La vez pasada:

Si $0 \to M_1 \to M_2 \to M_1 \to 0$ es una s.e.c. de A-módulos

Entonces se tiene una s.e.larga

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M_1, N) \arrow[r] & \mathrm{Hom}_A(M_2, N) \arrow[r] & \mathrm{Hom}_A(M_1, N) \arrow[dll, rounded corners] & \\
& \mathrm{Ext}^1_A(M_1, N) \arrow[r] & \mathrm{Ext}^1_A(M_2, N) \arrow[r] & \mathrm{Ext}^1_A(M_1, N) \arrow[dll, rounded corners] & \\
& \mathrm{Ext}^2_A(M_1, N) \arrow[r] & \mathrm{Ext}^2_A(M_2, N) \arrow[r] & \mathrm{Ext}^2_A(M_1, N) \arrow[r] & \cdots
\end{tikzcd}$$

(dem: resolver $P^1_\bullet \to M_1$, $P^1_\bullet \to M_1$ y construir una resolucion de M_2 que en cada lugar sea la suma de las anteriores)

La otra variable

Ahora M fijo, $0 \to N_1 \to N_2 \to N_1 \to 0$ es una s.e.c. de A-módulos, tenemos

$$0 \to \mathrm{Hom}_A(M, N_1) \to \mathrm{Hom}_A(M, N_2) \to \mathrm{Hom}_A(M, N_1)$$

Teorema: La sucesion anterior se extiende a una s.e. larga de la forma

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M, N_1) \arrow[r] & \mathrm{Hom}_A(M, N_2) \arrow[r] & \mathrm{Hom}_A(M, N_3) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^1_A(M, N_1) \arrow[r] & \mathrm{Ext}^1_A(M, N_2) \arrow[r] & \mathrm{Ext}^1_A(M, N_3) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^2_A(M, N_1) \arrow[r] & \mathrm{Ext}^2_A(M, N_2) \arrow[r] & \mathrm{Ext}^2_A(M, N_3) \arrow[r] & \cdots
\end{tikzcd}$$

$$0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \longrightarrow 0$$

dem: resolvemos M

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

Tomamos el complejo $P_\bullet$

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to 0$$

Aplicamos $\mathrm{Hom}_A(-, N_i)$, $i = 1, 2, 3$ y tenemos el diagrama

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_A(P_0, N_1) \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}_A(P_1, N_1) \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}_A(P_2, N_1) \arrow[r] \arrow[d, "f_*"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_A(P_0, N_2) \arrow[r] \arrow[d, "g_*"] & \mathrm{Hom}_A(P_1, N_2) \arrow[r] \arrow[d, "g_*"] & \mathrm{Hom}_A(P_2, N_2) \arrow[r] \arrow[d, "g_*"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_A(P_0, N_3) \arrow[r] & \mathrm{Hom}_A(P_1, N_3) \arrow[r] & \mathrm{Hom}_A(P_2, N_3) \arrow[r] & \cdots
\end{tikzcd}
$$

pero como P_i es proyectivo $\forall i$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_A(P_0, N_1) \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}_A(P_1, N_1) \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}_A(P_2, N_1) \arrow[r] \arrow[d, "f_*"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_A(P_0, N_2) \arrow[r] \arrow[d, "g_*"] & \mathrm{Hom}_A(P_1, N_2) \arrow[r] \arrow[d, "g_*"] & \mathrm{Hom}_A(P_2, N_2) \arrow[r] \arrow[d, "g_*"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_A(P_0, N_3) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_1, N_3) \arrow[r] \arrow[d] & \mathrm{Hom}_A(P_2, N_3) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\therefore\ 0 \longrightarrow \mathrm{Hom}_A(P_\bullet, N_1) \xrightarrow{f_*} \mathrm{Hom}_A(P_\bullet, N_2) \xrightarrow{g_*} \mathrm{Hom}_A(P_\bullet, N_3) \longrightarrow 0$$

es una s.e.c. de complejos $\rightsquigarrow$ s.e.larga en (co)homología s.e.larga en la cada variable: $\forall(0 \to X \to Y \to Z \to 0), P, I$:

$$0 \to \mathrm{Hom}_A(P,X) \to \mathrm{Hom}_A(P,Y) \to \mathrm{Hom}_A(P,Z) \to \mathrm{Ext}^1_A(P,X) \to \cdots$$

$$0 \to \mathrm{Hom}_A(Z,I) \to \mathrm{Hom}_A(Y,I) \to \mathrm{Hom}_A(X,I) \to \mathrm{Ext}^1_A(Z,I) \to \cdots$$

$\therefore$ P proyectivo $\iff \mathrm{Ext}^1_A(P,N) = 0\ \forall N$
I inyectivo $\iff \mathrm{Ext}^1_A(M,I) = 0\ \forall M$

[criterio de Baer] $J \subset A$ ideal, I inyectivo $\iff$

$$\begin{tikzcd} J \arrow[r, hook, "i"] \arrow[d, "\forall f"'] & A \arrow[dl, dotted, "\exists! \widetilde{f}"] \\ I & \end{tikzcd}$$

Coro: I inyectivo $\iff \mathrm{Ext}^1_A(A/J, I) = 0\ \forall$ ideal $J \subset A$

$\mathrm{Ext}^\bullet$ derivando la 2da variable:

Dados M, N, se toma

$$0 \to N \to I_0 \to I_{-1} \to I_{-2} \to I_{-3} \to \cdots$$

Notación: $0 \to N \to I^0 \to I^1 \to I^2 \to I^3 \to \cdots$

una resolución inyectiva. Se define
$I_\bullet := \left(0 \to I^0 \to I^1 \to I^2 \to I^3 \to \cdots\right)$
(con esa indexación es un compejo de CO-cadenas)

$$\widetilde{\mathrm{Ext}}_A^\bullet(M,N) := H^n(\mathrm{Hom}_A(M, I^\bullet))$$

Teo: $\widetilde{\mathrm{Ext}}^\bullet_A(M,N) \cong \mathrm{Ext}^\bullet(M,N)$
dem: Consideramos una resolución proyectiva $P_\bullet \to M$ y el complejo **doble**

$$C_{ij} := \mathrm{Hom}_A(P_i, I^j)$$

con diferenciales los que vienen de $P_\bullet$ y los que vienen de $I^\bullet$ (con signos)
A partir de

$$0 \longrightarrow N \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$$

y

$$\cdots \xrightarrow{\partial_2} P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

se tiene

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
\mathrm{Hom}_A(P_2,N) \arrow[r, "\eta_*"] \arrow[u, "\partial_2^*"] & \mathrm{Hom}_A(P_2,I^0) \arrow[r, "d_0"] \arrow[u, "-\partial_2^*"] & \mathrm{Hom}_A(P_2,I^1) \arrow[r, "d_1"] \arrow[u, "\partial_2^*"] & \mathrm{Hom}_A(P_2,I^2) \arrow[r, "d_2"] \arrow[u, "-\partial_2^*"] & \cdots \\
\mathrm{Hom}_A(P_1,N) \arrow[r, "\eta_*"] \arrow[u, "\partial_1^*"] & \mathrm{Hom}_A(P_1,I^0) \arrow[r, "d_0"] \arrow[u, "-\partial_1^*"] & \mathrm{Hom}_A(P_1,I^1) \arrow[r, "d_1"] \arrow[u, "\partial_1^*"] & \mathrm{Hom}_A(P_1,I^2) \arrow[r, "d_2"] \arrow[u, "-\partial_1^*"] & \cdots \\
\mathrm{Hom}_A(P_0,N) \arrow[r, "\eta_*"] \arrow[u, "\partial_0^*"] & \mathrm{Hom}_A(P_0,I^0) \arrow[r, "d_0"] \arrow[u, "-\partial_0^*"] & \mathrm{Hom}_A(P_0,I^1) \arrow[r, "d_1"] \arrow[u, "\partial_0^*"] & \mathrm{Hom}_A(P_0,I^2) \arrow[r, "d_2"] \arrow[u, "-\partial_0^*"] & \cdots \\
\mathrm{Hom}_A(M,N) \arrow[r, "\eta_*"] \arrow[u, "\epsilon^*"] & \mathrm{Hom}_A(M,I^0) \arrow[r, "d_0"] \arrow[u, "-\epsilon^*"] & \mathrm{Hom}_A(M,I^1) \arrow[r, "d_1"] \arrow[u, "\epsilon^*"] & \mathrm{Hom}_A(M,I^2) \arrow[r, "d_2"] \arrow[u, "-\epsilon^*"] & \cdots
\end{tikzcd}
$$

$$\mathrm{Ext}^\bullet_A(M, N) \to \widetilde{\widetilde{\mathrm{Ext}^\bullet}}_A(M, N) \leftarrow \widetilde{\mathrm{Ext}^\bullet}_A(M, N)$$

Ext y extensiones

$[f] \in \mathrm{Ext}^1(M, N)$, significa:

$$\cdots \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

y $f \in \mathrm{Hom}_A(P_1, N)$ tal que $d^*f = 0$, o sea, $f \circ d_1 = 0$, o bien $f|_{Imd_1} \equiv 0$. Se puede armar un diagrama

$$\begin{tikzcd}
{} \arrow[r, "d_1"] & P_1 \arrow[r, "d_0"] \arrow[d, "f"] & P_0 \arrow[r, "\epsilon"] & M \arrow[r] & 0 \\
& N & & &
\end{tikzcd}$$

que se puede completar como

$$\begin{tikzcd}
P_1 \arrow[r, "d"] \arrow[d, "f"] & P_0 \arrow[r, "\epsilon"] \arrow[d] & M \arrow[r] & 0 \\
N \arrow[r, "j"] & N \oplus_{P_1} P_0 & &
\end{tikzcd}$$

$$\begin{tikzcd} P_1 \arrow[r, "d_0"] \arrow[d, "f"] & P_0 \arrow[r, "\epsilon"] \arrow[d] & M \arrow[r] & 0 \\ N \arrow[r, "j"] & N \oplus_{P_1} P_0 & & \end{tikzcd}$$

donde $N \oplus_{P_1} P_0 = (N \oplus P_0)/\langle (f(p), 0) - (0, d(p)) : p \in P_1 \rangle$ es el push-out

Afirmación: j es mono:

$$j(n) = \overline{(n,0)} = 0 \iff \exists p : (n,0) = (fp, -dp)$$

$$\Rightarrow dp = 0, \Rightarrow p = d_1(p_2)$$

$$\Rightarrow n = f(p) = f(d_1(p_2)) = 0$$

(pues $d_1^* f = 0$)

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "d_0"] \arrow[d, "f"] & P_0 \arrow[r, "\epsilon"] \arrow[d] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r, "j"] & N \oplus_{P_1} P_0 & &
\end{tikzcd}$$

o bien, ya que $f|_{Im(d_1)} \equiv 0$,

$$\begin{tikzcd}
0 \arrow[r] & P_1/\mathrm{Ker}(d) \arrow[r, "d_0"] \arrow[d, "\overline{f}"] & P_0 \arrow[r, "\epsilon"] \arrow[d] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r, "j"] & N \oplus_{P_1} P_0 & &
\end{tikzcd}$$

Tiene un conúcleo:

$$\begin{tikzcd}
0 \arrow[r] & P_1/\mathrm{Ker}(d)f \arrow[r, "d_0"] \arrow[d, "\overline{f}"] & P_0 \arrow[r, "\epsilon"] \arrow[d] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r, "j"] & N \oplus_{P_1} P_0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & P_1/\mathrm{Ker}(d)f \arrow[r, "d_0"] \arrow[d, "\overline{f}"] & P_0 \arrow[r, "\epsilon"] \arrow[d] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r, "j"] & N \oplus_{P_1} P_0 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Afirmacion: $M \cong C$:

Hecho general: $\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "\phi"] & Y \arrow[r, "p"] \arrow[d] & M \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] & p.out \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$ $\Rightarrow C \cong M$.

dem:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "\phi"] & Y \arrow[r, "p"] \arrow[d] \arrow[dr, bend left, "p"] & M \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] \arrow[rr, bend right, "0"'] & p.out \arrow[r, dotted] & M \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "\phi"] & Y \arrow[r, "p"] \arrow[d] \arrow[dr, bend left, "p"] & M \arrow[r] & 0 \\
0 \arrow[r] & Z \arrow[r, "j"] \arrow[rr, bend right, "0"'] & p.out \arrow[r, dotted] & M \arrow[r] & 0
\end{tikzcd}$$

$$\overline{(0, y)} \mapsto p(y) \;\Rightarrow\; \text{epi}$$

$$\overline{(z, y)} \mapsto p(y) = 0 \Rightarrow y = i(x)$$

$$\Rightarrow \overline{(z, y)} = \overline{(z, i(x))} = \overline{(z - \phi(x), 0)}$$

$$= j(z - \phi(x))$$

$$\Rightarrow \mathrm{Ker}(\cdots >) = Im(j)$$

Concluimos

$$\begin{tikzcd} \arrow[r, "d_1"] & P_1 \arrow[r, "d_0"] \arrow[d, "f"] & P_0 \arrow[r, "\epsilon"] & M \arrow[r] & 0 \\ & N & & & \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} 0 \arrow[r] & P_1/\mathrm{Ker}(d)f \arrow[r, "d_0"] \arrow[d, "\overline{f}"] & P_0 \arrow[r, "\epsilon"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "j"] & E_f \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Supongamos ahora que tenemos una s.e.c.

$$0 \to N \to E \to M \to 0$$

y la comparamos con la resolucion:

$$\begin{tikzcd}
\cdots \arrow[r, "d_1"] & P_1 \arrow[r, "d_0"] & P_0 \arrow[r, "\epsilon"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

el lema de levantamiento implica

$$\begin{tikzcd}
\cdots \arrow[r, "d_1"] & P_1 \arrow[r, "d_0"] \arrow[d, "f"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$\rightsquigarrow f : P_1 \to N$ tal que $d_1^* f = 0$

$$
\begin{tikzcd}
\cdots \arrow[r, "d_1"] & P_1 \arrow[r, "d_0"] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] & P_0 \arrow[r, "\epsilon"] \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "f_0"] \arrow[dl, dotted, "h"'] & M \arrow[r] \arrow[d, equal] \arrow[dl, dotted] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}
$$

Ademas el levantado es único a menos de homotopia,

$$\Rightarrow \exists h : P_0 \to N \text{ tal que } f - g = hd_0 + 0 = d_0^*(h)$$

$$\Rightarrow f - g \in d^*\Big(\mathrm{Hom}(P_0, N)\Big) \Rightarrow [f] = [g] \in \mathrm{Ext}^1_A(M, N)$$

Def:

$$\mathcal{E}_1 = (\, 0 \longrightarrow N \xrightarrow{a_1} E_1 \xrightarrow{b_1} M \longrightarrow 0 \,)$$

$$\mathcal{E}_2 = (\, 0 \longrightarrow N \xrightarrow{a_1} E_1 \xrightarrow{b_1} M \longrightarrow 0 \,)$$

decimos $\mathcal{E}_1 \sim \mathcal{E}_2 \iff \exists \phi : E_1 \to E_2$ tal que

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "a_1"] \arrow[d, equal] & E_1 \arrow[r, "b_1"] \arrow[d, "\phi"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "a_2"] & E_2 \arrow[r, "b_2"] & M \arrow[r] & 0
\end{tikzcd}$$

Teorema: $\exists$ biyección entre $\mathrm{Ext}^1_A(M, N)$
y clases de equivalencia de extensiones $\mathcal{E}$
de la forma $0 \to N \to E \to M \to 0$

Teorema: $\exists$ biyección entre $\mathrm{Ext}^n_A(M,N)$
y clases de equivalencia de extensiones $\mathcal{E}$
de la forma $0 \to N \to E_n \to \cdots \to E_2 \to E_1 \to M \to 0$

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d, dotted, "f"'] & P_{n-1} \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dotted] & P_0 \arrow[r] \arrow[d, dotted] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E_n \arrow[r] & \cdots \arrow[r] & E_2 \arrow[r] & E_1 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$