Grupos y Álgebras de Lie
1er cuatrimestre 2021

Clase 23

Caracteres II

Caracteres como funciones generatrices

Caso $\mathfrak{sl}(2,\mathbb{C})$ Miramos autovalores de $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

V	$\bigoplus_i (V)_i$	$\mathrm{ch}_V = \sum_i \dim(V_i) q^i$
V_0	$\bullet^{0}_{v_0}$	$q^0 = 1$
V_1	$\bullet^{+1}_{v_0} \underset{X}{\overset{Y}{\rightleftarrows}} \bullet^{-1}_{v_1}$	$q + q^{-1}$
V_2	$\bullet^{+2}_{v_0} \underset{X}{\overset{Y}{\rightleftarrows}} \bullet^{0}_{v_1} \underset{X}{\overset{Y}{\rightleftarrows}} \bullet^{-2}_{v_2}$	$q^2 + 1 + q^{-2}$
V_3	$\bullet^{+3}_{v_0} \underset{X}{\overset{Y}{\rightleftarrows}} \bullet^{+1}_{v_1} \underset{X}{\overset{Y}{\rightleftarrows}} \bullet^{-1}_{v_2} \underset{X}{\overset{Y}{\rightleftarrows}} \bullet^{-3}_{v_3}$	$q^3 + q + q^{-1} + q^{-3}$

Caracteres como trazas

Caso general: $\{h_1, \ldots, h_\ell\}$ es base de $\mathfrak{h}$, si $H \in \mathfrak{h}$,

$$H = \tau_1 h_1 + \cdots + \tau_\ell h_\ell, \qquad , \tau_i \in \mathbb{C}$$

$$\mathrm{ch}(V) : \mathfrak{h} \to \mathbb{C}$$

$$\mathrm{ch}(V)(H) = \mathrm{tr}(e^H|_V) = \sum_\mu \dim(V_\mu)\mathrm{tr}(e^H|_{V_\mu})$$

$$v \in V_\mu \Rightarrow e^H \cdot v = e^{\mu(H)} v = e^{\sum_i \tau_i \mu(h_i)} v$$

$$= \prod_{i=1}^{\ell} (e^{\tau_i})^{\mu(h_i)} v = \prod_{i=1}^{\ell} q_i^{\mu(h_i)} v$$

$$\Rightarrow \mathrm{ch}(V)(H) = \sum_\mu \dim(V_\mu) \prod_{i=1}^{\ell} q_i^{\mu(h_i)} \in \mathbb{C}[q_1^\pm, \ldots, q_\ell^\pm]$$

El caracter como objeto algebraico

$V \in \mathfrak{g}$-mod, de pesos / dim $V_\mu < \infty \forall \mu$, ch=la suma formal:

$$\mathrm{ch}\, V := \sum_\mu \dim V_\mu e^\mu$$

Si dim $V < \infty$, entonces $\mu \in \Lambda$ retículo $\cong \mathbb{Z}^\ell$

$$\Lambda \cong \mathbb{Z}^\ell$$

$$\lambda_i(h_j) = \delta_{ij} \leftrightarrow e_i$$

$$\mathrm{ch}\, V \in \mathbb{C}[\Lambda] \cong \mathbb{C}[\mathbb{Z}^\ell] \cong \mathbb{C}[q_1^{\pm 1}, \ldots, q_\ell^{\pm 1}]$$

$$e^{\lambda_i} \quad \leftrightarrow \quad e_i \quad \leftrightarrow \quad q_i$$

Claramente valen $V \cong W \Rightarrow \mathrm{ch}(V) = \mathrm{ch}(W)$,

$$\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}\, W$$

$$\mathrm{ch}(V \otimes W) = \mathrm{ch}(V) \cdot \mathrm{ch}\, W$$

(con el producto $e^\mu e^\lambda = e^{\mu+\lambda}$)

Weyl character formula

Caso $\mathfrak{sl}(2,\mathbb{C})$, $V = V(m)$, observamos

$$\mathrm{ch}(V(m)) = q^{-m} + q^{-m+2} + \cdots q^{m-2} + q^{m} = \frac{q^{(m+1)} - q^{-(m+1)}}{q - q^{-1}}$$

Weyl character formula

Caso $\mathfrak{sl}(2,\mathbb{C})$, $\mathrm{ch}(V):\mathfrak{h}\to\mathbb{C}$, $q=e^{\tau}$, $\tau H\in\mathfrak{h}=\mathbb{C}H$

$$\mathrm{ch}(V(m))=\frac{q^{(m+1)}-q^{-(m+1)}}{q-q^{-1}}$$

$\alpha\in\{\alpha\}=\Delta\subset\mathfrak{h}^*=(\mathbb{C}H)^*$, $\alpha(H)=2$,
$w(\alpha)=s_\alpha(\alpha)=-\alpha$, $s_\alpha(H)=-H$,
$V(m)=L(\lambda)$ con $\lambda(H)=m\Rightarrow\lambda=\frac{m}{2}\alpha$

$$q^m=e^{\tau m}=e^{\tau\lambda(H)}=e^{\lambda(\tau H)},\ q^{m+1}=e^{(\lambda+\frac{\alpha}{2})(\tau H)}$$

$$\therefore\ \mathrm{ch}(V(m))=\mathrm{ch}(L(\lambda))=\frac{e^{(\lambda+\frac{\alpha}{2})(\tau H)}-e^{(\lambda+\frac{\alpha}{2})(w\tau H)}}{e^{\frac{\alpha}{2}\tau H}-e^{-\frac{\alpha}{2}\tau H}}$$

Weyl character formula

λ tal que $\dim L(\lambda) < \infty$, la rep. simple de peso máximo λ, $H \in \mathfrak{h}$:

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{\omega \in W} (-1)^{\omega} e^{w(\lambda+\rho)}}{\prod_{\alpha>0} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})} \in \mathbb{C}[\Lambda] \subset \mathbb{C}[\tfrac{1}{2}\Lambda]$$

donde $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$, $(-1)^{\omega} = (-1)^{\ell(\omega)}$,
$\ell(\omega)$ = longitud en W,

Sobre la definición de signo:

Sobre la definición de signo:

Consideremos $\Lambda := \frac{1}{2} < \Phi >_{\mathbb{Z}}$ y

$$D := \prod_{\alpha > 0} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \in \mathbb{C}[\Lambda]$$

Lema

$\alpha_0 \in \Delta \Rightarrow s_{\alpha_0}(\Phi^+ \setminus \{\alpha_0\}) = \Phi^+ \setminus \{\alpha_0\}$, *y claramente* $s_{\alpha_0}(\alpha_0) = -\alpha_0$

dem: $s_\alpha(\beta) = \beta - \frac{2\kappa(\beta,\alpha)}{\kappa(\alpha,\alpha)}\alpha = \beta + (..)\alpha$

Coro: $s_{\alpha_0}(D) = -D$, y por lo tanto

$$w \cdot (D) = \pm D$$

Queda definido $sgn(w) = (-1)^w$ via $w \cdot (\delta) = (-1)^w \delta$

Sobre la dem de la WCF

Llamemos M_λ= módulo de Verma de peso máx λ.
Comenzamos con un epi

$$M_\lambda \twoheadrightarrow L(\lambda)$$

$\therefore \mathrm{ch}(L(\lambda)) = \mathrm{ch}(M_\lambda) - \cdots$

Teorema (BGG=Berstein-Gelfand-Gelfand)

Existe una sucesión exacta

$$0 \to M_{w_0 * \lambda} \to \cdots \bigoplus_{w:\ell(w)=k} M_{w*\lambda} \to \cdots \bigoplus_{\alpha\in\Phi^+} M_{s_\alpha * \lambda} \to M_\lambda \to L(\lambda) \to 0$$

donde $w * \lambda = w(\lambda + \rho) - \rho$

dem: final Pablo??

BGG para $\mathfrak{sl}(2,\mathbb{C})$

El Verma $M_m = \mathbb{C}[Y] \otimes \mathbb{C}_\lambda$ con $\lambda(H) = m \Rightarrow \lambda = \frac{m}{2}\alpha$
Base:

$$
\begin{array}{ccccc|ccc}
\bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \cdots \\
1 & Y & \cdots & Y^{m-1} & Y^m & Y^{m+1} & Y^{m+2} & \cdots \\
m & m-2 & \cdots & -m+2 & -m & -m-2 & -m-4 & \cdots \\
\lambda & \lambda-\alpha & \cdots & & \lambda-m\alpha & \lambda-(m+1)\alpha & \lambda-(m+2)\alpha & \cdots \\
\frac{m\alpha}{2} & \frac{(m-2)\alpha}{2} & \cdots & & \frac{-m\alpha}{2} & \frac{-(m+2)\alpha}{2} & \frac{-(m+4)\alpha}{2} & \cdots \\
 & & & & & -(\lambda+\frac{\alpha}{2})-\frac{\alpha}{2} & & \\
 & & & & & w(\lambda+\rho)-\rho & &
\end{array}
$$

Pesos de $V(m) = L(\lambda)$:
m, m-2, m-4, $\ldots$, -m+2, -m

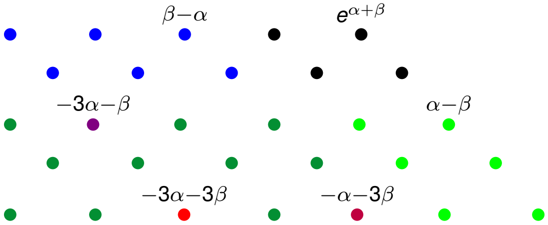
$$V(m) \cong M_m/M_{-m-2} = M_\lambda/M_{-\lambda-\alpha}$$

BGG para $\mathfrak{sl}(3,\mathbb{C})$

Dibujo $\mathfrak{sl}(3,\mathbb{C})$ con $V = \mathfrak{g}^{ad} = L(1,1)$

$$\begin{array}{ccccc} & x_\beta & & x_{\alpha+\beta} & \\ y_\alpha & & h_1, h_2 & & x_\alpha \\ & y_{\alpha+\beta} & & y_\beta & \end{array} \quad \text{dimensiones: } \begin{array}{ccccc} & 1 & & 1 & \\ 1 & & 2 & & 1 \\ & 1 & & 1 & \end{array}$$

En el Verma $M(\lambda)$, $\lambda = \alpha + \beta = \rho$



WCF a partir de BGG

Observamos que si

$$0 \to V_d \to V_{d-1} \to \cdots \to V_1 \to V_0 \to 0$$

es una sucesión exacta de esp. vect. de dim finita

$$\Rightarrow \sum_i (-1)^i \dim(V_i) = 0$$

Consecuencia: para μ fijo

$$\sum_k (-1)^k \dim\left(\bigoplus_{w:\ell(w)=k} (M_{w*\lambda})_\mu \right) = \dim L(\lambda)_\mu$$

$$\Rightarrow \sum_w (-1)^{\ell(w)} \mathrm{ch}(M_{w*\lambda}) = \mathrm{ch} L(\lambda)$$

Calculemos $\mathrm{ch}(M_\lambda)$:

Si $Y_{\alpha_1}, \ldots, Y_{\alpha_r}$ son vectores raíces ($0 \neq Y_\alpha \in \mathfrak{g}_{-\alpha}$, $\alpha > 0$), entonces como espacio vectorial

$$M_\lambda = \bigoplus_{(n_1,\ldots,n_k)\in\mathbb{N}_0^k} \mathbb{C}\, Y_{\alpha_1}^{n_1} Y_{\alpha_2}^{n_2} \cdots Y_{\alpha_k}^{n_k} \overline{1}$$

Como espacio vectorial y como $\mathfrak{h}$-módulo

$$M_\lambda \cong \mathbb{C}[Y_{\alpha_1}] \otimes \mathbb{C}[Y_{\alpha_2}] \otimes \cdots \otimes \mathbb{C}[Y_{\alpha_k}] \otimes \mathbb{C}_\lambda$$

$$\Rightarrow \mathrm{ch}(M_\lambda) = \Big(\prod_{\alpha>0} \mathrm{ch}(\mathbb{C}[Y_\alpha]) \Big) e^\lambda$$

$$\mathrm{ch}(\mathbb{C}[Y_\alpha]) = \sum_{n=0}^{\infty} e^{n\alpha} = \frac{1}{1 - e^\alpha}$$

$$\Rightarrow \mathrm{ch}(M_\lambda) = \frac{e^\lambda}{\prod_{\alpha>0} 1 - e^\alpha} = \frac{e^{\rho+\lambda}}{\prod_{\alpha>0} e^{\alpha/2} - e^{-\alpha/2}}$$

Expresiones antisimétricas

Usando BGG

$$\mathrm{ch}(L(\lambda)) = \sum_{w} (-1)^{\ell(w)} \mathrm{ch}(M_{w*\lambda})$$

$$+ \qquad \mathrm{ch}(M_\lambda) = \frac{e^{\rho+\lambda}}{\prod_{\alpha>0} e^{\alpha/2} - e^{-\alpha/2}}$$

(notar $\rho + w * \lambda = w(\rho + \lambda)$ concluimos WCF :

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_w (-1)^{\ell(w)} e^{w(\rho+\lambda)}}{\prod_{\alpha>0} e^{\alpha/2} - e^{-\alpha/2}}$$

Expresiones antisimétricas

Diremos que $\xi \in \mathbb{C}[\mathfrak{h}^*] = \sum_\mu n_\mu e^\mu$ es antisimétrico si

$$w \cdot \xi := \sum_\mu n_\mu e^{w\mu} = (-1)^w \xi$$

Ejemlo 1: $D = \prod_{\alpha>0}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})$

Ejemlo 2: $\forall \lambda \in \mathfrak{h}^*$,

$$A(\lambda) := \sum_{w \in W} (-1)^w e^{w\lambda}$$

Proposición (Weyl denominator formula)

$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$, *entonces*

$$D = \prod_{\alpha>0}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) = A(\rho)$$

Expresiones antisimétricas

dem:

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_w (-1)^{\ell(w)} e^{w(\rho+\lambda)}}{\prod_{\alpha>0} e^{\alpha/2} - e^{-\alpha/2}}$$

$L(\lambda = 0) = \mathbb{C}$

$$\Rightarrow 1 = \frac{\sum_w (-1)^{\ell(w)} e^{w(\rho)}}{\prod_{\alpha>0} e^{\alpha/2} - e^{-\alpha/2}}$$

$\dim(\lambda) < \infty$ entonces

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{w\in W}(-1)^w e^{w\cdot(\lambda+\rho)}}{\sum_{w\in W}(-1)^w e^{w\cdot\rho}}$$

$$= \frac{\sum_{w\in W}(-1)^w e^{w\cdot(\lambda+\rho)}}{\prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2})}$$

$$\dim_q(V) := \mathrm{tr}(q^{2\rho}) = \mathrm{ch}(V)(\tau H_0)$$

donde $H_0 = \sum_{\alpha>0}^{\ell} h_\alpha$

$$\dim_q(V) = \mathrm{tr}(q^{2\rho}) = \sum_{\mu} \dim(V_\mu) q^{2\kappa(\rho,\mu)}$$

$$\dim_q(V) = \sum_{\mu} \dim(V_\mu) q^{2\kappa(\rho,\mu)}$$

Recordamos $\mathrm{ch}(V) = \sum_\mu \dim(V_\mu) e^\mu$. Si definimos

$$\pi_\rho : \mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[q^\pm]$$
$$e^\mu \mapsto q^{2\kappa(\rho,\mu)}$$

$$\Rightarrow \dim_q V = \pi_\rho(\mathrm{ch}(V))$$

De WCF

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{w\in W}(-1)^w e^{w\cdot(\lambda+\rho)}}{\sum_{w\in W}(-1)^w e^{w\cdot\rho}}$$

se sigue

$$\dim_q L(\lambda) = \mathrm{ch}(L(\lambda)) = \frac{\sum_{w\in W}(-1)^w q^{2\kappa(w\cdot(\lambda+\rho),\rho)}}{\sum_{w\in W}(-1)^w q^{2\kappa(w\cdot\rho,\rho)}}$$

Usamos la fórmula

$$\boxed{\boxed{\sum_{w\in W}(-1)^w e^{w\cdot\rho} = \prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2})}}$$

Entonces el numerador

$$\sum_{w\in W}(-1)^w q^{2\kappa(w\cdot(\lambda+\rho),\rho)} = \sum_{w\in W}(-1)^w q^{2\kappa(\lambda+\rho, w\cdot\rho)}$$

$$= \pi_{\lambda+\rho}\Big(\sum_{w\in W}(-1)^w e^{w\cdot\rho}\Big) = \pi_{\lambda+\rho}\Big(\prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2})\Big)$$

$$= \prod_{\alpha>0}(q^{\kappa(\lambda+\rho,\alpha)} - q^{-\kappa(\lambda+\rho,\alpha)})$$

y de manera análoga ($\lambda = 0$) el denominador

$$\sum_{w\in W}(-1)^w q^{2\kappa(w\cdot(\rho),\rho)} = \prod_{\alpha>0}(q^{\kappa(\rho,\alpha)} - q^{-\kappa(\rho,\alpha)})$$

Hemos demostrado:

Corolario

$$\dim_q L(\lambda) = \prod_{\alpha>0} \frac{q^{\kappa(\rho+\lambda,\alpha)} - q^{-\kappa(\rho+\lambda,\alpha)}}{q^{\kappa(\rho,\alpha)} - q^{-\kappa(\rho,\alpha)}}$$

Lema

$m, n \in \mathbb{C}$, $q^z := e^{z \ln q}$, entonces

$$\lim_{q\to 1} \frac{q^m - q^{-m}}{q^n - q^{-n}} = \frac{m}{n}$$

dem:

$$\frac{q^m - q^{-m}}{q^n - q^{-n}} = \frac{q^m}{q^n}\frac{(1 - q^{-2m})}{(1 - q^{-2n})}$$

Como $\frac{q^m}{q^n} \to 1$ basta ver

$$\lim_{q \to 1} \frac{1 - q^{-2m}}{1 - q^{-2n}}$$

y por l'Hôpital

$$\lim_{q \to 1} \frac{1 - q^{-2m}}{1 - q^{-2n}} = \lim_{q \to 1} \frac{-(-\ln(q)2m)q^{-2m}}{-(-\ln(q)2n)q^{-2n}} = \frac{m}{n}$$

Corolario

$$\dim_{\mathbb{C}} L(\lambda) = \lim_{q\to 1} \dim_q L(\lambda) = \lim_{q\to 1} \prod_{\alpha>0} \frac{q^{\kappa(\rho+\lambda,\alpha)} - q^{-\kappa(\rho+\lambda,\alpha)}}{q^{\kappa(\rho,\alpha)} - q^{-\kappa(\rho,\alpha)}}$$

$$= \prod_{\alpha>0} \frac{\kappa(\rho+\lambda,\alpha)}{\kappa(\rho,\alpha)} = \prod_{\alpha>0} \frac{(\rho+\lambda,\alpha)}{(\rho,\alpha)}$$

En la última igualdad cambiamos $\kappa(-,-)$ por $(-,-) = cte\kappa$, total la formula no cambia. En las álgebras matriciales, podemos usar la traza del producto.

Caso $\mathfrak{sl}(3, \mathbb{C})$

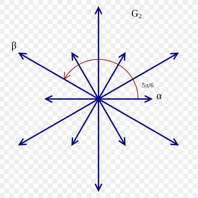
$$\dim_{\mathbb{C}} L(\lambda) = \prod_{\alpha > 0} \frac{\kappa(\rho + \lambda, \alpha)}{\kappa(\rho, \alpha)}$$

En $\mathfrak{sl}(3, \mathbb{C})$, hay 3 raíces positivas $\alpha, \beta, \alpha + \beta$, $\rho = \frac{1}{2}(\alpha + \beta + (\alpha + \beta)) = \alpha + \beta$. Si cambiamos κ a menos de múltiplo la fórmula no cambia:

$$= \frac{(\lambda + \alpha + \beta, \alpha)}{(\alpha + \beta, \alpha)} \frac{(\lambda + \alpha + \beta, \beta)}{(\alpha + \beta, \beta)} \frac{(\lambda + \alpha + \beta, \alpha + \beta)}{(\alpha + \beta, \alpha + \beta)}$$

Tomamos el pairing tal que $(\alpha_i, \alpha_j) = a_{ij}$, la matriz de Cartan con 2 y -1, y así $(\lambda, \alpha) = m_1$, $(\lambda, \beta) = m_2$

$$\dim(L(\lambda)) = (m_1 + 1)(m_2 + 1) \frac{(m_1 + m_2 + 2)}{2}$$



¿cuál es a rep. de dim. mínima de $\mathfrak{g}_2$?

Φ^+: $\alpha, \beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha, 2\beta+3\alpha$

α es corta β es larga. $\rho = 3\beta + 5\alpha$.

$$\dim_{\mathbb{C}} L(\lambda) = \prod_{\alpha>0} \frac{\kappa(\rho+\lambda, \alpha)}{\kappa(\rho, \alpha)}$$

Tomamos el pairing t.q. $(\alpha, \alpha) = 2$, $(\beta, \beta) = 6$, $(\alpha, \beta) = -3$

$$\lambda(h_\alpha) = m_1, \lambda(h_\beta) = m_2 \Rightarrow \lambda = (2m_1 + 3m_2)\alpha + (m_1 + 2m_2)\beta (\textit{chequear!})$$

$$\dim(L(\lambda)) = (..)(..)(..)(..)(..)(..)$$