ON THE TWISTED CONJUGACY PROBLEM FOR LARGE-TYPE ARTIN GROUPS

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ABSTRACT. We show that the twisted conjugacy problem is solvable for free-of-infinity large-type Artin groups; and for XXXL Artin groups whose defining graph is connected and does not have a cut-vertex or a separating edge.

1. INTRODUCTION

Let G be a group and $\varphi \in \mathrm{Aut}(G)$. The group G has solvable *φ-twisted conjugacy problem* ($TCP_\varphi(G)$) if there is an algorithm that given $u, v \in G$ can decide if there exists a $z \in G$ such that $u = \varphi(z)vz^{-1}$. It has solvable *twisted conjugacy problem* ($TCP(G)$) if there is an algorithm that given $\varphi \in \mathrm{Aut}(G)$ and $u, v \in G$ can decide if there exists a $z \in G$ such that $u = \varphi(z)vz^{-1}$. When φ is trivial, $TCP_{id}(G)$ is known as the *conjugacy problem* ($CP(G)$). If A is a subgroup of $\mathrm{Aut}(G)$, we say that A is *orbit decidable* if there is an algorithm which, for any two $u, v \in G$ decides whether there exists $\varphi \in A$ such that $\varphi(u)$ and v are conjugate in G.

The twisted conjugacy problem has proven to be much harder to solve than the conjugacy problem, and few positive results are known. In the context of Artin groups, it is only known for braid groups [GMV14], and for Artin groups whose defining graph has two vertices [Cro24a, Cro24b]. In this article we show that some large-type Artin groups have solvable twisted conjugacy problem. Our result is the first one to hold for generic Artin groups in a probabilistic sense (see [GV23, Theorem 1.2]). Here is the precise statement, for definitions see Section 2.

Theorem 1.1. *Let A_Γ be a free-of-infinity large-type Artin group; or an Artin group satisfying the COST property whose defining graph is connected and does not have a cut-vertex or a separating edge. Then $TCP(A_\Gamma)$ is solvable.*

As a corollary of Theorem 1.1 we get the following (see [BMV09, §2] for the definition of algorithmic and Section 2 for the statement of Theorem 2.4).

Corollary 1.2. *Let A_Γ be an Artin group as in Theorem 1.1. Then, for any algorithmic short exact sequence of groups*

$$1 \to A_\Gamma \to G \to H \to 1$$

with H finitely generated and satisfying the hypotheses of Theorem 2.4, the group G has solvable conjugacy problem.

In particular, if an Artin group A_Γ satisfies the assumptions of Theorem 1.1, then any group G containing A_Γ as a finite-index normal subgroup has solvable conjugacy problem. In fact, we can deduce from our proofs that unless Γ is an even edge, such groups G are systolic and therefore biautomatic [JŚ06, Theorem E]. In the case when Γ is not the even edge we can also drop the finite generation assumption in Corollary 1.2.

2020 *Mathematics Subject Classification.* 20F36 (primary), 20F10, 20F28.
Key words and phrases. Artin groups, twisted conjugacy problem, systolic groups, automorphism groups.

Acknowledgements. The first author wants to thank Enric Ventura for pointing out Theorem 2.4. The second author was partially supported by the National Science Centre (Poland) grant 2022/47/D/ST1/00779.

2. Preliminaries

A *presentation graph* is a finite simplicial graph Γ where every edge connecting vertices s and t is labelled by an integer $m_{st} \geq 2$. Given a presentation graph Γ, the associated *Artin group* A_Γ is the group given by the following presentation:

$$A_\Gamma := \langle s \in V(\Gamma) \mid \Pi(s,t;m_{st}) = \Pi(t,s;m_{st}) \ \text{ whenever } s,t \text{ are adjacent in } \Gamma \rangle,$$

where $\Pi(x,y;k)$ denotes the alternating product of $xyxy\cdots$ with k letters. Similarly the associated *Coxeter group* W_Γ is the group given by the following presentation:

$$W_\Gamma := \langle s \in V(\Gamma) \mid s^2 = 1 \text{ for all } s \in V(\Gamma), (st)^{m_{st}} = 1 \ \text{ whenever } s,t \text{ are adjacent in } \Gamma \rangle,$$

Given a subset $S \subseteq V(\Gamma)$, the subgroup $\langle S \rangle \leqslant A_\Gamma$ is called a *standard parabolic subgroup* of A_Γ. By a result of van der Lek [vdL83, Theorem II.4.13], the subgroup $\langle S \rangle \leqslant A_\Gamma$ is isomorphic to the Artin group whose defining presentation graph is the subgraph of Γ induced by S.

An Artin group A_Γ is:

- *large-type* if all labels in Γ are at least 3;
- *XXXL* if all labels in Γ are at least 6;
- *free-of-infinity* if Γ is a complete graph;
- *hyperbolic-type* if W_Γ is a hyperbolic group;
- *spherical-type* if W_Γ is finite.

Definition 2.1. The *Deligne complex* of an Artin group A_Γ is the simplicial complex D_Γ defined as follows:

- Vertices correspond to left cosets of standard parabolic subgroups of spherical type.
- For every $g \in A_\Gamma$ and for every chain of induced subgraphs $\Gamma_0 \subsetneq \cdots \subsetneq \Gamma_k$ with $A_{\Gamma_0}, \dots, A_{\Gamma_k}$ of spherical type, there is a k-simplex with vertices $gA_{\Gamma_0}, \dots, gA_{\Gamma_k}$.

Equivalently, the Deligne complex D_Γ is the geometric realisation of the poset of left cosets of standard parabolic subgroups of spherical type.

The group A_Γ acts on D_Γ by left multiplication on left cosets, and we denote by K_Γ the subcomplex induced by the vertices of the form $1 \cdot A_{\Gamma'}$. A *standard tree* of D_Γ is the fixed-point set of a conjugate of a standard generator of A_Γ (see [MP22]).

In [BMV24] the first author, Martin and Vaskou studied homomorphisms between large-type Artin groups, and to do so introduced the following notions.

Definition 2.2. A *cycle of standard trees* is a sequence $T_1, \dots, T_n$ of distinct standard trees of the Deligne complex D_Γ such that for every $i \in \mathbb{Z}/n\mathbb{Z}$, the intersection $T_i \cap T_{i+1}$ is a vertex, and such that:

- for every $i \in \mathbb{Z}/n\mathbb{Z}$, the generators x_i and x_{i+1} of $\mathrm{Fix}(T_i)$ and $\mathrm{Fix}(T_{i+1})$ respectively generate a dihedral Artin group,
- for every distinct i, j with $j \neq i \pm 1$, the generators of $\mathrm{Fix}(T_i)$ and $\mathrm{Fix}(T_j)$ generate a non-abelian free group.

An Artin group A_Γ satisfies the *Cycle of Standard Trees Property* (COST) if the following holds for any cycle of standard trees: let $T_1, \dots, T_n$ be a cycle of standard trees in D_Γ, and let γ be the loop of D_Γ obtained by concatenating, for $i \in \mathbb{Z}/n\mathbb{Z}$, the geodesic segments γ_i of T_i between the vertices $T_i \cap T_{i-1}$ and $T_i \cap T_{i+1}$. Then there exists an element $g \in A_\Gamma$ such that γ is contained in gK_Γ.

In [BMV24, Proposition 6.1] it was shown that XXXL Artin groups satisfy the COST property, but it is conjectured to hold for bigger families. Combining results from [BMV24] and [Vas23], we can obtain the following description of the automorphism group of some classes of large-type Artin groups. Here graph automorphisms are automorphisms of A_Γ induced by automorphisms of the labelled graph Γ, and the global inversion is the automorphism that sends every generator to its inverse. We also say that an edge e in a connected graph Γ is separating if there exist two induced connected subgraphs Γ_1, Γ_2 such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = e$.

Theorem 2.3 ([Vas23, Theorem A], [BMV24, Corollary 1.7])**.** *Let A_Γ be a free-of-infinity large-type Artin group with $|V(\Gamma)| \geq 3$; or an Artin group satisfying the COST property and whose defining graph is connected, not an even edge, and does not have a cut-vertex or a separating edge. Then $\mathrm{Aut}(A_\Gamma)$ is generated by the conjugations, the graph automorphisms, and the global inversion. In particular, $\mathrm{Out}(A_\Gamma)$ is finite.*

Large-type Artin groups also enjoy the property of being systolic, and as a consequence they have solvable conjugacy problem [ECH+92, Theorem 2.5.7]. This was shown by Huang and Osajda in [HO20], where they constructed a thickening of the Cayley complex of a large-type Artin group and proved that it is systolic.

If G is a group, F is a normal subgroup of G and $g \in G$, we denote by φ_g the automorphism of F induced by conjugation by g. In [BMV09] Bogopolski, Martino and Ventura proved the following theorem (see [BMV09, §2] for the definition of algorithmic).

Theorem 2.4 ([BMV09, Theorem 3.1])**.** *Let*

$$1 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 1$$

be an algorithmic short exact sequence of groups such that

(1) $CP(H)$ is solvable, and
(2) for every $1 \neq h \in H$, the subgroup $\langle h \rangle$ has finite index in its centralizer $C_H(h)$, and there is an algorithm which computes a finite set of coset representatives, $z_{h,1}, \dots, z_{h,t_h} \in H$,

$$C_H(h) = \langle h \rangle z_{h,1} \sqcup \cdots \sqcup \langle h \rangle z_{h,t_h}.$$

Moreover, suppose that $TCP(F)$ is solvable. Then, the following are equivalent:

(a) $CP(G)$ is solvable,
(b) the action subgroup $A_G = \{\varphi_g \mid g \in G\} \leqslant \mathrm{Aut}(F)$ is orbit decidable.

3. Proofs

Proof of Theorem 1.1. In the case when Γ is an even edge, the result was proved by Crowe in [Cro24b, Theorem 3.28].

Suppose Γ is not an even edge. Let $\varphi \in \mathrm{Aut}(A_\Gamma)$, and consider the group $A_\Gamma \rtimes \langle \varphi \rangle$, where $\langle \varphi \rangle$ is a cyclic group of order k if φ has order $k < \infty$ and an infinite cyclic group otherwise. By [BMV09, Proposition 4.1], solvability of $CP(A_\Gamma \rtimes \langle \varphi \rangle)$ implies solvability of $TCP_\varphi(A_\Gamma)$. So in order to prove Theorem 1.1, it suffices to show that $A_\Gamma \rtimes \langle \varphi \rangle$ is systolic. We will prove that the action of A_Γ on the thickening of the Cayley complex extends to an action of $A_\Gamma \rtimes \langle \varphi \rangle$. If φ has finite order, then $A_\Gamma \rtimes \langle \varphi \rangle$ is a finite extension of A_Γ and therefore the latter action is geometric.

We proceed to describe the thickening of the Cayley complex constructed in [HO20, Definitions 3.7 and 5.3]. Let $\tilde{K}_\Gamma$ be the Cayley complex of A_Γ associated to the standard presentation. For each edge in Γ between vertices $s, t \in V(\Gamma)$ there is a relation of length $2m_{st}$ that lifts to copies in $\tilde{K}_\Gamma$. Subdivide each of these lifts to a $2m_{st}$-gon by adding $m_{st} - 2$ interior vertices, and call each of these subdivided 2-cells a precell (see Figure 1). Now, if two precells C_1, C_2 intersect at more than one edge, first connect interior vertices

$v_1 \in C_1$, $v_2 \in C_2$ such that both $\{v_1\} \cup e$ and $\{v_2\} \cup e$ span triangles for some edge $e \subset C_1 \cap C_2$, and then add edges between interior vertices of C_1 and C_2 to form a zigzag as in Figure 2. The flag completion of this complex is the desired thickening.

It is clear that if $\varphi \in \mathrm{Aut}(A_\Gamma)$ is a graph automorphism or the global inversion, then it induces an automorphism of the thickening of $\tilde{K}_\Gamma$, and since φ has finite order we get that $A_\Gamma \rtimes \langle\varphi\rangle$ acts geometrically. Hence $A_\Gamma \rtimes \langle\varphi\rangle$ is systolic and has solvable conjugacy problem, so $TCP_\varphi(A_\Gamma)$ is solvable. Since the classes of such automorphisms φ generate $\mathrm{Out}(A_\Gamma)$ by Theorem 2.3, it follows that $TCP(A_\Gamma)$ is solvable as well, as required. □

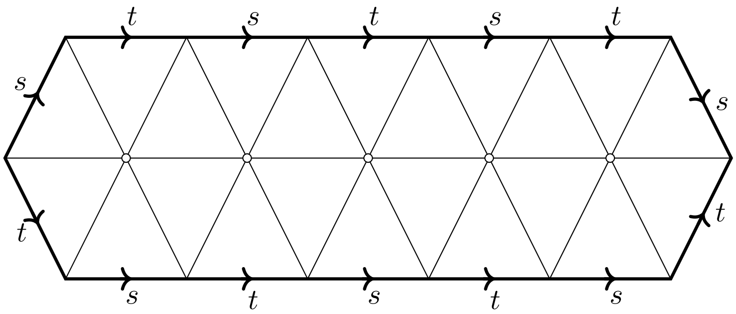

FIGURE 1. Subdivision of a precell with $m_{st} = 7$.

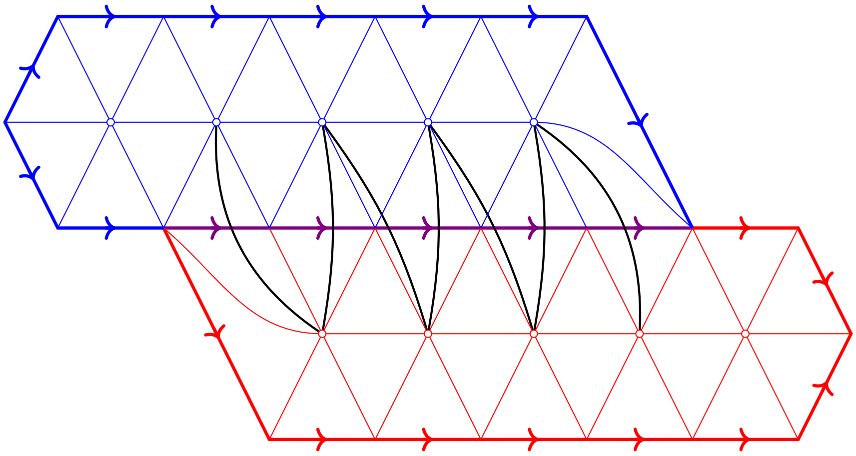

FIGURE 2. Adding a zigzag (black) between two intersecting precells (red and blue).

Proof of Corollary 1.2. In the case when Γ is an even edge, the result was proved by Crowe in [Cro24b, Theorem 4.4].

Suppose Γ is not an even edge. By Theorem 2.4, to prove Corollary 1.2 it is enough to show that the action subgroup $A_G = \{\varphi_g \mid g \in G\} \leqslant \mathrm{Aut}(A_\Gamma)$ is orbit decidable. Let $O_G \leqslant \mathrm{Aut}(A_\Gamma)$ be a set of unique representatives of the projection of A_G in $\mathrm{Out}(A_\Gamma)$. Then given $u, v \in A_\Gamma$, checking whether there is an automorphism in A_G that sends u to v is equivalent to checking whether v is conjugate to some element in $\{\varphi(u) \mid \varphi \in O_G\}$. Since $\mathrm{Out}(A_\Gamma)$ is finite (by Theorem 2.3), this problem is algorithmically solvable. □

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