tenga el mismo borde!

$\vec{v}(x,y,z) = (1,0,1/2)$ un campo constante. Cuál es su flujo a través de la mitad superior de la esfera?



$$Flujo = \iint_{\mathcal{S}} \langle \vec{v}, N \rangle dA = ?$$

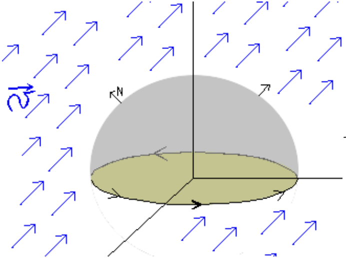
Parametrizo la parte superior de la esfera y calculo...

Es $\vec{v} = rot(F)$? En ese caso

$$\iint_{\mathcal{S}} \langle rot(F), N \rangle dA = \int_{\partial \mathcal{S}^+} \langle F, d\vec{\ell} \rangle = \iint_{\widetilde{\mathcal{S}}} \langle rot(F), N \rangle dA$$

$$\vec{v}(x,y,z) = (1,0,1/2)$$

$$(1,0,1/2) = (\partial_y F_3 - \partial_z F_2 \ , \ \partial_z F_1 - \partial_x F_3 \ , \ \partial_x F_2 - \partial_y F_1)?$$



$$F = (0,0,y) \Rightarrow rot(F) = (1,0,0)$$

$$F = (0,x/2,0) \Rightarrow rot(F) = (0,0,\tfrac{1}{2})$$

$$\therefore (1,0,1/2) = rot(0,x/2,y)$$

$$\iint_{\mathcal{S}} \langle \vec{v}, N\rangle dA = \iint_{\mathcal{S}} \langle rot((0,x/2,y)), N\rangle dA$$

$$= \int_{\partial\mathcal{S}^+} \langle (0,x/2,y), d\vec{\ell}\rangle = \iint_{\widetilde{\mathcal{S}}} \langle rot((0,x/2,y)), N\rangle dA$$

$$= \iint_{\widetilde{\mathcal{S}}} \langle (1,0,\tfrac{1}{2}), (0,0,1)\rangle dA = \iint_{\widetilde{\mathcal{S}}} \tfrac{1}{2} dA = \tfrac{1}{2}\pi$$


REAL HOUSEMOMS

$$T : D \to \mathcal{S}$$

param. reg. orientada, F un campo vectorial C^1, entonces

$$\iint_{\mathcal{S}} \langle \nabla \times F, N \rangle dA = \iint_D \langle (\nabla \times F)(T(u,v)), T_u \times T_v \rangle dudv$$

Sea $\sigma : [a,b] \to \partial D^+$ una parametrización regular a trozos de ∂D^+, $\sigma(t) = (u(t), v(t))$. Entonces

$$T \circ \sigma : [a,b] \to \partial \mathcal{S}^+$$

es una parametrización regular a trozos de $\partial \mathcal{S}^+$,

$$\int_{\partial \mathcal{S}^+} \langle F, d\vec{\ell} \rangle = \int_a^b \langle F(T(\sigma(t))), (T \circ \sigma)'(t) \rangle dt$$

$$= \int_a^b (\ldots) u'(t) dt + (\ldots) v'(t) dt = \int_{\partial D^+} P du + Q dv = \iint_D (\partial_u Q - \partial_v P) dudv$$

Caso particular: $\mathcal{S} = graf(g)$

$$T : D \to \mathcal{S}$$

$$T(x, y) = (x, y, g(x, y))$$

$$T_x = (1, 0, g_x), \quad T_y = (0, 1, g_y), \quad T_x \times T_y = (-g_x, -g_y, 1)$$

$$\iint_{\mathcal{S}} \langle rot(F), N \rangle dA =$$

$$= \iint_D \langle rot(F(T(u, v)), T_u \times T_v \rangle dudv$$

$$= \iint_D \Big\langle \Big(F_{3y} - F_{2z}, F_{1z} - F_{3x}, F_{2x} - F_{1y} \Big), (-g_x, -g_y, 1) \Big\rangle dudv$$

$$= \iint_D -(F_{3y} - F_{2z})g_x - (F_{1z} - F_{3x})g_y + F_{2x} - F_{1y} \; dudv$$

Sea $\sigma : [a, b] \to \partial D^+$ parametrización de ∂D^+.

$$\sigma(t) = (x(t), y(t))$$

$$\rightsquigarrow T(\sigma(t)) = \Big(x(t), y(t), g\big(x(t), y(t)\big)\Big)$$

$$(T \circ \sigma(t))' = (x', y', g_x x' + g_y y')$$

$$\int_{\partial \mathcal{S}^+} \langle F, d\vec{\ell} \rangle = \int_a^b \langle F(T(\sigma(t))), (T \circ \sigma)'(t) \rangle dt$$

$$= \int_a^b \Big\langle F\Big(x(t), y(t), g\big(x(t)), y(t)\big)\Big), (x', y', g_x x' + g_y y') \Big\rangle dt$$

$$= \int_a^b F_1 x' + F_2 y' + F_3 (g_x x' + g_y y')\ dt$$

$$= \int_a^b \underbrace{(F_1 + F_3 g_x)}_{P} x' + \underbrace{(F_2 + F_3 g_y)}_{Q} y'\ dt$$

Resumen:

$$\iint_{\mathcal{S}} \langle rot(F), N \rangle dA =$$

$$= \iint_D -(F_{3y} - F_{2z})g_x - (F_{1z} - F_{3x})g_y + F_{2x} - F_{1y} \; dudv$$

$$\int_{\partial \mathcal{S}^+} \langle F, d\vec{\ell} \rangle = \int_a^b \underbrace{(F_1 + F_3 g_x)}_{P} x' + \underbrace{(F_2 + F_3 g_y)}_{Q} y' \; dt$$

Green nos dice

$$\int_{\partial D^+} Pdx + Qdy = \iint_D (Q_x - P_y) dxdy$$

Hay que comparar la integral doble con $Q_x - P_y$. Es un ejercicio de regla de la cadena:

$$\int_{\partial\mathcal{S}^+} \langle F, d\vec{\ell}\rangle = \int_a^b \underbrace{(F_1 + F_3 g_x)}_{P} x' + \underbrace{(F_2 + F_3 g_y)}_{Q} y' \; dt$$

$$Q_x - P_y = \Big(F_2\big(x,y,g(x,y)\big) + F_3\big(x,y,g(x,y)\big) g_y\Big)_x$$

$$-\Big(F_1\big(x,y,g(x,y)\big) + F_3\big(x,y,g(x,y)\big) g_x\Big)_y$$

$$= F_{2x} + F_{2z} g_x \; + \; F_{3x} g_y + F_{3z} g_x g_y + F_3 g_{yx}$$

$$-F_{1y} - F_{1z} g_y \; - \; F_{3y} g_x - F_{3z} g_y g_x - F_3 g_{xy}$$

Comparamos con

$$\iint_D -(F_{3y} - F_{2z}) g_x - (F_{1z} - F_{3x}) g_y + F_{2x} - F_{1y} \; dudv \; \checkmark$$