Lema de levantamiento: $f : M \to N \rightsquigarrow \exists \, \{f_n : P_n \to Q_n\}_{n \geq 0}$

$$
\begin{array}{ll}
P_i \textit{ proyectivos} & \\
 & \\
\textit{exacto} &
\end{array}
\quad
\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d, dotted, "?"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dotted, "?"] & P_0 \arrow[r] \arrow[d, dotted, "?"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_n \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}
$$

$$
f_0 : \qquad
\begin{tikzcd}
P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \to 0 \arrow[d, "f"] \\
Q_1 \arrow[r, "\partial_1"'] & Q_0 \arrow[r, "\partial_0"'] & N \to 0
\end{tikzcd}
$$

inducción:

$$
\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r, "d_n"] \arrow[d, "f_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[d, "\cdots", "\cdots"'] \\
\cdots \arrow[r, "\partial_{n+2}"'] & Q_{n+1} \arrow[r, "\partial_{n+1}"'] & Q_n \arrow[r, "\partial_n"'] & Q_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

Lema de unicidad del levantado a menos de homotopía:
$\{f_n\}_{n\geq 0}$ levanta al $0 \Rightarrow f \sim 0$

$$
\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] \arrow[dl, dotted] & P_n \arrow[r] \arrow[d, "f_n"] \arrow[dl, dotted, "h_n"'] & \cdots \arrow[r] \arrow[dl, dotted] & P_1 \arrow[r] \arrow[d, "f_1"] \arrow[dl, dotted, "h_1"'] & P_0 \arrow[r] \arrow[d, "f_0"] \arrow[dl, dotted, "h_0"'] & M \arrow[r] \arrow[d, "0"] \arrow[dl, dotted, "0"'] & 0 \\
\cdots \arrow[r] & Q_{n+1} \arrow[r] & Q_n \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}
$$

caso 0

$$
\begin{tikzcd}
P_1 \arrow[r, "d_1"] & P_0 \arrow[r] \arrow[d, "f_0"] \arrow[dl, dotted, "h_0?"'] & M \arrow[r] \arrow[d, "0"] \arrow[dl, "h_{-1}=0"'] & 0 \\
Q_1 \arrow[r, "\partial_1"'] & Q_0 \arrow[r, "\partial_0"'] & N \arrow[r] & 0
\end{tikzcd}
$$

paso inductivo

$$\begin{tikzcd}
{} \arrow[r] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] \arrow[dl, dotted, "h_{n+1}?"'] & P_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, "h_n"'] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"'] & {} \\
Q_{n+2} \arrow[r, "\partial_{n+2}"'] & Q_{n+1} \arrow[r, "\partial_{n+1}"'] & Q_n \arrow[r, "\partial_n"'] & Q_{n-1} \arrow[r] & {}
\end{tikzcd}$$

$$f_n = \partial_{n+1} h_n + h_{n-1} d_n, \qquad f_{n+1} = \partial_{n+2}(*?*) + h_n d_{n+1}$$

$$\underbrace{f_{n+1} - h_n d_{n+1}}_{g} = \partial_{n+2}(*?*)$$

Corolario: Unicidad de resolución a menos de equvalencia homotópica:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & P_n \arrow[r] \arrow[d, "f_n"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\cdots \arrow[r] & Q_{n+1} \arrow[r] \arrow[d, "g_{n+1}"] & Q_n \arrow[r] \arrow[d, "g_n"] & \cdots \arrow[r] & Q_1 \arrow[r] \arrow[d, "g_1"] & Q_0 \arrow[r] \arrow[d, "g_0"] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\cdots \arrow[r] & P_{n+1} \arrow[r] & P_n \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, "g_{n+1}f_{n+1}"'] & P_n \arrow[r] \arrow[d, "g_n f_n"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "g_1 f_1"] & P_0 \arrow[r] \arrow[d, "g_0 f_0"] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\cdots \arrow[r] & P_{n+1} \arrow[r] & P_n \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow gf \sim Id_{P_\bullet}$.

Obs: (caso $M = 0$)

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to 0$$

exacto $\Rightarrow$ contráctil

Concluimos, $P_\bullet = P(M)$ bien definido a menos de equivalencia homotópica (i.e. a menos de iso en $\mathcal{H}(A)$), y fijadas resoluciones $P_\bullet$ y $Q_\bullet$ de M y N, está bien definido

$$f : M \to N \rightsquigarrow$$

$$\rightsquigarrow f_\bullet : P_\bullet \to Q_\bullet \in \mathrm{Hom}_{\mathcal{H}(A)}(P_\bullet, Q_\bullet) = \mathrm{Hom}_{\mathrm{Chain}(A)}(P, Q)/\sim$$

Es decir, tomar una resolución da un funtor, definido a menos de iso único

$$A - Mod \to \mathcal{H}(A)$$

$$M \mapsto P(M)$$

$$f \mapsto \{f_n\}_{n \geq 0}$$

Definición:

N_A, ${}_AM \rightsquigarrow$

$$\mathrm{Tor}_n^A(N, M) = H_n(N \otimes_A P_\bullet)$$

donde $P_\bullet \to M \to 0$ es una resolución de M como A-módulo.

Como $P_1 \to P_0 \to M \to 0$ es exacta $\Rightarrow$
$N \otimes_A P_1 \to N \otimes_A P_0 \to N \otimes_A M \to 0$ también

$$\Rightarrow \mathrm{Tor}_0^A(N, M) = H_0(N \underset{A}{\otimes} P_\bullet) =$$

$$= H_0(\cdots \to N \underset{A}{\otimes} P_2 \to N \underset{A}{\otimes} P_1 \to N \underset{A}{\otimes} P_0 \to 0) \cong N \underset{A}{\otimes} M$$

Pero $\mathrm{Tor}_n^A(N, M)$ con $n > 0$ son funtores "nuevos"

Ejemplo: $A = k[x,y]$, $M = N = k$ con $p(x,y)\cdot 1 = p(0,0)$.
Para calcular $\mathrm{Tor}_n^A(k,k)$:

$$P_\bullet \to k \to 0: \quad \begin{array}{l} 0 \to A \longrightarrow A \oplus A \longrightarrow A \longrightarrow k \longrightarrow 0 \\ \qquad p \mapsto (yp, -xp) \qquad p \longmapsto p(0) \\ \qquad\qquad (f,g) \longmapsto xf + yg \end{array}$$

$k \otimes_A P_\bullet$:

$$\begin{tikzcd} 0 \arrow[r] & k \otimes_A A \arrow[r] \arrow[d, "\cong"] & k \otimes_A A \oplus k \otimes_A A \arrow[r] \arrow[d, "\cong"] & k \otimes_A A \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & k \arrow[r] & k \oplus k \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

Ejemplo: $A = k[x, y]$, $M = N = k$ con $p(x, y) \cdot 1 = p(0, 0)$.
Para calcular $\mathrm{Tor}_n^A(k, k)$:

$$P_\bullet \to k \to 0: \quad 0 \to A \longrightarrow A \oplus A \longrightarrow A \longrightarrow k \longrightarrow 0$$

$$p \mapsto (yp, -xp) \qquad p \longmapsto p(0)$$

$$(f, g) \longmapsto xf + yg$$

$k \otimes_A P_\bullet$:

$$\begin{array}{ccccccccc} 0 & \to & k \otimes_A A & \to & k \otimes_A A \oplus k \otimes_A A & \to & k \otimes_A A & \to & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & k & \longrightarrow & k \oplus k & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

$\therefore \mathrm{Tor}_1^A(k, k) = k \oplus k, \qquad \mathrm{Tor}_2^A(k, k) = k.$

Vendrá:

- $0 \to X \to Y \to Z \to 0$ s.e.c. de A-mod $\Rightarrow \forall N_A$:

$$\cdots \to \mathrm{Tor}_1^A(N,X) \to \mathrm{Tor}_1^A(N,Y) \to \mathrm{Tor}_1^A(N,Z) \to N \underset{A}{\otimes} X \to N \underset{A}{\otimes} Y \to N \underset{A}{\otimes} Z \to 0$$

- Tor deriva $\underset{A}{\otimes}$ en las dos variables:

$$\mathrm{Tor}_n^A(N,M) = H_n(N \underset{A}{\otimes} P(M)) \cong H_n(P(N) \underset{A}{\otimes} M) \cong H_n(P(N) \underset{A}{\otimes} P(M))$$

- Cálculo de algunas resoluciones funtoriales
  $(P(-) : A\text{-Mod} \to \mathrm{Chain}(A))$