INTRODUCTION TO THE THEORY OF MOTIVES

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1. Some equivalence relations on cycles

We fix a field k and write $V(k)$ for the category of connected, proper and smooth k-schemes. The objects of $V(k)$ will be called *varieties*.

For $X \in V(k)$, let $Z(X) = \bigoplus_{r \geq 0} Z^r(X)$ be the group of algebraic cycles on X, graded by codimension (this is actually a finite sum since $Z^r(X) = 0$ for $r > \dim X$). By definition, $Z^r(X)$ is the free abelian group generated by the closed integral subschemes of X of codimension r (these generating cycles will be called *prime cycles* (of codimension r)). We note that $Z^0(X)$ is the free group generated by the cycle X.

Two prime cycles Z, W of X are said to *intersect properly* if, for every irreducible component Y_α of $Z \cap W$, $\operatorname{codim} Y_\alpha = \operatorname{codim} Z + \operatorname{codim} W$. When Z and W intersect properly, their intersection product is the cycle defined by

$$Z \cdot W = \sum_\alpha I(Z \cdot W, Y_\alpha; X)\, Y_\alpha\,,$$

where the coefficients are the intersection multiplicities. Extending this definition by linearity we obtain a partial product on $Z(X)$ which is commutative and associative.

Let T be a smooth, connected and quasi-projective scheme over k and let Z be a prime cycle on $X \times T$ of codimension p. Let t be a closed point of T such that $Z \not\subset X \times \{t\}$, and let $W_1, \dots W_r$ be the irreducible components of $Z \cap (X \times \{t\})$. Define the intersection cycle of Z and the fiber $X \times \{t\}$ to be

$$Z(t) = \sum \operatorname{length}\left(\mathcal{O}_{Z,W_i}/(f_i)\right) W_i\,,$$

where f_i is the image in $\mathcal{O}_{Z,W_i}$ of a local parameter at t. (The coefficient appearing in this sum should be interpreted as the "order of vanishing of f_i along W_i".)

If Z is any cycle in $Z^p(X)$ with no fibral components, then the above definition extends by linearity and $\{Z(t)\}_{t \in T(k)}$ is called an "algebraic family parametrized by T".

Now two cycles W and W' in $Z^p(X)$ are said to be *algebraically equivalent* if there exists an algebraic family $\{Z(t)\}$ such that $W = Z(t)$ and $W' = Z(t')$ for

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suitable t, t' (note the analogy with the definition of two paths being homotopically equivalent). It is not difficult to show that algebraic equivalence is compatible with substraction, so it is an equivalence relation. If the parameter scheme T is the projective line $\mathbb{P}^1 = \mathbb{P}^1_k$, then algebraic equivalence is called *rational equivalence*. Clearly, if two cycles are rationally equivalent, then they are algebraically equivalent.

We now consider the abelian group

$$\mathrm{CH}(X) = Z(X)/\sim$$

of classes of cycles modulo rational equivalence. This group can be given the structure of a commutative ring (with unit) by means of the intersection product on cycles. This will follow from

Proposition 1.1.

(1) *("Chow's moving lemma") If Z, W are cycles on a variety X, then there exists a cycle Y rationally equivalent to W such that the intersection product $Z \cdot Y$ is defined.*

(2) *Let Z, W, Y be three cycles on a variety X. Suppose that Y is rationally (resp. algebraically) equivalent to W and that $Z \cdot Y$ and $Z \cdot W$ are defined. Then $Z \cdot Y$ is rationally (resp. algebraically) equivalent to $Z \cdot W$*

We can now define a ring structure on $\mathrm{CH}(X)$. Given two cycle classes $[Z], [W]$ in $\mathrm{CH}(X)$, we find a cycle Y in $[W]$ such that $Z \cdot Y$ is defined (which is possible by part (1) of the lemma). Then we set $[Z][W] = [Z \cdot Y]$. By part (2) of the lemma, this definition is independent of the choice of Y. The product of cycle classes so defined makes $\mathrm{CH}(X)$ into a (graded) commutative ring, called *Chow ring* of X.

Similarly, the group of cycle classes modulo *algebraic equivalence* is a ring under intersection product. Other equivalence relations on cycles having the property that the corresponding class group becomes a ring under intersection product will be given below.

For any variety X of dimension n we define a *degree map* $< \cdot >: Z(X) \to \mathbb{Z}$ as follows: if $Z = \sum m_i P_i$ is in $Z^n(X)$ (so the P_i are closed points of X), we define $< Z >= \sum m_i [k(P_i) : k]$. Otherwise we set $< Z >= 0$.

We say that two cycles Z, W on X are *numerically equivalent* if for every cycle Y such that $Z \cdot Y$ and $W \cdot Y$ are defined, we have $< Z \cdot Y >=< W \cdot Y >$. They are said to be *torsion equivalent* if there exists an integer $m \neq 0$ such that $m(Z - W)$ is algebraically equivalent to 0. Then both numerical and torsion equivalence define corresponding rings of cycle classes. Later we shall define homological equivalence on $Z(X)$ (with respect to a fixed Weil cohomology) and will then have another interesting example of a group of cycle classes which becomes a ring under intersection product. For now we observe that the equivalence relations defined so far are related through the following sequence of inclusions

$$\mathrm{Rat}_0(X) \subset \mathrm{Alg}_0(X) \subset \mathrm{Alg}_0^{\tau}(X) \subset \mathrm{Num}_0(X),$$

where $\mathrm{Rat}_0(X)$ (resp. $\mathrm{Alg}_0(X), \mathrm{Alg}_0^{\tau}(X), \mathrm{Num}_0(X)$) is the group of cycles on X rationally (resp. algebraically, torsion, numerically) equivalent to 0. For example torsion equivalence is *finer* than numerical equivalence.

Let "eq" stand for any one of the above four equivalence relations and let $Z_{\text{eq}}(X)$ denote the corresponding ring of cycle classes. Let $f\colon X \to Y$ be a morphism of varieties. Then f induces a functorial ring homomorphism $f^*\colon Z^r_{\text{eq}}(Y) \to Z^r_{\text{eq}}(X)$ and a functorial *group* homomorphism $f_*\colon Z^r_{\text{eq}}(X) \to Z^{r+m-n}_{\text{eq}}(Y)$, where $n = \dim X$ and $m = \dim Y$. These are defined as follows: f^* is induced by the map which sends a closed, integral subscheme W of Y to its intersection-theoretic inverse image $f^{-1}(W)$ (whenever this is defined), while f_* is induced by the map which assigns to a closed, integral subscheme Z of X the cycle $d\,[f(Z)]$, where d equals $[k(Z) : k(f(Z))]$ if $\dim f(Z) = \dim Z$ and is 0 if $\dim f(Z) < \dim Z$. The homomorphisms f^* and f_* are related through the following *projection formula* (also known as "Frobenius Reciprocity")

$$f_*(f^*W \cdot Z) = W \cdot f_*(Z).$$

We now introduce some additional notation. If $X \in V(k)$, $p_1, p_2\colon X \times X \to X$ will denote the projection maps. Further, we will write $\Delta_X\colon X \to X \times X$ for the diagonal morphism. We now make the following general definition.

Definition 1.2. *An equivalence relation "eq" on cycles is called adequate if the following conditions hold.*

(1) *The residue classes $Z_{\text{eq}}(X)$ form a ring under intersection product for any $X \in V(k)$.*
(2) *The operations on cycles corresponding to a morphism $f\colon X \to Y$ induce a functorial ring homomorphism $f^*\colon Z^r_{\text{eq}}(Y) \to Z^r_{\text{eq}}(X)$ and a functorial group homomorphism $f_*\colon Z^r_{\text{eq}}(X) \to Z^{r+m-n}_{\text{eq}}(Y)$ ($n = \dim X$, $m = \dim Y$) which satisfy the projection formula*

$$f_*(f^*W \cdot Z) = W \cdot f_*(Z),$$

where $W \in Z_{\text{eq}}(Y)$ and $Z \in Z_{\text{eq}}(X)$.
(3) *If $X \in V(k)$, the multiplication map $Z_{\text{eq}}(X) \otimes Z_{\text{eq}}(X) \to Z_{\text{eq}}(X)$ factors as*

$$Z_{\text{eq}}(X) \otimes Z_{\text{eq}}(X) \to Z_{\text{eq}}(X \times X) \xrightarrow{\Delta_X^*} Z_{\text{eq}}(X),$$

where the left-hand map is induced by $a \otimes b \mapsto p_1^*a \cdot p_2^*b$ ($a, b \in Z_{\text{eq}}(X)$).
(4) *If $X_1, X_2 \in V(k)$ and $\varphi_i\colon X_i \to X_1 \amalg X_2$ are the natural maps ($i = 1, 2$), then $(\varphi_1^*, \varphi_2^*)\colon Z_{\text{eq}}(X_1 \amalg X_2) \to Z_{\text{eq}}(X_1) \oplus Z_{\text{eq}}(X_2)$ is an isomorphism with inverse $(\varphi_{1*}, \varphi_{2*})$.*
(5) *For any $X \in V(k)$, there is an augmentation homomorphism $Z_{\text{eq}}(X) \to \mathbb{Z}$ which is an isomorphism for $X = \operatorname{Spec} k$.*

The four equivalence relations on cycles defined so far (rational, algebraic, torsion and numerical equivalence) are all adequate equivalence relations in the sense of the above definition.

Homological equivalence (to be defined in the next section) is also an adequate equivalence relation on cycles.

2. Weil cohomologies

Let F be a field of characteristic zero and write $\mathcal{H}$ for the category of augmented, finite dimensional, graded anti-commutative F-algebras (here "anti-commutative"

means that if the elements x, y have degrees r, s respectively, then $xy = (-1)^{rs}yx$).
A *Weil cohomology* is a contravariant functor

$$H\colon V(k) \to \mathcal{H}$$

which satisfies the following conditions.

(1) ("Poincaré duality") For any $X \in V(k)$ of dimension n, $H^i(X) = 0$ for $i \notin [0, 2n]$. Also, there exists an "orientation morphism" (or "trace map") $H^{2n}(X) \simeq F$. Finally, the canonical pairings

$$H^i(X) \times H^{2n-i}(X) \to H^{2n}(X), (a, b) \mapsto ab,$$

are non-degenerate.

We shall write ϵ for the (augmentation) map $H(X) \to F$ which is the orientation isomorphism on $H^{2n}(X)$ and zero elsewhere.

(2) ("Kunneth formula") The canonical map

$$H(X) \otimes_F H(X) \to H(X \times Y),$$

induced by $a \otimes b \mapsto p_1^* a\, p_2^* b$, is an isomorphism.

(3) ("Cycle map") There exist group homomorphisms

$$\zeta_X : Z^p(X) \to H^{2p}(X)$$

with the following properties:

(a) (Functoriality) For a morphism of varieties $f\colon X \to Y$, we write f^* for the ring homomorphism $H(f) : H(Y) \to H(X)$ and $f_*\colon H(X) \to H(Y)$ for its F-linear transpose, after identifying (for each i) the dual of $H^i(X)$ with $H^{2n-i}(X)$ via Poincaré duality and the map ϵ. (We note that these maps satisfy the projection formula: $f_*((f^*(a)\, b) = a\, f_* b$, $a \in H(Y), b \in H(X)$.) Then

$$f^*\zeta_Y = \zeta_X f^* \quad \text{and} \quad f_*\zeta_X = \zeta_Y f_*.$$

(Of course the maps f^* and f_* appearing on the left-hand sides of these formulas are the maps just defined, whereas the maps so denoted which appear on the right-hand sides are the maps on cycles defined earlier).

(b) (Multiplicativity) We have

$$\zeta_{X\times Y}(Z \times W) = \zeta_X(Z) \otimes \zeta_Y(W).$$

(c) (Non-triviality) If $e = \operatorname{Spec} k$, the diagram

$$\begin{array}{ccc} Z(e) & \xrightarrow{\zeta_e} & H(e) \\ {\scriptstyle <\cdot>}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ \mathbb{Z} & \xrightarrow{\subset} & F \end{array}$$

commutes.

Examples of Weil cohomologies.

(1) (Singular cohomology) If X is a variety over $\mathbb{C}$, i.e. $k = \mathbb{C}$, let $H(X)$ to be the singular cohomology of X regarded as a complex manifold ($H(X)$ is a $\mathbb{C}$-algebra, so $F = \mathbb{C}$ in this case). Then H is a Weil cohomology. For a proof of Poincaré duality, construction of the cycle map, etc., see [Greenberg].

(2) (l-adic cohomology) If char $k = p \geq 0$ and $l \neq p$ is a prime, take

$$H^*(X) = \left(\varprojlim H^*_{\text{et}}(X \otimes \bar{k}, \mathbb{Z}/l^n\mathbb{Z})\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \ ,$$

where $\bar{k}$ denotes separable closure. Then this is a Weil cohomology, with $F = \mathbb{Q}_l$. If $k = \mathbb{C}$, it is equivalent to singular cohomology. For proofs of the various properties that make H a Weil cohomology, see [Milne].

(3) (Cristalline cohomology) [To be added later]

Assume now that we have at our disposal a Weil cohomology H, so in particular we have a cycle map

$$\zeta_X : Z^p(X) \to H^{2p}(X).$$

We say that two cycles Z, W on X are *homologically equivalent* (with respect to the given Weil cohomology) if $\zeta_X(Z) = \zeta_X(W)$. This defines an equivalence relation on cycles which is called *homological equivalence.* The following holds.

Proposition 2.1. *Homological equivalence is an adequate equivalence relation, lying between torsion equivalence and numerical equivalence.*

Proof. See [Kleiman, §1.2.1-1.2.3].

Thus we can extend the sequence of inclusions displayed on page 2 to

$$\mathrm{Rat}_0(X) \subset \mathrm{Alg}_0(X) \subset \mathrm{Alg}_0^\tau(X) \subset \mathrm{Hom}_0(X) \subset \mathrm{Num}_0(X),$$

where $\mathrm{Hom}_0(X)$ denotes the group of cycles homologically equivalent to zero.

In the next section we shall develop a "theory of correspondences" on $V(k)$ using any one of the equivalence relations defined until now. This will enable us to define various categories of motives.

3. Correspondences

Let X, Y, Z be varieties and let α, β be cycles on $X \times Y, Y \times Z$, respectively. Consider

$$\beta \circ \alpha = p_{XZ*}(p^*_{XY}\alpha \cdot p^*_{YZ}\beta),$$

where $p_{XY} : X \times Y \times Z \to X \times Y$, $p_{XZ} : X \times Y \times Z \to X \times Z$ and $p_{YZ} : X \times Y \times Z \to Y \times Z$ are the projection maps. If $p^*_{XY}\alpha$ and $p^*_{YZ}\beta$ intersect properly, then $\beta \circ \alpha$ is a well-defined cycle on $X \times Z$. In general it may be necessary to "move" β (as in Chow's moving lemma, see Proposition 1.1 above) in order to define $\beta \circ \alpha$, that is, $\beta \circ \alpha$ is defined up to rational equivalence.

Let $\sim$ denote any adequate equivalence relation on cycles which is coarser than rational equivalence, for example any one of the five equivalence relations we have defined so far (rational, algebraic, torsion, homological and numerical equivalences). Then, for any variety X, let

$$C(X) = Z(X)/\sim$$

be the corresponding ring of cycle classes. This is a graded, commutative and associative ring under intersection product. Then the composition rule $\circ$ defined above induces a bilinear homomorphism

$$C(X \times Y) \times C(Y \times Z) \xrightarrow{\circ} C(X \times Z).$$

The elements of $C(X \times Y)$ are called *correspondences from X to Y*, and the above composition law is called *composition of correspondences.*

We have a *transpose*

$$C(X \times Y) \xrightarrow{\ '\ } C(Y \times X),$$

given by $\alpha \mapsto \alpha' := s_*(\alpha)$, where $s\colon X \times Y \to Y \times X$ is the morphism that switches the factors (i.e. $s(P,Q) = (Q,P)$ on points) and $s_*\colon C(X \times Y) \to C(Y \times X)$ is induced by s.

Now let $f\colon X \to Y$ be a morphism of varieties and let $\Gamma_f\colon X \to X \times Y$ be the graph morphism. The cycle ${\Gamma_f}_*(X) \in Z(X \times Y)$ represents a class in $C(X \times Y)$, i.e., a correspondence. In order to simplify the notation, we will write Γ_f for this correspondence. If $n = \dim X$ and $m = \dim Y$, then ${\Gamma_f}_*(X)$ is a cycle on $X \times Y$ of codimension $(m+n) - n = m$. Thus we have a map

$$\Gamma\colon \mathrm{Mor}(X,Y) \to C^m(X \times Y), \quad f \mapsto \Gamma_f.$$

Proposition 3.1. *Let $\alpha \in C(X \times Y)$, $\beta \in C(Y \times Z)$ be correspondences and let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms. We have*

(1) *If $\gamma \in C(Z \times W)$, then $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.*
(2) *$(\beta \circ \alpha)' = \alpha' \circ \beta'$ and $(\alpha')' = \alpha$.*
(3) *$\Gamma_g \circ \alpha = (1_X \times g)_*(\alpha)$.*
(4) *$\beta \circ \Gamma_f = (f \times 1_Z)^*(\alpha)$.*
(5) *$\Gamma_g \circ \Gamma_f = \Gamma_{g \circ f}$.*

Proof. See [Fulton, Prop. 16.1.1].

Corollary 3.2. *The composition law $\circ$ makes $C(X \times X)$ into an associative ring with unit $\Delta_X = \Gamma_{1_X}$, and with an involution $\alpha \mapsto \alpha'$.*

We call the ring $(C(X \times X), \circ)$ the *ring of correspondences on X*. Do not confuse this ring with the ring $(C(X \times X), \cdot)$, where $\cdot$ denotes intersection product. The latter is always commutative, whereas the former seldom is.

Also, we see from the proposition that the map

$$\Gamma\colon \mathrm{Mor}(X,X) \to C^n(X \times X), \quad f \mapsto \Gamma_f,$$

where $n = \dim X$, is a homomorphism of monoids which maps 1_X to Δ_X.

Definition 3.3. *For $\alpha \in C(X \times Y)$, define a homomorphism $\alpha_*\colon C(X) \to C(Y)$ by the formula*

$$\alpha_*(a) = p_{Y*}(\alpha \cdot p_X{}^*(\alpha)),$$

and a homomorphism $\alpha^*\colon C(Y) \to C(X)$ by the formula

$$\alpha^*(b) = p_{X*}(\alpha \cdot p_Y{}^*(\beta)).$$

Here $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ are the projection maps.

Proposition 3.4.

(a) *If $\alpha \in C(X \times Y)$ and $\beta \in C(Y \times Z)$ are correspondences, then*

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_* \quad \textit{and} \quad (\beta \circ \alpha)^* = \alpha^* \circ \beta^*.$$

(b) $(\alpha')_* = \alpha^*$ *and* $(\alpha')^* = \alpha_*$.

(c) *If $f\colon X \to Y$ is a morphism of varieties, then $(\Gamma_f)_* = f_*$ and $(\Gamma_f)^* = f^*$.*

Proof. See [Fulton, Prop. 16.1.2].

Next we define a grading on the ring of correspondences $C(X \times X)$.

A correspondence from X to Y is said to be *homogeneous of degree p* if $\alpha \in C^{m+p}(X \times Y)$, where $m = \dim Y$. It is not difficult to check that if $\alpha \in C(X \times Y)$ and $\beta \in C(Y \times Z)$ are homogeneous of degrees p and q, respectively, then $\beta \circ \alpha$ is homogeneous of degree $p + q$. In particular, if we compose two correspondences of degree 0 then we obtain a correspondence of degree 0. Thus there is a bilinear form

$$C^m(X \times Y) \times C^l(Y \times Z) \xrightarrow{\circ} C^l(X \times Z),$$

where $l = \dim Z$. It follows that $C^n(X \times X)$, $n = \dim X$, is a subring of the ring of correspondences $C(X \times X)$, which we call the *ring of correspondences of degree 0*. Note that each morphism $f\colon X \to X$ gives rise to a correspondence of degree 0 on X, namely $\Gamma_f \in C^n(X \times X)$.

Now let $\alpha \in C^{m+p}(X \times Y)$ be a correspondence of degree p. Then $\alpha^*\colon C(Y) \to C(X)$ maps $C^k(Y)$ to $C^{k+p}(X)$ and $\alpha_*\colon C(X) \to C(Y)$ maps $C^k(X)$ to $C^{k+p+(m-n)}(Y)$. In particular if α has degree 0, then α^* preserves degrees (i.e. codimensions), while α_* shifts them by $m-n$ (i.e. α_* preserves *dimensions*). This is an easy exercise. Now recall that $\alpha' = s_*(\alpha)$, where $s\colon X \times Y \to Y \times X$ is the morphism that switches the factors. Since $X \times Y$ and $Y \times X$ have the same dimension, $\alpha' \in C^{m+p}(Y \times X) = C^{n+p+(m-n)}(Y \times X)$ (see §1). Consequently α' has degree $p + (m-n)$, i.e. transposing shifts degrees by $m-n$. In particular Γ_f' has degree $m-n$ for any $f \in \mathrm{Mor}(X,Y)$. Now if $X = Y$, then α' has the same degree as α, whence $'$ respects the grading on $C(X \times X)$. Therefore the ring of correspondences of degree 0 on X, $C^n(X \times X)$, is stable under $'$.

4. The category of correspondences

For many purposes, correspondences behave very much like morphisms. We can compose them (when they are composable!) and, moreover, every morphism of varieties $f\colon X \to Y$ gives rise in a natural way to a correspondence (of degree 0) from X to Y, namely $\Gamma_f \in C^n(X \times Y)$. This prompts the following

Definition 4.1. *The category of C-correspondences, $CV(k)$, is defined as follows.*

(a) *The objects of $CV(k)$ are those of $V(k)$. The variety X as an object of $CV(k)$ will be denoted by $\overline{X}$.*

(b) *For $\overline{X}, \overline{Y} \in CV(k)$, we define*

$$\mathrm{Hom}(\overline{X}, \overline{Y}) = C(X \times Y).$$

Composition of morphisms in $CV(k)$ is composition of correspondences.

An important variant of $CV(k)$ is the *category of correspondences of degree* 0, denoted $CV^0(k)$, which differs from $CV(k)$ only in that the morphisms allowed in $CV^0(k)$ are correspondences of degree 0, i.e., if $\overline{X}, \overline{Y}$ are objects of $CV^0(k)$, then

$$\mathrm{Hom}(\overline{X}, \overline{Y}) = \mathrm{Hom}^0(\overline{X}, \overline{Y}) = C^n(X \times Y),$$

where $n = \dim X$.

There is a (covariant) functor

$$\Gamma\colon V(k) \to CV(k)$$

which is the identity on objects and maps $f \in \mathrm{Mor}(X, Y)$ to $\Gamma_f \in C(X \times Y) = \mathrm{Hom}(\overline{X}, \overline{Y})$. Note that Γ actually factors through $CV^0(k)$. A *contravariant* functor $\Gamma'\colon V(k) \to CV(k)$ is obtained by mapping $f \in \mathrm{Mor}(X, Y)$ to $\Gamma'_f \in C(Y \times X) = \mathrm{Hom}(\overline{Y}, \overline{X})$. Note that, unlike the functor Γ, Γ' does *not* factor through $CV^0(k)$.

Since $CV(k)$ is an additive category, it might be said that Γ "converts" the category $V(k)$ into an additive one.

Now recall the category $\mathcal{H}$ of augmented, finite dimensional, graded anti-commutative F-algebras, and let $H\colon V(k)^{\mathrm{op}} \to \mathcal{H}$ be a Weil cohomology. Then, at least if the equivalence relation on cycles being considered is finer than homological equivalence, there is a commutative diagram of functors

$$\begin{array}{ccc} V(k)^{\mathrm{op}} & \xrightarrow{\Gamma'} & CV(k) \\ \Big\| & & \Big\downarrow{\scriptstyle \varsigma} \\ V(k)^{\mathrm{op}} & \xrightarrow{H} & \mathcal{H} \end{array}$$

where the vertical map at right is the cycle map. Now the category $\mathcal{H}$ is abelian, but $CV(k)$ is far from being abelian. As we will see, it contains many projectors that do not correspond to a direct-sum decomposition of an object. To partially remedy this, we shall formally add to $CV(k)$, or rahter to $CV^0(k)$, the kernels and images of all projectors, thereby obtaining a pseudo-abelian category $CV^0(k)^{\sim} = \mathcal{M}_k^+$, the category of effective C-motives. (The full category of motives $\mathcal{M}_k$ will be obtained from $\mathcal{M}_k^+$ by formally inverting a particular object T in $\mathcal{M}_k^+$, and then considering all products of objects of $\mathcal{M}_k^+$ by positive or negative powers of T). Conjecturally, the functor $h\colon V(k)^{\mathrm{op}} \to \mathcal{M}_k^+$ arising from this construction is itself a Weil cohomology theory, and is therefore *universal* among such theories. This feat, the construction of a universal cohomological theory of algebraic varieties, was one of Grothendieck's main reasons for introducing the theory of motives. In connection with this conjecture, Grothendieck calls the functor h *the functor of motivic cohomology.* Since for any variety X the object $H(X)$ depends only on $h(X)$, $H(X)$ is usually called the *realization of the motive* $h(X)$.

We have already noted that $CV(k)$ is an additive category. In addition, it is possible to define a direct sum and a tensor product in $CV(k)$. Indeed, if X and Y are varieties, we set

$$\overline{X} \oplus \overline{Y} = \overline{X \amalg Y} \quad \text{and} \quad \overline{X} \otimes \overline{Y} = \overline{X \times Y}.$$

Now we define, for any $T \in V(k)$, $\overline{X}(\overline{T}) = \operatorname{Hom}(\overline{T}, \overline{X})$. Further, if $\varphi \in \operatorname{Hom}(\overline{X}, \overline{Y})$ is any correspondence, $\varphi_T \colon \overline{X}(\overline{T}) \to \overline{Y}(\overline{T})$ is the map $g \mapsto \varphi \circ g$. The following lemma is not difficult to prove.

Lemma 4.2.

(i) *If* $T = e = \operatorname{Spec} k$, *then* $\varphi_T \colon C(X) \to C(Y)$ *agrees with the map* φ_* *defined previously.*

(ii) *More generally, let* $\mathrm{id}_{\overline{T}} \in \operatorname{Hom}(\overline{T}, \overline{T}) = C(T \times T)$ *be the identity correspondence, i.e.,* $\mathrm{id}_{\overline{T}} = \Delta_T$ *is the diagonal class. Define*

$$\mathrm{id}_{\overline{T}} \otimes \varphi = s_{23*}(p_{12}^*(\mathrm{id}_T) p_{34}^*(\varphi)),$$

which is an element of $\operatorname{Hom}(\overline{T} \otimes \overline{X}, \overline{T} \otimes \overline{Y}) = C(T \times X \times T \times Y)$. *Here* $s_{23} \colon T \times T \times X \times Y \to T \times X \times T \times Y$ *denotes the morphism that switches the second and third factors and* $p_{12} \colon T \times T \times X \times Y \to T \times T$, $p_{34} \colon T \times T \times X \times Y \to X \times Y$ *are projection maps. Then*

$$\varphi_T = (\mathrm{id}_{\overline{T}} \otimes \varphi)_* \colon C(T \times X) \to C(T \times Y).$$

(iii) *If* $f \colon X \to Y$ *is a morphism of varieties, then*

$$(\Gamma_f)_T = (1_T \times f)_* \quad \textit{and} \quad (\Gamma_f')_T = (1_T \times f)^*.$$

(By (i), the above generalizes Proposition 3.4(c).)

(iv) *Finally, for any morphism of varieties* $f \colon X \to Y$,

$$\mathrm{id}_T \otimes \Gamma_f = \Gamma_{1_T \times f}.$$

Theorem 4.3 (Manin's Identity Principle) *Let* $\varphi, \psi \in \operatorname{Hom}(\overline{X}, \overline{Y})$ *be correspondences. Then the following are equivalent.*

(i) $\varphi = \psi$.

(ii) $\varphi_T = \psi_T$ *for all* T.

(iii) $\varphi_X = \psi_X$.

Proof. It suffices to check that (iii) implies (i). We have $\varphi_X(\Delta_X) = \varphi \circ \Delta_X = \varphi$, and the theorem follows.

Example. Let $C(X)$ be the ring of cycles modulo rational equivalence, and let $f\colon X \to Y$ be a finite morphism of degree d. (Such a morphism between varieties is necessarily surjective.) Writing $[X]$ for the class of X in $C(X)$ (this is the unit of the ring $C(X)$ under intersection product), and similarly for Y, we have

$$f_*([X]) = d\,[f(X)] = d\,[Y].$$

Consequently, for any $y \in C(Y)$,

$$f_*\, f^*(y) = f_*([X] \cdot f^*(y)) = f_*([X])\, y = d\, y$$

by the projection formula, i.e. $f_*\, f^* = d\, \mathrm{id}_{C(Y)}$. By Lemma 4.2 this can be written as

$$(\Gamma_f \circ \Gamma_f')_e = d\,(\mathrm{id}_{\overline{Y}})_e.$$

Similarly, since $1_T \times f\colon T \times X \to T \times Y$ is finite of degree d for any T, we have

$$(\Gamma_f \circ \Gamma_f')_T = d\,(\mathrm{id}_{\overline{Y}})_T \quad \text{for all } T.$$

Thus, by Manin's Identity Principle (Theorem 4.3), the following is an identity in $CV(k)$:

$$\Gamma_f \circ \Gamma_f' = d\, \mathrm{id}_{\overline{Y}}.$$

We now set $p = \frac{1}{d}(\Gamma_f \circ \Gamma_f') \in CV(k)_{\mathbb{Q}} = CV(k) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $p \in \mathrm{Hom}(\overline{X}, \overline{X})$ is a projector in $CV(k)_{\mathbb{Q}}$. Indeed,

$$p^2 = \frac{1}{d^2}\, \Gamma_f' \circ (\Gamma_f \circ \Gamma_f') \circ \Gamma_f = \frac{1}{d}\, \Gamma_f' \circ \Gamma_f = p.$$

But $CV(k)_{\mathbb{Q}}$ is a non-abelian category and, generally speaking, p does not have kernel in it. Therefore we cannot decompose $\overline{X}$ as the direct sum of the kernel of p and the image of p, contrary to what happens in linear algebra, for example, where such a decomposition is automatically available every time one has an idempotent linear map. When we define the category of (effective) motives in the next section by formally adding the kernels and images of such projectors, we will see that the image of p is isomorphic to $\overline{Y}$, so that $\overline{Y}$ is a "direct summand of $\overline{X}$".

5. The category of effective motives

We begin with a general definition.

Definition 5.1. *An additive category $\mathcal{D}$ is* pseudo-abelian *if the following conditions hold. For any projector $p \in \mathrm{Hom}(X, X)$, $X \in \mathrm{Ob}(\mathcal{D})$, there exists a kernel* Ker p *in $\mathcal{D}$ and the canonical morphism*

$$\mathrm{Ker}\ p \oplus \mathrm{Ker}\ (1_X - p) \to X$$

is an isomorphism.

Note that since $\mathrm{Ker}\ (1_X - p) = \mathrm{Im}\ p$, the above can be written as

$$\mathrm{Ker}\ p \oplus \mathrm{Im}\ p \xrightarrow{\sim} X.$$

We now make the

Definition 5.2. *Let $\mathcal{D}$ be an additive category. Then the* pseudo-abelian completion *of $\mathcal{D}$ is the category $\tilde{\mathcal{D}}$ defined as follows.*

(a) *The objects of $\tilde{\mathcal{D}}$ are pairs (X,p), where $X \in \mathrm{Ob}(\mathcal{D})$ and $p \in \mathrm{Hom}_{\mathcal{D}}(X,X)$ is a projector.*

(b) *For objects $(X,p),(Y,q)$ in $\tilde{\mathcal{D}}$,*

$$\mathrm{Hom}_{\tilde{\mathcal{D}}}((X,p),(Y,q)) = \{f \in \mathrm{Hom}_{\mathcal{D}}(X,Y), fp = qf\}/\{\ldots, fp = qf = 0\}.$$

Composition of morphisms in $\tilde{\mathcal{D}}$ is induced by composition of morphisms in $\mathcal{D}$.

There is a natural functor $\mathcal{D} \to \tilde{\mathcal{D}}, X \mapsto \tilde{X} = (X, \mathrm{id}_X), (f\colon X \to Y) \mapsto \tilde{f} =$ class of f in $\mathrm{Hom}_{\tilde{\mathcal{D}}}(\tilde{X},\tilde{Y}) = \mathrm{Hom}_{\mathcal{D}}(X,Y)$. Then, if $p \in \mathrm{Hom}_{\mathcal{D}}(X,X)$ is a projector, we have

$$\mathrm{Im}\ \tilde{p} = (X,p), \quad \mathrm{Ker}\ \tilde{p} = (X, 1_X - p).$$

Proposition 5.3.

(i) *The category $\tilde{\mathcal{D}}$ is pseudo-abelian and the functor $\mathcal{D} \to \tilde{\mathcal{D}}$ defined above is fully faithful.*

(ii) *$\tilde{\mathcal{D}}$ is* universal *in the following sense: given a pseudo-abelian category $\mathcal{E}$ and an additive functor $\mathcal{D} \to \mathcal{E}$, there exists an additive functor (unique up to isomorphism) $F\colon \tilde{\mathcal{D}} \to \mathcal{E}$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d, equal] & \mathcal{E} \\ \mathcal{D} \arrow[r] & \tilde{\mathcal{D}}. \arrow[u, "F"'] \end{tikzcd}$$

We now apply the above construction to the additive category $CV^0(k)_{\mathbb{Q}}$, whose morphisms are

$$\mathrm{Hom}^0(\overline{X},\overline{Y}) = \mathrm{Hom}_{CV^0(k)_{\mathbb{Q}}}(\overline{X},\overline{Y}) = C^n(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $n = \dim X$. The resulting pseudo-abelian category $(CV^0(k)_{\mathbb{Q}})^{\sim}$ is called *the category of (effective) C-motives*. It is denoted by $\mathcal{M}_k^+$.

If $\overline{X} \in \mathrm{Ob}(CV^0(k)_{\mathbb{Q}})$, we write $\tilde{X} = (X, \mathrm{id}_X)$ (rather than $\tilde{X} = (\overline{X}, \mathrm{id}_X)$) and call $\tilde{X}$ *the motive of the variety X*. Every effective C-motive has the form (X,p) for some projector $p \in \mathrm{Hom}^0(\overline{X},\overline{X})$, and given such an X and p we have a decomposition

$$\tilde{X} = (X,p) \oplus (X, 1_X - p) = \mathrm{Ker}\ \tilde{p} \oplus \mathrm{Im}\ \tilde{p}.$$

Proposition 5.4.

(i) *$\mathcal{M}_k^+$ is a $\mathbb{Q}$-linear, pseudo-abelian, tensor category, with tensor product given by*

$$(X,p) \otimes (Y,q) = (X \times Y, p \times q).$$

(ii) *Let h denote the (contravariant) functor $V(k) \to \mathcal{M}_k^+, X \mapsto \tilde{X}, f \mapsto \tilde{\Gamma}'_f$. Then every object of $\mathcal{M}_k^+$ is a direct factor of an $h(X)$, and we have*

$$h(X \amalg Y) = h(X) \oplus h(Y),$$

$$h(X \times Y) = h(X) \otimes h(Y).$$

(The last formula is sometimes called "the motivic Kunneth formula".)

(iii) *The morphism*

$$h(\Delta_X) = \tilde{\Delta}_X \colon h(X) \otimes h(X) \to h(X)$$

defines a multiplicative structure on the motive $h(X)$.

(iv) *Set $\mathbf{1} = h(e) = \tilde{e}$. Then, for any $M \in \mathcal{M}_k^+$, there is a canonical isomorphism*

$$M = \mathbf{1} \otimes M.$$

Examples. The following examples are very important for what follows.

(a) Let $f \colon X \to X$ be an idempotent morphism of varieties, i.e., $f^2 = f$. Then $p = \Gamma_f \in \mathrm{Hom}^0(\overline{X}, \overline{X})$ is a projector and we have the following decomposition in the category of motives $\mathcal{M}_k^+$:

$$\tilde{X} = \text{Ker } \tilde{p} \oplus \text{Im } \tilde{p}.$$

(b) Let $f \colon X \to Y$ be a morphism of varieties having a global section, i.e., there is a morphism $g \colon Y \to X$ such that $f \circ g = 1_Y$. Then $g \circ f \colon X \to X$ is an idempotent morphism, whence $p = \Gamma_{g \circ f} \in \mathrm{Hom}^0(\overline{X}, \overline{X})$ is a projector and we have (by (a))

$$\tilde{X} = \text{Ker } \tilde{p} \oplus \text{Im } \tilde{p} = \text{Ker } \tilde{p} \oplus (X, p).$$

We will show that the morphisms in $\mathcal{M}_k^+$

$$\tilde{\Gamma}_f \colon (X, p) \to \tilde{Y}$$

and

$$\tilde{\Gamma}_g \colon \tilde{Y} \to (X, p)$$

are mutually reciprocal isomorphisms. First note that $\tilde{\Gamma}_g(\tilde{Y})$ is indeed contained in $(X, p) = \text{Ker } (1_X - \tilde{p})$ since

$$(1_X - \tilde{p}) \circ \tilde{\Gamma}_g = \tilde{\Gamma}_g - \tilde{\Gamma}_g = 0.$$

Now $\tilde{\Gamma}_f \circ \tilde{\Gamma}_g = \tilde{\Gamma}_{f \circ g} = \mathrm{id}_{\tilde{Y}}$, and

$$\tilde{\Gamma}_g \circ \tilde{\Gamma}_f|_{(X,p)} = \tilde{p}|_{\text{Im } \tilde{p}} = \mathrm{id}_{\text{Im } \tilde{p}} = \mathrm{id}_{(X,p)},$$

which proves our assertion. Thus we have a canonical decomposition

$$\tilde{X} = \text{Ker } \tilde{p} \oplus \tilde{Y}.$$

This example shows that every time we have a morphism $f\colon X \to Y$ which admits a global section, then "Y becomes a direct factor of X in the category of motives".

(c) Let $X \in V(k)$ be any variety and let $\varphi\colon X \to e$ be its structural morphism. Suppose that X has a k-rational point, i.e. there exists a section $\psi : e \to X$ of φ. Then by example (b) above, we have $\mathrm{Im}\ \tilde{p} \simeq \tilde{e}$, and there is a canonical decomposition

$$\tilde{X} = \tilde{e} \oplus \mathrm{Ker}\ \tilde{p},$$

where $p = \Gamma_{\psi\circ\varphi}$.

(d) We return to the example following Proposition 4.3, i.e. $C(X)$ is the Chow ring and $f\colon X \to Y$ is a finite morphism of degree d. As we have seen, there are correspondences $a = \Gamma'_f$ and $b = \frac{1}{d}\Gamma_f$ of degree 0 (note that a has degree 0 since X and Y are of the same the same dimension) such that $b \circ a = 1$ and $a \circ b = p$ is a projector. Arguing as in the previous example, we get reciprocal isomorphisms

$$\tilde{b}\colon \mathrm{Im}\,\tilde{p} \to \tilde{Y} \quad \text{and} \quad \tilde{a}\colon \tilde{Y} \to \mathrm{Im}\,\tilde{p}$$

and a decomposition

$$\tilde{X} = \tilde{Y} \oplus \mathrm{Ker}\ \tilde{p}.$$

(e) (quasi-sections. To be added)

The reader may have noticed that the above examples have something in common. The general fact that underlies examples (b) through (d) above is the following

Lemma 5.5. *Let $\mathcal{D}$ be an additive category, let X, Y be objects of $\mathcal{D}$ and let $a\colon X \to Y$ and $a'\colon Y \to X$ be morphisms in $\mathcal{D}$ such that $aa' = \mathrm{id}_Y$. Then the morphism $p = a'a\colon X \to X$ is a projector, and the morphisms in $\tilde{\mathcal{D}}$*

$$\tilde{a}\colon (X,p) \to \tilde{Y} \quad \textit{and} \quad \tilde{a}'\colon \tilde{Y} \to (X,p)$$

are mutually reciprocal isomorphisms.

Proof. This is an exercise. Use the preceeding examples as a guide.

6. Calculation of basic motives

A variety $X \in V(k)$ is called *C-special* if it admits a global section over k, i.e. $X(k) \neq \emptyset$, and if for any such section $\psi\colon e \to X$ the image $\psi_*(e) \in C^n(X)$ is independent of ψ (here $n = \dim X$).

Example. If k is algebraically closed and C is constructed from numerical equivalence, then all varieties are C-special. Indeed, if ψ and ψ' are two k-rational points, then $\psi_*(e)$ and $\psi'_*(e)$ are numerically equivalent since $< \psi_*(e) >= [k(\psi_*(e)) : k] = 1$ for any ψ. In Chow theory and in certain homological theories, the varieties $\mathbb{P}^n = \mathbb{P}^n_k$ and the various products of them are special.

Definition 6.1. *Assume that X is C-special and write $\psi_*(e) = e_X$ for any k-rational point ψ (e_X is really independent of ψ so our notation is justified). Define*

$$p_X = p_1^*(e_X) \quad \textit{and} \quad q_X = p_2^*(e_X),$$

where $p_1, p_2 \colon X \times X \to X$ *are the projections.*

Clearly p_X and q_X are elements of $C^n(X \times X) = \mathrm{Hom}^0(\overline{X}, \overline{X})$. Also, if $\varphi \colon X \to e$ is the structural morphism of X, then it is not difficult to see that

$$p_X = \Gamma'_{\varphi \circ \psi} \quad \text{and} \quad q_X = \Gamma_{\varphi \circ \psi}.$$

Thus since ψ is a global section of φ, p_X and q_X are projectors. Further, as seen in Example(c) at the end of the previous section, we have Im $q_X \simeq \tilde{e}$ and

$$\tilde{X} = \tilde{e} \oplus \mathrm{Ker}\ \tilde{q}_X.$$

Proposition 6.2. *The motive* Im $\tilde{p}_X = (X, p_X)$ *depends only on the dimension of* X*. Further,*

$$\mathrm{Hom}((X, p_X), (Y, p_Y)) = \begin{cases} \mathbb{Q} & \textit{if } \dim X = \dim Y \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. See [..., p.445].

Example. Let $X = \mathbb{P}^1$. In this case $\tilde{p}_X + \tilde{q}_X = 1$ (exercise), so that

$$\tilde{\mathbb{P}}^1 = \tilde{e} \oplus \mathbf{L}$$

where $\mathbf{L} = \mathrm{Ker}\ \tilde{q}_X = \mathrm{Im}\ \tilde{p}_X$ is (by definition) *the Tate motive.*

Lemma 6.3. *For all* C*-special varieties* X*,*

$$\mathrm{Im}\ \tilde{p}_X = (X, p_X) \simeq \mathbf{L}^n,$$

where $\mathbf{L}^n$ *denotes* $\mathbf{L} \otimes \cdots \otimes \mathbf{L}$ *(n factors).*

Proof. By Proposition 6.2,

$$(X, p_X) \simeq (Y, p_Y),$$

where $Y = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (n factors). On the other hand, $\tilde{Y} = (\tilde{\mathbb{P}}^1)^n$ and $\tilde{p}_Y = (\tilde{p}_{\mathbb{P}^1})^n$, so

$$\mathrm{Im}\ \tilde{p}_Y \simeq (\mathrm{Im}\ \tilde{p}_{\mathbb{P}^1})^n = \mathbf{L}^n,$$

as claimed.

Remark. Note that $\tilde{Y} = (\tilde{\mathbb{P}}^1)^n = (\tilde{e} \oplus \mathbf{L})^n = \oplus_{i=0}^n \binom{n}{i} \mathbf{L}^i$. Therefore, since $\tilde{e} = \mathrm{Im}\ \tilde{q}_Y$ and Im $\tilde{p}_Y = \mathbf{L}^n$ by the lemma, we have a decomposition

$$\tilde{Y} = \mathrm{Im}\ \tilde{q}_Y \oplus \left(\bigoplus_{i=1}^{n-1} \binom{n}{i} \mathbf{L}^i \right) \oplus \mathrm{Im}\ \tilde{p}_Y.$$

Theorem 6.4. (The motive of a projective space) *We have*

$$\tilde{\mathbb{P}}^n = \tilde{e} \oplus \mathbf{L} \oplus \cdots \oplus \mathbf{L}^n.$$

Proof. To be added.

Theorem 6.5. (The motive of a projective bundle) *Let $\mathcal{E}$ be a locally free sheaf of rank r on a variety X. Then*

$$h(\mathbb{P}(\mathcal{E})) = \tilde{X} \oplus \tilde{\mathbb{P}}^r = \bigoplus_{i=0}^{r} (\tilde{X} \otimes \mathbf{L}^i).$$

Proof. To be added.

Theorem 6.6. (The motive of a blow-up) *Let X be a variety, let Y a closed subscheme of X of codimension r, and let X' be the blow-up of X along Y. Then*

$$\tilde{X}' = \tilde{X} \oplus \left(\bigoplus_{i=1}^{r-1} (\tilde{Y} \otimes \mathbf{L}^i) \right).$$

Proof. To be added.

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