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Non-positive curvature and Artin groups

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Abstract. In this thesis we introduce different generalizations and variants of the notion of non-positive curvature in the context of geometric group theory. We present new small cancellation conditions (T'), $\mathcal{T}'$ and $\mathcal{T}'_<$ and study their properties. We obtain results concerning hyperbolicity, diagrammatic reducibility, equations over groups, and solvability of the word and conjugacy problems for groups satisfying these conditions. In the process we define (strictly) systolic angled complexes, which generalize systolic complexes by allowing angles different from $\frac{\pi}{3}$.

The second focal point of the thesis are Artin groups. While studying condition $\mathcal{T}'$, we show that Artin groups are two-dimensional (i.e. they have geometric dimension at most 2) if and only if their standard presentation satisfies condition $\mathcal{T}'$. An important conjecture regarding Artin groups is that any intersection of parabolic subgroups is a parabolic subgroup. By introducing systolic-by-function complexes (another generalization of systolic complexes) and using their geometry, we solve this conjecture in the case of two-dimensional (2,2)-free Artin groups.

Another open question for Artin groups was to decide whether a parabolic subgroup P_1 of an Artin group A contained in another parabolic subgroup P_2 of A is a parabolic subgroup of P_2. We finish this thesis by answering this question in the positive for all Artin groups. In contrast to the rest of our work, the techniques used in this case are mostly algebraic instead of geometric.

Key words: non-positive curvature, hyperbolic groups, small cancellation, systolicity, Artin groups, parabolic subgroups.

Introduction

Geometric group theory concerns the study of groups by means of their actions on objects with an interesting geometry in a broad sense. Its origins can be traced back to the works of many mathematicians, particularly those of Max Dehn. Among his numerous contributions, he formulated the following problems [35] (see also [68]):

The word problem: a group G with presentation $\langle S \mid R \rangle$ has solvable word problem if there exists an algorithm that decides, given a word $w \in F(S)$, if w is trivial in G.

The conjugacy problem: a group G with presentation $\langle S \mid R \rangle$ has solvable conjugacy problem if there exists an algorithm that decides, given two words $w_1, w_2 \in F(S)$, if w_1 and w_2 are conjugate in G.

The isomorphism problem: given two finite group presentations, decide if they present isomorphic groups.

While algebraic or algorithmic at first sight, as we will see in this thesis, these problems have a deep connection with the geometry of the groups and the presentations involved. However, even though the work of Dehn dates back to the beginning of the twentieth century, the approach of studying groups geometrically did not rise to prominence until much later. Between Dehn's work and geometric group theory lie combinatorial group theory and small cancellation theory. The original procedure was to work with group presentations combinatorially. By looking at their presentation complex and their Cayley graph, these combinatorial ideas became more geometric. Research in this direction was carried out by Magnus, Baumslag, Solitar, Greendlinger, Lyndon, Schupp and Rips among others.

It is the seminal work of Gromov and his article [50] that decidedly established geometric group theory as an active field. Gromov introduced the two most important families of groups of non-positive or negative curvature: hyperbolic groups and CAT(0) groups. These groups are characterized by acting nicely (geometrically) on hyperbolic and respectively CAT(0) metric spaces. Both of these families of metric spaces enjoy coarse curvature properties. By the works of Gromov and subsequent authors, all three of Dehn's questions have a positive answer for hyperbolic groups, and the word and conjugacy problems are solvable for CAT(0) groups.

Motivated by Gromov's contributions many notions of non-positive curvature for groups have arisen in recent years. One of them is systolic simplicial complexes and groups. The complexes were first introduced by Chepoi under the name of *bridged complexes* in [23]. Systolic complexes were later rediscovered and studied by Januszkiewicz and Świątkowski in [61] and by Haglund in [52]. We will follow the viewpoint of Januszkiewicz and Świątkowski (see Section 1.3 for definitions). The idea behind systolic complexes is to find an easy-to-check combinatorial condition on simplicial complexes resembling that of CAT(0) cube complexes. Though systolic simplicial complexes are not necessarily CAT(0), groups acting geometrically on them share many properties with CAT(0) groups.

In this thesis we will explore two generalizations of systolic complexes and groups. The first one is (strictly) systolic angled complexes. Instead of thinking that all triangles are equilateral with angles of $\frac{\pi}{3}$, we allow for different angles, gaining more flexibility. The second one is systolic-by-function complexes. In this case what we change are the lengths of the sides of the triangles. We do so in a combinatorial rather than a metric fashion. Both of these definitions were inspired by metrically systolic complexes, a metric generalization of systolic complexes introduced by Huang and Osajda in [57] to study two-dimensional Artin groups.

Small cancellation theory consists on studying groups given by presentations where the relators have small overlaps. These small overlaps are formalized in what are called small cancellation conditions. Groups satisfying strong enough small cancellation conditions have good algebraic, algorithmic and geometric properties, such as solvable word an conjugacy problems or being hyperbolic. Small cancellation conditions are stated in purely algebraic terms, but have a very clear geometric interpretation. Due to the work of van Kampen [62] and Lyndon [67] it is known that a trivial word in a group presented by a presentation P is equivalent to having a disk diagram over the presentation complex K_P whose boundary reads said word (i.e. a combinatorial map from a combinatorial structure of a singular disk to K_P). As we will see in Section 1.5, small cancellation conditions over the presentation translate to geometric conditions over these diagrams.

In Chapter 1 we recall the basic ideas and results on spaces and groups of non-positive curvature. We start with hyperbolic and CAT(0) spaces and groups, and in the process review some concepts of independent interest (quasi-isometries, geodesic metric spaces, Dehn functions and metric simplicial complexes among others). Then we present systolicity and a more recent generalization, metric systolicity. These two notions play a key role in the rest of the thesis, as they serve as inspiration for systolic angled complexes (see Section 3.1) and systolic-by-function complexes (see Section 4.2). Finally, we give a short introduction to small cancellation theory.

In Chapter 3 we introduce new small cancellation conditions that both unify and expand on the classical ones. We start by defining strictly systolic angled complexes (see Definition 3.1.1). Using curvature techniques and a combinatorial version of the Gauss–Bonnet theorem we prove the following:

Theorem 3.1.9. *Let X be a strictly systolic angled complex. Then there exists a constant $K > 0$ such that*

$$\mathrm{Area}(\gamma) \leq Kl(\gamma),$$

for every closed edge-path γ in X.

That is, we show that strictly systolic angled complexes satisfy a linear isoperimetric inequality, and thus we get the subsequent corollary. In particular it follows that groups acting geometrically on these complexes are hyperbolic.

Corollary 3.1.10. *The 1-skeleton $X^{(1)}$ of a strictly systolic angled complex X with its standard geodesic metric is hyperbolic. More generally, if we endow X with a piecewise Euclidean metric with* $\mathrm{Shapes}(X)$ *finite, then X is hyperbolic.*

As an application of this corollary, we investigate when one-relator groups act properly and cocompactly by simplicial automorphisms on a strictly systolic angled complex. Given a one-relator group, we start by taking its Cayley complex and thicken it in a dual fashion, similarly to what Huang and Osajda do for two-dimensional Artin groups in [57]. While analyzing this complex, a natural small cancellation condition arises. We call this small cancellation condition (T'). This condition together with condition $C'(\frac{1}{4})$ are sufficient to obtain our desired result.

Theorem 3.2.1. *Let Γ be a one-relator group with presentation $P = \langle \mathcal{A} \mid R \rangle$. If P satisfies the metric small cancellation condition $C'(\frac{1}{4})$ and Condition (T'), then Γ is hyperbolic.*

Motivated by this result, we pose the obvious question: can condition (T') be studied combinatorially as the classical small cancellation conditions? That is, independently of strictly systolic angled complexes. We show that the answer is positive. Furthermore, we define more general conditions $\mathcal{T}'$ and $\mathcal{T}'_<$. Condition $\mathcal{T}'_<$ encompasses conditions $C'(\frac{1}{4})-(T'), C'(\frac{1}{6}), C'(\frac{1}{4})-T(4)$ and $C'(\frac{1}{3}) - T(6)$, while condition $\mathcal{T}'$ is the non-strict version. Essentially, these conditions allow large overlapping among relators as long as, when seen in disk diagrams, these large overlappings do not concentrate at a single vertex.

We state here some of the results that we obtained with these new conditions. The first one is related to diagrammatic reducibility. Diagrammatic reducibility (DR) is a combinatorial condition stronger than asphericity, which has applications to equations over groups (see Section 3.4).

Theorem 3.4.2. *If a presentation P satisfies condition $\mathcal{T}'$ and has no proper powers, then it is DR.*

Then we prove theorems related to non-positive curvature properties of the groups. The first one is a strengthening of Theorem 3.2.1.

Theorem 3.4.4. *Let G be a group which admits a finite presentation satisfying conditions $\mathcal{T}'_<$ and $C(3)$. Then G is hyperbolic.*

The second one is related to the non-strict condition. Note that in this theorem, we require that all relators have the same length. We believe the result holds without requiring this condition, although we do not have a proof yet.

Theorem 3.5.2. *Let P be a presentation satisfying conditions $\mathcal{T}'-C'(\frac{1}{2})$ and such that all its relators have length r, then P has a quadratic Dehn function. Moreover, if P is finite the group G presented by P has solvable conjugacy problem.*

The third one is about examples of groups with presentations that satisfy condition $\mathcal{T}'$. The ones that stand out are two-dimensional Artin groups. In fact, we show the following:

Theorem 3.3.2. *An Artin group A_Γ is two-dimensional if and only if its standard presentation P_Γ satisfies condition $\mathcal{T}'$.*

Most of the results of Chapter 3 are part of the articles [9] and [10] written in collaboration with Minian, and Minian and Sadofschi-Costa respectively.

Artin groups constitute one of the most studied families in geometric group theory. They are deeply connected to Coxeter groups and are also a generalization of braid groups. In Chapter 2 we recall their definition and basic results, and briefly survey some well-known facts and open problems. As we will see, many questions related to Artin groups are geometric in nature. Hence, having a nice geometric structure to work with seems promising. Theorem 3.3.2 gives said a nice geometric structure to two-dimensional Artin groups. We posed ourselves the question of whether these ideas could be used to tackle open problems for Artin groups in the case of two-dimensional Artin groups. That is the content of Chapter 4. Artin groups have a natural family of subgroups called parabolic subgroups. They play a central role in their study and can be thought of as the building blocks of Artin groups. One open question which has received much attention in recent years is whether an intersection of parabolic subgroups is once again a parabolic subgroup. In Chapter 4 we give a positive answer to this question for $(2,2)$-free two-dimensional Artin groups.

In a recent article by Cumplido, Martin and Vaskou [31] they solve this problem for large-type Artin groups. They do so by studying a simplicial complex called the Artin complex. Then they show that when the Artin group is large-type, this complex is systolic. They prove that the intersection of parabolic subgroups is a parabolic subgroup when action of the Artin group on the Artin complex satisfies certain path fixing condition. Systolic complexes satisfy the needed path fixing condition and their result follows. We wish to recover the result for a broader family of Artin groups, so we need a more encompassing geometric notion. We achieve this by considering systolic-by-function complexes, which generalize systolic complexes. Systolic-by-function complexes have a more flexible structure than systolic complexes since we allow the edges to have different lengths. At the same time, their geometry is rigid enough to satisfy an analogue of the Cartan–Hadamard theorem and other geometric properties similar to those of systolic complexes. In particular, they satisfy a path fixing condition (see Theorem 4.3.1) and thus we are able to prove the following result.

Theorem 4.3.2. *Let A_Σ be a $(2,2)$-free two-dimensional Artin group. Then the intersection of an arbitrary family of parabolic subgroups is a parabolic subgroup.*

As a consequence of this theorem and an algorithm introduced by Cumplido [29] we solve the conjugacy stability problem for $(2,2)$-free two-dimensional Artin groups. A subgroup H of a group G is *conjugacy stable* if, for every pair $h, h' \in H$ such that there exists $g \in G$ with $g^{-1}hg = h'$, there is $\tilde{h} \in H$ such that $\tilde{h}^{-1}h\tilde{h} = h'$. The conjugacy stability problem consists in deciding which of the parabolic subgroups of an Artin group are conjugacy stable.

Theorem 4.3.9. *Let A_Γ be a $(2,2)$-free two-dimensional Artin group and A_{Γ_X} a standard parabolic subgroup. Then A_{Γ_X} is not conjugacy stable if and only if there exist vertices x, y in Γ_X that are connected by an odd-labeled path in Γ, but are not connected by an odd-labeled path in Γ_X.*

Some of the results of Chapter 4 appear in the recent article [8].

Chapter 5 is somewhat independent of the previous chapters. Even though we continue investigating Artin groups and their parabolic subgroups, the techniques used are completely different. While they do have a geometric background (see Section 5.2), the arguments are mostly algebraic. The contents of Chapter 5 correspond to an article in collaboration with Paris [11]. In [47] Godelle conjectures that a parabolic subgroup P_1 of an Artin group A which is contained in another parabolic subgroup P_2 of A is a parabolic subgroup of P_2. This conjecture had already been proven for some families of

Artin groups. We show that it is true for all Artin groups. More precisely we prove the following.

Theorem 2.3.1. *Let* Γ *be a finite simplicial graph, let* $m : E(\Gamma) \to \mathbb{N}_{\geq 2}$ *be a labeling, and let* $A = A_\Gamma$ *be the Artin group of* Γ*. Let* $X, Y \subset V(\Gamma)$ *and* $\alpha \in A$ *such that* $\alpha A_Y \alpha^{-1} \subset A_X$*. Then there exist* $Y' \subset X$ *and* $\gamma \in A_X$ *such that* $\alpha A_Y \alpha^{-1} = \gamma A_{Y'} \gamma^{-1}$*.*

Contents

Chapter 1

Non-positively curved spaces and groups

1.1 Hyperbolic spaces and groups

Hyperbolic metric spaces and groups were introduced by Gromov in [50]. In essence, a group is hyperbolic if, when equipped with its word metric, it satisfies certain geometric properties analogous to those of classical hyperbolic geometry. Apart from the classical theory of manifolds of negative curvature, Gromov's definition draws inspiration from the works of Dehn and Rips. Unlike the CAT(κ) conditions that we will present later, hyperbolicity encapsulates the large scale geometry of a space. More precisely, hyperbolicity is preserved under quasi-isometries. Intuitively speaking, two spaces are quasi-isometric if when looked from afar, they look the same. Gromov's article [50] can be considered the starting point of modern geometric group theory. Since their introduction, hyperbolic groups have continuously played a central role in the area. We refer the reader to Bridson and Haefliger's book [15] and the series of notes edited by Short [80] for a more detailed exposition. Here we only recall the basic definitions and results that we will use later on.

1.1.1 Geodesic metric spaces and quasi-isometries

Before defining hyperbolic groups we need some basic concepts on geodesic spaces.

Let (X, d) be a metric space. We define the *length* of a continuous curve $\gamma : [a, b] \to X$ as

$$l(\gamma) = \sup_{a=t_o \le t_1 \cdots \le t_n = b} \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})).$$

If the length of a curve γ is finite, γ is said to be *rectifiable*.

Definition 1.1.1. A metric space (X, d) is a *length space* if the distance between every pair of points $x, y \in X$ is equal to the infimum of the lengths of the curves joining them.

We wish for a slightly stronger notion, where these infimums are attained. For that, we need geodesics.

Definition 1.1.2. Let (X, d) be a metric space. A *geodesic* from $x \in X$ to $y \in X$ is a continuous function $\gamma : [0, l] \to X$ such that $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, l]$. We usually denote the image of γ by $[x, y]$ and call it a *geodesic segment*.

It is clear from the definition that the length of a geodesic $\gamma : [0, l] \to X$ is equal to l. A metric space is said to be *geodesic* if each pair of points is joined by a geodesic. Examples of geodesic metric spaces include normed vector spaces (in particular Euclidean spaces), model spaces of constant curvature (see Definition 1.2.1) and metric graphs whose edges have finitely many different lengths.

Length spaces need not be geodesic in general. However, under reasonable hypotheses they are. This is known as the Hopf–Rinow theorem.

Theorem 1.1.3 (Hopf–Rinow Theorem)**.** *Let X be a complete and locally compact length space. Then X is geodesic, and every bounded and closed subset of X is compact.*

Another notion we will need is that of quasi-isometry. As mentioned earlier, a standout feature of hyperbolic groups is that they are stable under quasi-isometries. Thus, hyperbolicity reflects the large scale geometry of a space.

Definition 1.1.4. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A function $f : X_1 \to X_2$ is a *quasi-isometric embedding* if there exist constants $A \geq 1$ and $B \geq 0$ such that for every $x, y \in X_1$,

$$\frac{1}{A} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A d_1(x, y) + B.$$

If there also exists a constant $C \geq 0$ such that for every $z \in X_2$ there is an $x \in X_1$ satisfying

$$d_2(z, f(x)) \leq C,$$

we say that f is a *quasi-isometry*. When this last condition holds, we say that the image of f is *quasi-dense* in X_2. Two metric spaces are *quasi-isometric* if there exists a quasi-isometry between them. It is easy to see that being quasi-isometric is an equivalence relation

For example, any two bounded metric spaces are quasi-isometric. One can think of bounded metric spaces as a point when looked from afar. Another standard example of quasi-isometric spaces are $\mathbb{Z}$ and $\mathbb{R}$ with their standard norm metrics, where the quasi-isometry is given by the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{R}$. As we will see later, this is a particular case of a more general phenomenon that we will allow us to compare the geometry of a group to that of a geodesic metric space.

1.1.2 Hyperbolic metric spaces

In [50], Gromov attributes the following definition to Rips.

Definition 1.1.5. Let $\delta > 0$. A geodesic metric space X is δ-*hyperbolic* if for every $x, y, z \in X$ and geodesic segments $[x, y], [x, z], [y, z]$ we have that

$$[x, y] \subseteq \bar{B}_\delta([x, z]) \cup \bar{B}_\delta([y, z]),$$

where $\bar{B}_\delta(Y)$ denotes the closed δ-ball around a subset $Y \in X$ (i.e., $\bar{B}_\delta(Y) = \{x \in X \mid d(x, Y) \leq \delta\}$). The space X is *hyperbolic* if it is δ-hyperbolic for some δ.

The previous definition requires the metric space to be geodesic and the condition is usually called the “slim triangles condition”. However, there is another characterization of hyperbolicity that does not require the metric space to be geodesic and is equivalent to the previous one when the space is geodesic. For this, we need the notion of Gromov product. Let X be a metric space. Given three points $x, y, z \in X$, the *Gromov product* of x and y at z is

$$\langle x, y \rangle_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y)).$$

Definition 1.1.6. Let $p \in X$ and $\delta > 0$. The pair (X, p) is *δ-hyperbolic* if for every $x, y \in X$

$$\langle x, y \rangle_p \geq \min_{z \in X} \{\langle x, z \rangle_p, \langle y, z \rangle_p\} - \delta.$$

A metric space X is *hyperbolic* if there exist $p \in X$ and $\delta > 0$ such that (X, p) is δ-hyperbolic.

Definitions 1.1.6 and 1.1.5 coincide when the metric space is geodesic. There are more equivalent definitions. A good treatment of these equivalences can be found in [80].

A remarkable property of hyperbolicity is that it is invariant under quasi-isometries in geodesic metric spaces. As mentioned before, this is telling us that hyperbolicity captures the large scale geometry of the space.

Theorem 1.1.7. *Let X_1 and X_2 be quasi-isometric geodesic metric spaces. Then X_1 is hyperbolic if and only if X_2 is hyperbolic.*

1.1.3 Hyperbolic groups

We wish to understand groups geometrically. One way of achieving this is viewing them as metric spaces. To do so, we need the notions of word metric and Cayley graph.

Let G be a group with generating set S. Given $g, h \in G$, we set $d_S(g, h) = 0$ if $g = h$. Otherwise, we set $d_S(g, h)$ as the smallest $n \in \mathbb{N}$ such that there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ with $s_1 \cdots s_n = g^{-1}h$. We call d_S the *word metric* in G associated to S.

Let G be a group with generating set S. The *Cayley graph* $\Gamma(G, S)$ of G with respect to S is the graph whose vertices are the elements of G and has an edge joining g to gs for each $g \in G$ and $s \in S$. It can be made into a metric graph by declaring each edge to be isometric to the unit interval.

Notice that the natural inclusion $i : (G, d_S) \to \Gamma(G, S)$ is a quasi-isometry. A particular case of this fact is the previously given example of a quasi-isometry between $\mathbb{Z}$ and $\mathbb{R}$. It is a well known fact that if G is finitely generated and S_1 and S_2 are finite generating sets, then (G, d_{S_1}) and (G, d_{S_2}) are quasi-isometric. Thus the following definition does not depend on the chosen (finite) generating set.

Definition 1.1.8. A finitely generated group G is *hyperbolic* if there is a finitely generating set such that (G, d_S) (or equivalently $\Gamma(G, S)$) is a hyperbolic metric space.

A common theme in geometric group theory is to study groups by their actions on metric spaces. It is convenient to study actions that, in some sense, are good.

Definition 1.1.9. A (left) action of a group G on a metric space X is said to be

- *cocompact* if there exist a compact set $K \subseteq X$ such that $X = GK$;
- *proper* if for each $x \in X$ there exists $r > 0$ such that the set $\{g \in G \mid gB_r(x) \cap B_r(x) \neq \emptyset\}$ is finite;
- *geometric* if G acts by isometries and the action is cocompact and proper.

The following is a fundamental result in geometric group theory. It enables us to understand a group in terms of the geometry of a metric space on which it acts geometrically.

Theorem 1.1.10 (Švarc–Milnor Lemma)**.** *Let G be a group and X be a length space. If G acts geometrically on X, then G is finitely generated, and for any choice of basepoint $x_0 \in X$, the map $g \mapsto gx_0$ is a quasi-isometry.*

Corollary 1.1.11. *A group G acting geometrically on a hyperbolic length space X is a hyperbolic group.*

1.1.4 Dehn functions

We will present another characterization of hyperbolic groups in terms of their presentations. A key tool for understanding group presentations geometrically are Dehn functions.

Let G be a finitely presented group with finite presentation $\langle S \mid R\rangle$. Given two words w and w' in the free group $F(S)$, we write $w =_G w'$ to indicate that they represent the same element in G. Let $w \in F(S)$ be a cyclically reduced word in the free group over S that is trivial in G. Then w admits an expression of the form

$$w = \prod_{i=1}^{k} u_i r_i^{\pm 1} u_i^{-1},$$

where $r_i \in R$ and $u_i \in F(S)$ for all i. The *area* of a word w that is trivial in G is the least amount of relators needed to express w as a product of conjugates of relators and their inverses (i.e. the minimal k in the above formula). It is denoted by $A(w)$. The term "area" has a clear geometric interpretation in terms of diagrams over the presentation, which will be explored in Section 1.5. The following definition allows us to bound the area of a word in terms of its word length (when looking at diagrams, this translates to bounding their area by their perimeter).

Definition 1.1.12. The *Dehn function* of the group G with respect to the finite presentation $P = \langle S \mid R\rangle$ is

$$Dehn_{G,P}(l) = \max_{l(w)\leq l,\ w =_G 1} \{A(w)\},$$

where $l(w) = d_S(w, 1)$.

To see that the Dehn function does not depend (up to an equivalence relation) on the finite presentation chosen, we will need the following definition.

Definition 1.1.13. Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be two non-decreasing functions. We say that f is *Dehn dominated* by g if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$f(n) \leq Cg(Cn + C) + Cn + C.$$

We denote it by $f \prec g$. If $f \prec g$ and $g \prec f$, we say that f and g are *Dehn equivalent*, and we note it $f \sim g$.

Proposition 1.1.14. *If $\langle S_1 \mid R_1 \rangle$ and $\langle S_2 \mid R_2 \rangle$ are finite presentations for a group G, then the Dehn functions with respect to $\langle S_1 \mid R_1 \rangle$ and $\langle S_2 \mid R_2 \rangle$ are Dehn equivalent.*

When $Dehn_G \prec f$ and f is a linear (resp. quadratic, polynomial, exponential, etc) function, we say that G satisfies a linear (resp. quadratic, polynomial, exponential, etc) isoperimetric inequality (this terminology will be made clearer in Section 1.5). Now we can give a new characterization of hyperbolic groups.

Theorem 1.1.15 ([44, 50])**.** *A group G is hyperbolic if and only if it is finitely presentable and satisfies a linear isoperimetric inequality.*

Note that in particular, all hyperbolic groups admit a finite presentation. As with hyperbolicity, the Dehn function of a finitely presented group is a quasi-isometry invariant.

1.1.5 Dehn's problems

In 1911 Max Dehn proposed the following problems [35]. Though, at first sight, not geometric in nature, they motivated much of the development of modern combinatorial and geometric group theory, and continue to be widely studied. For a more in-depth exposition of these questions we recommend the book by Brady, Riley and Short [14].

The word problem: a group G with presentation $\langle S \mid R \rangle$ has solvable word problem if there exists an algorithm that decides, given a word $w \in F(S)$, if w is trivial in G.

The conjugacy problem: a group G with presentation $\langle S \mid R \rangle$ has solvable conjugacy problem if there exists an algorithm that decides, given two words $w_1, w_2 \in F(S)$, if w_1 and w_2 are conjugate in G.

The isomorphism problem: given two finite group presentations, decide if they present isomorphic groups.

Notice that having solvable conjugacy problem implies having solvable word problem, since a word is trivial if and only if it is conjugate to the trivial word. These problems are known to not always be solvable, even for finitely presented groups.

There is a close connection between these problems and the Dehn function of a presentation. For example, if the Dehn function is recursive, we can bound the area of a word representing the trivial element in terms of its length. It can be seen then that, if the word were trivial, a finite time algorithm gives an expression for the word as a product of conjugates of relators and their inverses. Furthermore, finitely presented groups have solvable word problem if and only if their Dehn function is recursive (see Gersten [42]). Hence, hyperbolic groups have solvable word problem. Additionally, they have solvable conjugacy problem [15] and isomorphism problem by Dahmani and Groves, Dahmani and Guiradel, and Sela [32, 33, 79].

1.2 CAT(0) spaces and groups

We will present some standard definitions and results concerning CAT(0) geometry and groups. As before, we refer the reader to [15] for a detailed exposition.

1.2.1 CAT(0) metric spaces

In order to talk about CAT(0) metric spaces, we will first introduce CAT(κ) metric spaces, and then concentrate on the non-positively curved case.

Definition 1.2.1. Let $\kappa \in \mathbb{R}$ and $n \in \mathbb{N}$. The n-dimensional *model space* of curvature κ, denoted M^n_κ, is defined as follows:

- if $\kappa > 0$, then M^n_κ is the n-dimensional unit sphere $\mathbb{S}^n$ with its usual distance multiplied by $\frac{1}{\sqrt{\kappa}}$;
- if $\kappa = 0$, then M^n_0 is the n-dimensional Euclidean space $\mathbb{E}^n$;
- if $\kappa < 0$, then M^n_κ is the n-dimensional hyperbolic space $\mathbb{H}^n$ with is usual distance multiplied by $\frac{1}{\sqrt{-\kappa}}$.

A CAT(κ) geodesic metric space is one where geodesic triangles are "slimmer" than in the corresponding model space M^2_κ. We make this more precise. Let X be a geodesic metric space, $x, y, z \in X$ and $\kappa \in \mathbb{R}$. If $\kappa > 0$, we also assume that $d(x,y)+d(y,z)+d(z,x) < \frac{2\pi}{\sqrt{\kappa}}$ (which is the diameter of M^2_κ). Let

$\Delta = \Delta([x,y],[y,z],[z,x])$ a geodesic triangle with vertices x, y, z. A *comparison triangle* for Δ in M^2_κ is a geodesic triangle $\Delta' = \Delta([x',y'],[y',z'],[z',x']) \subseteq M^2_\kappa$ with $d(x,y) = d(x',y')$, $d(y,z) = d(y',z')$ and $d(z,x) = d(z',x')$. Let $p \in [x,y]$. The *comparison point* for p in Δ' is the unique point $p' \in [x',y']$ such that $d(x,p) = d(x',p')$. Comparison points in the other two sides are defined analogously.

Definition 1.2.2. Let X be a geodesic metric space and $\kappa \in \mathbb{R}$. The space X is *CAT(κ)* if for every geodesic triangle $\Delta = \Delta([x,y],[y,z],[z,x])$ (where $d(x,y)+d(y,z)+d(z,x) < \frac{2\pi}{\sqrt{\kappa}}$ if $\kappa > 0$), and comparison triangle Δ', we have that if $p, q \in \Delta$ and $p', q' \in \Delta'$ are their comparison points, $d(p,q) \leq d(p',q')$.

It is a well known fact that if a space is CAT(κ), then it is CAT(κ') for all $\kappa < \kappa'$. Unlike hyperbolicity, the CAT(κ) conditions are not a quasi-isometry invariant. We state some properties of the CAT(0) case (and hence of all negatively curved cases).

Proposition 1.2.3 ([15])**.** *Let X be a $CAT(0)$ space. Then:*

- *X is uniquely geodesic, and geodesics vary continuously with its endpoints;*

- *the distance function is convex. That is, given geodesics $\gamma : [0,T] \to X$ and $\gamma' : [0,T] \to X$, we have that for all $t \in [0,T]$*

$$d(\gamma(t), \gamma'(t)) \leq (1-t)d(\gamma(0), \gamma'(0)) + td(\gamma(T), \gamma'(T));$$

- *X is contractible;*

- *a function $\gamma : I \to X$ is a geodesic if and only if it is a local geodesic.*

Theorem 1.2.4. *Let X be a $CAT(0)$ space and G a finite group acting by isometries on X. Then the action of G has a fixed point. Moreover, the fixed-point set is a non-empty convex subspace.*

The CAT(κ) condition is a global condition. This makes it hard to check in a general setting. Fortunately, in the non-positive curvature case there is an analogous statement to the Cartan–Hadamard theorem for Riemannian manifolds of non-positive sectional curvature. That is, we can go from a local condition to a global condition when our space is simply connected. A metric space X is of *curvature* $\leq \kappa$ if it is locally CAT(κ). That is, if every point in X has a CAT(κ) neighborhood.

Theorem 1.2.5 (Cartan–Hadamard Theorem, [15]). *Let X be a complete connected metric space. If X is of curvature $\leq \kappa$ with $\kappa \leq 0$, then its universal covering $\tilde{X}$ is CAT(κ). In particular, a simply connected space of non-positive curvature is CAT(0).*

As it happens with hyperbolic metric spaces and groups, we are interested in groups acting nicely on CAT(κ) spaces. A group G acting geometrically on a CAT(κ) space is a *CAT(κ) group*. Groups that are CAT(κ) with $\kappa < 0$ are hyperbolic. However, not all CAT(0) groups are hyperbolic (they may act on flat,"Euclidean", spaces). Nonetheless they exhibit many similar properties.

Proposition 1.2.6 ([15]). *Let G be a CAT(0) group. Then:*

- *G is finitely presented;*
- *there is a bound on the rank of free abelian subgroups of G;*
- *G satisfies a quadratic isoperimetric inequality;*
- *the conjugacy problem is solvable for G.*

1.2.2 CAT(0) simplicial complexes

Showing that a metric space is CAT(0) is usually a very hard task, since it is a global condition. A solution to this problem is to restrict oneself to a more manageable family of spaces. In this direction, Bridson introduced the following class of simplicial complexes [15].

Definition 1.2.7. Let $\kappa \in \mathbb{R}$. A simplicial complex X is called an M_κ-*simplicial complex* if

- for each simplex σ of X, there is a bijection $p_\sigma : \sigma \to M_\kappa^n$ from σ to a geodesic simplex in M_κ^n of the corresponding dimension;

- whenever two simplices σ_1, σ_2 of X share a face τ, the composition $p_{\sigma_2} \circ p_{\sigma_1}^{-1}$ is an isometry from $p_{\sigma_1}(\tau)$ to $p_{\sigma_2}(\tau)$.

The set of isometry classes of the simplices of X is denoted $\mathrm{Shapes}(X)$. When $\kappa = 0$ ($\kappa < 0, \kappa > 0$), we say that X is piecewise Euclidean (resp. hyperbolic, spherical).

To study these complexes geometrically, one endows them with a pseudometric. Let X be an M_κ-simplicial complex and $x, y \in X$. An *m-string* from x to y is a sequence $\Sigma = (x_0, \ldots, x_m)$ of points in X such that $x_0 = x$, $x_m = y$, and for each $i = 0, \ldots, m-1$, there exists a simplex σ_i containing x_i and x_{i+1}. We define the length of Σ as

$$l(\Sigma) = \sum_{i=0}^{m-1} d_{\sigma_i}(x_i, x_{i+1}),$$

where d_{σ_i} is the distance in σ_i induced by the bijection p_{σ_i}. The *intrinsic pseudometric* on X is defined by

$$d(x, y) = \inf\{l(\Sigma) \mid \Sigma \text{ a string from } x \text{ to } y\}.$$

A remarkable result by Bridson is the following.

Theorem 1.2.8 ([15])**.** *Let X be an M_κ-simplicial complex. If* $\mathrm{Shapes}(X)$ *is finite, then X with its intrinsic pseudometric is a complete geodesic metric space.*

A nice feature of M_κ-simplicial complexes is that, when they are simply connected, there is a local criterion to see if they are CAT(0). This is in the same spirit as the Cartan–Hadamard theorem for metric spaces of non-positive curvature.

Let X be an M_κ-simplicial complex with $\mathrm{Shapes}(X)$ finite and $v \in X^{(0)}$. We recall that given a simplex σ in X, its *link* $\mathrm{lk}_X(\sigma)$ is the subcomplex of X consisting of the simplices that are disjoint from σ and such that, together with σ, span a simplex of X. Let τ be a simplex in $\mathrm{lk}_X(v)$, and $x, y \in \tau$. The *angular distance* between x and y, denoted $\angle(x, y)$, is defined as the angle at v between geodesic segments $[v, x]$ and $[v, y]$ in σ. This makes $\mathrm{lk}_X(v)$ into an M_1-simplicial complex with $\mathrm{Shapes}(\mathrm{lk}_X(v))$ finite. The *angular metric* on $\mathrm{lk}_X(v)$ is an intrinsic pseudometric.

The following local criterion is usually known as Gromov's link condition.

Theorem 1.2.9 ([50]). *Let X be a simply connected M_κ-simplicial complex with* $\mathrm{Shapes}(X)$ *finite. Then X is CAT(κ) if and only if the link of every vertex is CAT(1) with the angular metric.*

This local condition is still not easy to verify in general, since links of vertices may be complicated high dimensional simplicial complexes. However, when X is 2-dimensional, the links of its vertices are simplicial graphs and the CAT(1) condition becomes simpler to check.

Corollary 1.2.10. *Let X be a 2-dimensional simply connected M_κ-simplicial complex with* $\mathrm{Shapes}(X)$ *finite. Then X is CAT(κ) if and only if every simple cycle in the link of every vertex has length greater than or equal to 2π with the angular metric.*

1.3 Systolicity

Another family of CAT(0) spaces are CAT(0) cube complexes. They are very prominent because the CAT(0) condition is easy to verify: a cube complex is CAT(0) if and only if the links of its vertices are flag simplicial complexes (i.e. every finite set of pairwise adjacent vertices spans a simplex).

Motivated by this characterization, one could ask if such a simple combinatorial condition exists for simplicial complexes. A possible answer are systolic complexes. While systolic complexes are not CAT(0), they behave very similarly to CAT(0) complexes and constitute a partial answer to this question. They were first defined by Chepoi under the name of *bridged complexes* in [23]. Systolic complexes were later rediscovered and studied by Januszkiewicz and Świątkowski in [61] and by Haglund in [52]. We now turn to the definitions, following mainly Januszkiewicz and Świątkowski [61].

A *cycle* in a simplicial complex X is a subcomplex σ homeomorphic to S^1. We denote by $|\sigma|$ the number of edges in σ, and call it its *length*. A subcomplex K of a simplicial complex X is *full* if any simplex of X spanned by a set of vertices in K is a simplex of K. A *diagonal* in a cycle σ in a simplicial complex X is an edge of X connecting two nonconsecutive vertices of σ. Thus, a cycle is full if and only if it has no diagonals and does not span a simplex.

Definition 1.3.1. Given a natural number $k \geq 4$, a simplicial complex X is *k-large* if it is flag and if every full cycle has length greater than or equal to k. It is *locally k-large* if the link of every vertex is large.

It is clear from the definitions that a k-large complex is locally k-large. This is because, since the complex is flag, the links of its vertices are flag and full cycles in the links are full cycles in the complex. When X is simply connected and $k \geq 6$ the converse holds. That is, a simply connected locally k-large complex with $k \geq 6$ is k-large [23, 61]. This is a local-to-global theorem analogous to the classical result for CAT(0) spaces, which motivates the following definition.

Definition 1.3.2. A simplicial complex X is *k-systolic* if it is connected, simply connected and locally k-large.

By the local-to-global theorem, it holds that a k-systolic complex is k-large if $k \geq 6$. In particular it is also flag. This means that they are determined by their 1-skeleton.

A group acting properly, cocompactly and by simplicial automorphisms on a k-systolic complex is called a *k-systolic* group. Since 6-systolic complexes and groups are the most studied, they are conventionally called *systolic* complexes and groups. For $k \geq 6$, these groups satisfy nice properties, which we summarize in the following proposition.

Proposition 1.3.3. *Let G be a k-systolic group with $k \geq 6$. Then:*

- *[61, 82] G is biautomatic. In particular it is finitely presentable, and has a quadratic Dehn function and solvable conjugacy problem;*
- *[61] if $k \geq 7$, G is hyperbolic;*
- *[53, 89] finitely presented subgroups of G are k-systolic;*
- *[58] virtually solvable subgroups of G are virtually cyclic or virtually $\mathbb{Z}^2$;*
- *[71] the centralizer of an infinite order element of G is commensurable with $\mathbb{Z}$ or $\mathbb{Z} \times F_n$.*

Systolic complexes also satisfy a fixed point theorem for finite groups, as in the CAT(0) case. This was proved by Chepoi and Osajda in [24].

Theorem 1.3.4. *[24] Let G be a finite group acting by simplicial automorphisms on a systolic complex X. Then there exists a simplex $\sigma \in X$ that is stabilized by G.*

1.4 Metric systolicity

Systolic complexes and groups satisfy numerous nice properties, as exhibited in the previous section. However, their structure is sometimes too rigid and, in order to get more examples, a more relaxed structure is needed. In this direction, Huang and Osajda introduced metrically systolic complexes, which are a metric generalization of systolic complexes [57]. In their previous article [58] they had shown that large-type Artin groups are systolic (see Chapter 2 for a definition of these groups). Metrically systolic complexes were introduced in [57] in order to obtain similar results (see Proposition 1.4.2) for two-dimensional Artin groups (which contain large-type Artin groups).

The idea of defining more lenient geometries to obtain more general results is a common theme in this thesis. Chapters 3 and 4 follow this line of action in different directions.

Let X be a flag simplicial complex whose 2-skeleton $X^{(2)}$ is an M_κ-simplicial complex with finite shapes. We call these complexes *metric simplicial complexes*. In this context, the link of a vertex will be the link in $X^{(2)}$ with the angular metric.

Let $k \geq 4$. A simple cycle σ with k edges in a simplicial complex is 2-full if there is no edge connecting any two vertices in σ having a common neighbor in σ.

Definition 1.4.1. The link of a vertex in a metric simplicial complex is 2π-large if every 2-full simple cycle in the link has angular length at least 2π. A metric simplicial complex X is locally 2π-large if the links of all of its vertices are 2π-large. A simply connected locally 2π-large metric complex is a *metrically systolic complex*. *Metrically systolic groups* are groups acting geometrically by isometries on metrically systolic complexes.

Notice that systolic complexes are metrically systolic when their 2-skeleton is endowed with the metric where every triangle is an Euclidean equilateral triangle of side length 1. Though certainly more flexible, metrically systolic groups and complexes do not have properties as strong as those in the systolic case. However, they are still very well behaved and show features of non-positive curvature.

Proposition 1.4.2 ([57]). *Let X be a metrically systolic complex and G a metrically systolic group. Then:*

- *every cycle in $X^{(1)}$ can be filled with a simplicial map from a simplicial CAT(0) disk;*
- *as a direct consequence, G has quadratic Dehn function;*
- *finitely presented subgroups of G are metrically systolic;*
- *if G is torsion-free and for every $g \in G$ g^m is conjugated to g^n only when $m = n$, then G has solvable conjugacy problem.*

In [57], Huang and Osajda pose many questions regarding metrically systolic complexes and groups. Among them, they ask if metrically systolic complexes are contractible and if metrically systolic groups are biautomatic. They also ask whether there is a fixed point theorem for finite groups acting on metrically systolic complexes, analogous to Theorem 1.3.4. In Chapter 4 we introduce another generalization of systolicity, which we call systolicity-by-function. We believe that the answer to these questions is affirmative for systolic-by-function complexes and groups. As we will see in Chapter 4, many systolic-by-function complexes are metrically systolic.

1.5 Small cancellation theory

Small cancellation theory studies group presentations in terms of the overlap of their relators (i.e. their cancellations). It is stated in purely algebraic terms, but it has a rich underlying geometric interpretation that captures many of the ideas that would later become standard in geometric group theory. Its origin goes back to the work of Max Dehn [36] in 1912, where he solved the word problem for fundamental groups of closed orientable surfaces of genus at least two. During the second half of the previous century the classical small cancellation theory was developed. A standard and comprehensive reference for this theory is in the book by Lyndon and Shupp [68].

We assume we are working with presentations $\langle S \mid R\rangle$ where relators are cyclically reduced and no relator is a cyclic permutation of another relator or of the inverse of another relator. Let R^* be the set of all cyclic permutations of the elements of R and their inverses. A word in $F(S)$ is a *piece* if it is a common prefix of two different elements in R^*. Given a word $w \in F(S)$ we note its number of letters by $|w|$. Now we state the classical small cancellation conditions.

Definition 1.5.1. Let $P = \langle S \mid R\rangle$ be a presentation, $p, q \in \mathbb{N}_{\geq 3}$ and $0 < \lambda < 1$. Then:

- P satisfies condition $C'(\lambda)$ if for every piece u, if u is a subword of some $r \in R^*$, then $|u| < \lambda |r|$;

- P satisfies condition $C(p)$ if no element of R^* can be written as the product of less than p pieces;

- P satisfies condition $T(q)$ if whenever $3 \leq l \leq q$ and $r_1, \ldots, r_l$ in R^* are such that $r_1 \neq r_2^{-1}, \ldots, r_l \neq r_1^{-1}$ then at least one of the products $r_1r_2, \ldots, r_{l-1}r_l, r_lr_1$ is freely reduced.

Condition $C'(\lambda)$ is called the metric small cancellation condition, and the other two are the non-metric conditions. Pieces represent the possible cancellation between relators and conditions $C'(\lambda)$ and $C(p)$ are telling us that these cancellations are small relative to the length of the relators. Notice that $C'(\frac{1}{n})$ implies $C(n+1)$. Condition $T(q)$ is not as intuitive, but it has a very clear interpretation in terms of diagrams.

We will give a geometric treatment of small cancellation theory in terms of diagrams first introduced by van Kampen in 1933 [62] and later rediscovered by Lyndon in 1966 [67]. A cellular map $f : L \to K$ between CW-complexes is *combinatorial* if its restriction to each open cell of L is a homeomorphism onto a cell of K, and a *combinatorial* 2*-complex* is a CW-complex for which the attaching map $\psi : S^1 \to K^{(1)}$ of each 2-cell is combinatorial after a suitable subdivision of S^1 (see [41, 88]). Let K be a combinatorial 2-complex. A diagram Δ in K is a combinatorial map $\varphi : M \to K$ where M is a combinatorial structure on the sphere to which, perhaps, we remove some open 2-cells. This includes spherical diagrams (when M is the whole sphere), (singular) disk diagrams (when M is a sphere with one 2-cell removed), and annular diagrams (when M is a sphere with two 2-cells removed). As usual, 0-cells, 1-cells and 2-cells are called respectively vertices, edges and faces.

Let $P = \langle S \mid R \rangle$ be a presentation of a group G. Its presentation complex K_P is a combinatorial 2-complex consisting of a bouquet of $|S|$ oriented circles (one for each generator in S) and a 2-cell for each relation in R attached along the corresponding word. Its fundamental group is G, and the 1-skeleton of its universal cover is the Cayley graph $\Gamma(G, S)$. A diagram over P is a diagram $\varphi : M \to K_P$ where K_P is the standard 2-complex associated to the presentation P. Since M is orientable we can fix an orientation in the usual way, so that when traversing the boundaries of the 2-cells the edges in the intersection of two faces f, f' are traversed twice, once in each possible orientation. The map $\varphi : M \to K_P$ induces a labeling on the edges of M by elements of S and their inverses. The label on the boundary of any oriented face of the diagram (starting at any vertex) is called a boundary label. Note that boundary labels are elements in R^*.

A diagram Δ is *reducible* if it contains two faces f, f' such that the intersection of their boundaries $\partial f \cap \partial f'$ contains an edge such that the boundary labels of f and f' read with opposite orientations and starting at a vertex of this edge coincide, otherwise Δ is called *reduced* (see [68, Chapter V, Section 2] for more details). The degree $d(v)$ of a vertex v in a diagram Δ is the number of edges incident to v (the edges with both boundary vertices at v are counted twice). A vertex v is called interior if $v \notin \partial M$.

Given a reduced diagram Δ over a presentation P, we can remove all interior vertices of degree 2 and label the new edges with the corresponding words. Observe that in this new diagram, labels of the interior edges correspond to *pieces* in R^*. The length $\ell(e)$ of an interior edge e in this new diagram is defined as the length of the corresponding word (equivalently, it is the number of edges of the original diagram that were glued together to obtain e).

With these diagrams, we can give new meaning to the small cancellation conditions. For a proof of the following affirmations, see [68, Chapter V, Section 2]. If $P = \langle S \mid R \rangle$ is a presentation and λ, p, q are as before, then

- P satisfies condition $C'(\lambda)$ if and only if for every reduced diagram over P the length of every interior edge is less than λ times the length of any of the relators labeling a face that contains that edge;

- P satisfies condition $C(p)$ if and only if faces with no edges in the boundary in reduced diagrams over P have at least p sides;

- P satisfies condition $T(q)$ if and only if interior vertices in reduced diagrams over P have degree at least q.

We started by saying that the origin of small cancellation theory goes back to a solution by Dehn of the word problem for fundamental groups of closed orientable surfaces of genus at least two. The following theorem, known as the van Kampen lemma gives a close relationship between disk diagrams and trivial words.

Theorem 1.5.2 (van Kampen's lemma [68]). *Let $P = \langle S \mid R \rangle$ be a presentation of a group G. Then a word $w \in F(S)$ is trivial in G if and only if there exists a reduced disk diagram over P having the word w as the label of its boundary path.*

We call such a disk diagram a *diagram for w*. The term "area" for the minimal number of relators needed to write a trivial word introduced previously now becomes much clearer. It is exactly the minimal number of faces of a diagram for w. Likewise the use of "isoperimetric inequality" to talk about Dehn functions becomes more transparent. The Dehn function bounds the number of faces of minimal disk diagrams in terms of their perimeter.

The question now is how the small cancellation conditions come into play. Due to our previous observations, small cancellation conditions determine the geometry of these disk diagrams. Therefore, when the diagrams curvature is non-positive, we will obtain a controlled isoperimetric function (and hence Dehn function). The classic conditions are $C(6)$, $C(4)-T(4)$ and $C(3)-T(6)$ (notice that every presentation is $T(3)$). This examples correspond intuitively to the regular tessellations of the plane by hexagons, squares and triangles respectively. For example, conditions $C(4) - T(4)$ mean that faces without edges in the boundary in diagrams have at least 4 sides, and that interior vertices have degree at least 4. In Chapter 3 we will interpret (generalizations of) these conditions by introducing combinatorial curvature and a combinatorial version of the Gauss–Bonnet theorem.

Now we state some of the standard results in small cancellation theory.

Theorem 1.5.3 ([68]). *Let $P = \langle S \mid R \rangle$ be a presentation of a group G. If P is $C(6)$, $C(4) - T(4)$ or $C(3) - T(6)$ then:*

- *if P is finite, then P has a quadratic Dehn function and the conjugacy problem is solvable;*
- *K_P is aspherical.*

Theorem 1.5.4 ([68]). *If a finite presentation P of a group G is $C'(\frac{1}{6})$, $C'(\frac{1}{4}) - T(4)$ or $C'(\frac{1}{3}) - T(6)$ then G is hyperbolic.*

Theorem 1.5.5 ([87]). *Let P be a finite presentation of a group G. If P is $C(6)$ then G is systolic.*

Theorem 1.5.6 (Greendlinger's lemma [49, 68]). *Let $P = \langle S \mid R \rangle$ be a presentation of a group G satisfying condition $C'(\lambda)$ with $1 < \lambda \leq \frac{1}{6}$. Let $w \in F(S)$ be a nontrivial freely reduced word that represents the trivial element in G. Then there exist a subword v of w and a word $r \in R^*$ such that v is also a subword of r and $|v| > (1 - 3\lambda)|r|$.*

Chapter 2

Artin groups

Artin groups are one of the most studied families of groups in geometric group theory. In this section we will give some basic definitions, results, and present open questions regarding Artin groups. They will be our main object of study during the second half of this thesis. Before introducing them, we need to present Coxeter groups. Though we are not going to study Coxeter groups in this thesis, they are closely intertwined with Artin groups.

2.1 Coxeter groups

Coxeter groups were introduced by Coxeter in [27], and developed in great depth by Tits in [84], and in Bourbaki's book [12]. They are formal groups of reflections that generalize finite Euclidean reflection groups. Coxeter groups are prevalent in many areas of mathematics, such as Lie groups and algebras theory, representation theory, combinatorics and geometric group theory.

Coxeter groups can be defined in terms of a labeled graph. Let Γ be a finite simplicial graph. We denote by $V(\Gamma)$ its set of vertices and by $E(\Gamma)$ its set of edges. We endow $E(\Gamma)$ with a labeling $m : E(\Gamma) \to \mathbb{N}_{\geq 2}$ and we take an abstract set $S = \{s_x \mid x \in V(\Gamma)\}$ in one-to-one correspondence with $V(\Gamma)$. We note $m_{xy} = m_{yx} = m(\{x,y\})$ and say that $m_{xy} = \infty$ if x and y are not connected in Γ. Then the *Coxeter group* W_Γ of Γ is defined by the presentation

$$\langle S \mid (s_x s_y)^{m_{xy}} \text{ for } e = \{x,y\} \in E(\Gamma), s_x^2 \text{ for } x \in V(\Gamma)\rangle.$$

When the graph Γ is clear form context we may denote the Coxeter group by W_S.

Examples of Coxeter groups are the dihedral groups and finite symmetric groups. Coxeter groups may be finite or infinite, and finite Coxeter groups are classified in terms of their defining graphs (see [12]).

Remark 2.1.1. *The labeling of the graph we have described is most commonly used when studying Artin groups. The convention for the labeling when working with Coxeter groups is usually different. Most times vertices x and y are not connected if $m_{xy} = 2$, and if $m_{xy} = \infty$ they are connected by an edge labeled by infinity.*

A Coxeter group W_Γ acts on $\mathbb{C}^n$ for some $n \in \mathbb{N}$ as a reflection group. This action preserves an open cone known as the Tits-cone, where W_Γ acts properly. This cone is delimited by the reflecting hyperplanes. The complement of these hyperplanes is the set of regular points of the cone (those with trivial isotropy). Let $\mathcal{H}_{W_\Gamma}$ be this hyperplane complement. Then $\mathcal{H}_{W_\Gamma}/W_\Gamma$ has fundamental group given by the following presentation

$$P_\Gamma = \langle \Sigma \mid \underbrace{\sigma_x\sigma_y\sigma_x\cdots}_{m_{xy}\text{ letters}} = \underbrace{\sigma_y\sigma_x\sigma_y\cdots}_{m_{xy}\text{ letters}} \text{ for } e = \{x,y\} \in E(\Gamma)\rangle,$$

where $\Sigma = \{\sigma_x \mid x \in V(\Gamma)\}$ is an abstract set in one-to-one correspondence with $V(\Gamma)$ (see van der Lek [65]). The group presented by P_Γ is the *Artin group* A_Γ of Γ. As with Coxeter groups, we may note it by A_Σ if the graph is clear form context. One of the main open problems regarding Artin groups is whether $\mathcal{H}_{W_\Gamma}/W_\Gamma$ is a $K(A_\Gamma, 1)$-space (i.e. a space with fundamental group A_Γ and all other homotopy groups trivial). This is usually known as the $K(\pi, 1)$ conjecture. Since this connection was made, Artin groups have been intensively studied.

Another source of motivation to study Artin groups is that they are a natural and vast generalization of braid groups. In fact, they are named Artin groups after the work of Emil Artin on braid groups and their presentations [4]. Apart from braid groups, basic examples of Artin groups are free groups (when there are no edges in the graph) and free abelian groups (when the graph is complete and every label is a 2).

In Chapter 5 we are going to go more into detail on Coxeter groups and some of their properties. We will use them and their close connection to Artin groups as an auxiliary tool to prove facts about Artin groups. For a modern treatment of Coxeter groups from a geometric and combinatorial group theory point of view, we recommend Davis' book [34].

2.2 Families of Artin groups and open problems

Unlike Coxeter groups, which are very well understood, Artin groups have a more mysterious nature. There are hardly any results known for all Artin groups. However, some particular families have proved to be more lenient and easy to study. Here we present the most relevant ones. An Artin group A_Γ is said to be

- *right-angled* if all the edges in Γ are labeled by 2;
- *spherical* if the corresponding Coxeter group W_Γ is finite;
- *FC-type* if the Artin groups corresponding to all clique subgraphs of Γ are spherical;
- *large-type* if every label in Γ is greater than or equal to 3;
- *two-dimensional* if it has geometric dimension at most 2. By results of Charney and Davis [19, 20] an Artin group is two-dimensional if and only if for every triangle in the graph Γ with edges labeled by p, q and r we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$.

Example 2.2.1. The deceptively simple graph in Figure 2.1 defines an Artin group which is not in any of the families above. Almost nothing is known about such Artin groups.

There are many open problems related to Artin groups, and each of these families has numerous properties. Here we will limit ourselves to a small fraction of all the body of work on Artin groups. We state some problems and questions for Artin groups, and say for which of the previous families

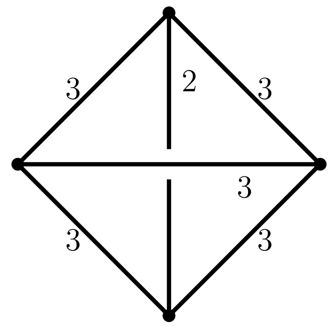

Figure 2.1: A graph for a not well understood Artin group.

the answer is known. We also give references to some of the solutions of the mentioned problems. Our lists are far from exhaustive since the literature is extensive. None of the following is known for all Artin groups:

- The $K(\pi, 1)$ conjecture: it is known to be true for all the families mentioned above. See [16, 17, 19, 20, 37, 54, 72] for proofs of various instances of this conjecture. Charney and Davis have shown that Artin groups for which the $K(\pi, 1)$ conjecture holds admit a finite classifying space [19]. In particular this would imply that they are torsion-free. As simple as it may sound, torsion-freeness has still not been proved for all Artin groups.

- Are Artin groups CAT(0)? They are conjectured to be CAT(0) and are known to be CAT(0) in few cases. Most notably, right-angled Artin groups are CAT(0) [19]. There are some other known examples, but no other family of the ones presented above is known to be CAT(0).

- Word problem and conjugacy problem: the word problem is known to be solvable for all the families described and more (see [1, 4, 7, 16, 25, 37]), and the conjugacy problem for all families mentioned (see [3, 16, 18, 28, 37, 57]).

- Are Artin groups biautomatic? Biautomaticity has been established for some families of Artin groups, including FC-type and large-type Artin groups (see [13, 18, 44, 55, 58, 75]).

2.3 Parabolic subgroups

A key tool in the study of Artin groups are their parabolic subgroups. Let Γ be a labeled graph defining an Artin group and let X be a subset of $V(\Gamma)$. We denote by Γ_X the full subgraph of Γ spanned by X with the induced labeling. We set $\Sigma_X = \{\sigma_x \mid x \in X\}$ and we denote by A_X the subgroup of A generated by Σ_X. A remarkable result by van der Lek [65] tells us that A_X is naturally isomorphic to A_{Γ_X}, hence we will not differentiate A_X from A_{Γ_X}. The subgroup A_X is called a *standard parabolic subgroup* of A_Γ and a subgroup conjugate to A_X is called a *parabolic subgroup* of A_Γ.

The fact that parabolic subgroups of Artin groups are themselves Artin groups makes them play a central role. Most of the proofs of the problems and questions mentioned in the previous sections rely on the parabolics of Artin groups. They can be thought of as smaller "building blocks" and used to do induction-like proofs. Their structure is also used in many geometric constructions such as the Deligne complex, the Artin complex, the Salvetti complex, the clique-cube complex and more (see [19, 20, 21, 31, 78]).

Being so important, understanding their algebraic structure becomes a natural thing to do. One of the problems that has attracted more interest in recent years is whether they are stable under intersection. More precisely, is the intersection of two parabolic subgroups a parabolic subgroup? In [69] Möller, Paris and Varghese show that if this is the case, then any arbitrary intersection of parabolic subgroups of an Artin group A_Γ is a parabolic subgroup of A_Γ. As a direct corollary one obtains that for any subset $B \subseteq A_\Gamma$ there exists a unique minimal (with respect to the inclusion) parabolic subgroup containing B. This is usually called the *parabolic closure* of B.

The answer to this question was already known to be affirmative for the intersection of standard parabolic subgroups by van der Lek [65] and in the case of braid groups. A braid group on n strands can be thought of as the mapping class group of a punctured disk D_n with n punctures. Its parabolic subgroups are in bijection with isotopy classes of nondegenerate, simple closed multicurves in D_n. The complement of each of these multicurves is a disjoint union of punctured disks in D_n. In Farb and Margalit [39] an intersection between these families of punctured disks is defined. This intersection corresponds, via the bijection, to the intersection between parabolic subgroups, and can be used to give an affirmative answer to the question. The analogous question was also known to be true for all Coxeter groups (see Solomon [81]).

All of the previous results motivated the study of this question for general Artin groups. It has been proved for graph products of groups (they generalize right-angled Artin groups, see Antolin and Minasyan [2]), and in particular for right-angled Artin groups (see Duncan, Kazachkov and Remeslennikov [38]). More recently, Cumplido, Gebhardt, González-Meneses and Wiest [30] generalized the case of braid groups to Artin groups of spherical type using Garside theory. Combining this previous result with the structure of the Deligne complex, Morris-Wright [70] showed that the intersection of two parabolic subgroups of spherical type inside an FC-type Artin group is a parabolic subgroup of spherical type. This last result has been extended by Möller, Paris and Varghese [69], who showed that if the intersection of parabolic subgroups corresponding to subgraphs of a clique subgraph of Γ is a parabolic subgroup, then the intersection of a parabolic subgroup with a parabolic subgroup corresponding to a clique subgraph is a parabolic subgroup.

In [31] Cumplido, Martin and Vaskou used a geometric approach to solve this problem for Artin groups of large-type. They introduced a simplicial complex associated to an Artin group, called the Artin complex, on which the Artin group acts cocompactly and without inversions. It turns out this complex is systolic if the Artin group is large-type. Using geometric properties of systolic complexes they gave a positive answer to the question. In Chapter 4 we will introduce systolic-by-function complexes and use their geometry to answer the question in the (2,2)-free two-dimensional case ((2,2)-free Artin groups are those whose defining graph does not have two consecutive edges labeled by 2).

A question related to the above is whether a parabolic subgroup P_1 of an Artin group A_Γ contained in another parabolic subgroup P_2 is a parabolic subgroup of P_2. More precisely, if P_1 and P_2 are parabolic subgroups of an Artin group A_Γ such that $P_1 \subseteq P_2$, we say that P_1 is a parabolic subgroup of P_2 if they are conjugate to standard parabolic subgroups in an inclusion $A_{X_1} \subseteq A_{X_2}$. An Artin group satisfying this property is called *standardisable* in [29]. This result is a preliminary to the above question, and it was a question posed by Godelle [47, Conjecture 2]. Additionally, it is a central step towards solving the conjugacy stability problem for Artin groups (see Cumplido [29]).

The question seems obvious but is not. It is also related to the study of normalizers and centralizers of parabolic subgroups. In Chapter 5 we prove this conjecture. We state the theorem more accurately.

Theorem 2.3.1. *Let Γ be a finite simplicial graph, let $m : E(\Gamma) \to \mathbb{N}_{\geq 2}$ be a labeling, and let $A = A_\Gamma$ be the Artin group of Γ. Let $X, Y \subset V(\Gamma)$ and $\alpha \in A$ such that $\alpha A_Y \alpha^{-1} \subset A_X$. Then there exist $Y' \subset X$ and $\gamma \in A_X$ such that $\alpha A_Y \alpha^{-1} = \gamma A_{Y'} \gamma^{-1}$.*

Corollary 2.3.2. *Let $P_1 \subseteq \cdots \subseteq P_n$ be a chain of parabolic subgroups of an Artin group A_Γ. Then there exist $X_1 \subseteq \cdots \subseteq X_n \subseteq V(\Gamma)$ and $g \in A_\Gamma$ such that $P_i = g A_{X_i} g^{-1}$ for all $1 \leq i \leq n$.*

Chapter 3

Strictly systolic angled complexes and generalized small cancellation

The first part of this chapter concerns the contents of [9] (joint work with G. Minian). We introduce the notion of strictly systolic angled complexes. They generalize 7-systolic complexes [61] and their metric counterparts, which appear as natural analogues to Huang and Osajda's metrically systolic simplicial complexes [57] in the context of negative curvature (see Chapter 1 for definitions). We prove that strictly systolic angled complexes and the groups that act on them geometrically, together with their finitely presented subgroups, are hyperbolic. Finally, we find a small cancellation condition for one-relator groups without torsion that ensures that they act geometrically on a strictly systolic angled complex (and hence are hyperbolic).

In the second part of this chapter we cover the contents of [10] (joint work with G. Minian and I. Sadofschi Costa). We present a metric condition $\mathcal{T}'$ which describes the geometry of classical small cancellation groups and applies also to other known classes of groups such as two-dimensional Artin groups. We prove that presentations satisfying condition $\mathcal{T}'$ are diagrammatically reducible in the sense of Sieradski and Gersten. In particular we deduce that the standard presentation of an Artin group is aspherical if and only if it is diagrammatically reducible. We show that, under some extra hypotheses, $\mathcal{T}'$-

groups have quadratic Dehn functions and solvable conjugacy problem. In the spirit of Greendlinger's lemma, we prove that if a presentation $P = \langle X \mid R\rangle$ of group G satisfies conditions $\mathcal{T}' - C'(\frac{1}{2})$, the length of any nontrivial word in the free group generated by X representing the trivial element in G is at least that of the shortest relator. We also introduce a strict metric condition $\mathcal{T}'_<$, which implies hyperbolicity and expands upon the work of the beginning of the chapter. These two conditions arose as natural generalizations of condition (T'), with the intent of studying it in combinatorial terms.

3.1 Strictly systolic angled complexes

Essentially, a strictly systolic angled complex is a combinatorial complex whose cells are simplices, such that the 2-skeleton is a nonnegative angled 2-complex in the sense of Wise [88] and with a link condition similar to Huang and Osajda's 2π-large condition for metrically systolic simplicial complexes [57]. This new notion is flexible enough to include objects of combinatorial nature, such as Januszkiewicz and Świątkowski's 7-systolic simplicial complexes [61], and also of geometric nature, such as a variation, for negative curvature, of Huang and Osajda's metrically systolic simplicial complexes.

A *quasi-simplicial complex* is a combinatorial complex X whose closed cells are simplices and such that different 2-simplices do not have two or more edges in common. Note that quasi-simplicial complexes do not have loops since the closed cells are simplices. This notion is less rigid than that of a simplicial complex, since we admit multiple edges between vertices. All the complexes that we deal with in this chapter are assumed to be locally finite. We say that a quasi-simplicial complex X is *3-flag* if every time X has three faces of a tetrahedron, then the whole tetrahedron is in X (and in particular, the fourth face is in X). Note that a flag simplicial complex is, in particular, a 3-flag quasi-simplicial complex.

Similarly as in [41, 88], we define a *weight* function ω on the *corners* of the 2-simplices of X. Given a vertex $v \in X$ we denote by $\mathrm{lk}_X(v)$ the *geometric link* of v in the 2-skeleton $X^{(2)}$. Recall that $\mathrm{lk}_X(v)$ is the graph corresponding to an epsilon sphere about the vertex v in $X^{(2)}$. The corners of the 2-simplices correspond to the edges of the link, and the function ω assigns a nonnegative value to each edge in $\mathrm{lk}_X(v)$ for every vertex $v \in X$ (i.e. the 2-skeleton $X^{(2)}$ is a nonnegative angled 2-complex in the sense of [88]). We require that the image of ω is finite and that ω satisfies a weak triangle inequality: for any vertex v, if α_{ij} is an edge in $\mathrm{lk}_X(v)$ from v_i to v_j then $\omega(\alpha_{13}) \leq \omega(\alpha_{12}) + \omega(\alpha_{23})$. The complex X together with a fixed weight function is called an *angled complex*.

Following Huang and Osajda [57], we say that a simple cycle σ of length greater than 3 in the link of a vertex $v \in X$ is *2-full* if there is no edge in $\mathrm{lk}_X(v)$ that connects two vertices having a common neighbor in σ. The angular length of a path in the link of a vertex in an angled complex is the sum of the weights of its edges, counted with multiplicity. An angled complex X is *locally 2π-large* if every 2-full cycle in every vertex link has angular length greater than or equal to 2π.

Definition 3.1.1. A simply connected, locally 2π-large, 3-flag, angled complex in which the sum of the internal weights of each triangle is (strictly) less than π is called a *strictly systolic angled complex*.

We will show that strictly systolic angled complexes satisfy a linear isoperimetric inequality. As a consequence, groups acting geometrically on strictly systolic angled complexes are hyperbolic. The proof will follow ideas of Gersten [41], Huck and Rosebrock [59], Wise [88], Januszkiewicz and Świątkowski [61], and Huang and Osajda [57].

In analogy with diagrams in small cancellation theory, we work with diagrams in this context. A *singular disk* is a simply connected and planar combinatorial 2-complex whose cells are simplices. We call a map simplicial if it takes simplices onto simplices (not necessarily of the same dimension). Explicitly, a map is simplicial if it maps vertices to vertices, whenever vertices span a simplex then their images do so, and the restriction of the map to each simplex is linear. Let X be a strictly systolic angled complex, and let $\gamma : S \to X^{(1)}$ be a combinatorial map from a triangulation of S^1 to the 1-skeleton of X. Its image is a closed edge-path in X which we will also denote by γ. A *singular diagram* for γ is a simplicial map $f : D \to X$ from a singular disk such that $f|_{\partial D} = \gamma$.

Note that since X is simply connected, every closed edge-path in X admits a singular diagram. This is a direct consequence of the relative simplicial approximation theorem (see [90]).

We define the area of a closed edge-path $\gamma : S \to X$ as

$$\mathrm{Area}(\gamma) = \min\{|D| : f : D \to X \text{ is a singular diagram for } \gamma\},$$

where $|D|$ denotes the number of faces (2-simplices) in the corresponding singular disk D. The length $l(\gamma)$ of a closed edge-path $\gamma : S \to X$ is the number of edges in S. Our aim is to show that there exists a constant $K > 0$ such that $\mathrm{Area}(\gamma) \leq Kl(\gamma)$ for every closed edge-path γ. To prove this we will need well behaved diagrams. A singular diagram is said to be *nondegenerate* if it is injective on every simplex.

Lemma 3.1.2 ([61], Lemma 1.6)**.** *Let γ be a homotopically trivial closed edge-path in a quasi-simplicial complex X. Then there exists a nondegenerate singular diagram for γ.*

Januszkiewicz and Świątkowski proved this lemma in the case where X is a simplicial complex and γ is a simple path, but the proof holds for γ an arbitrary closed edge-path as defined above. We will use it in its original fashion in Chapter 4. In the first part of their proof it is shown that a singular diagram, which they call *almost simplicial*, can be modified to be nondegenerate. The proof relies on the fact that the map from the disk to the complex is simplicial. The same proof works when X is a quasi-simplicial complex, since the defining maps of our singular diagrams are also simplicial (in the sense that they take simplices onto simplices). In addition, they showed that, in the simplicial case, nondegenerate diagrams can be modified further to become simplicial. In our case, since X is quasi-simplicial, we cannot modify nondegenerate diagrams to simplicial ones.

By Lemma 3.1.2 we may assume that the diagrams $f : D \to X$ are nondegenerate. We now adopt terminology from Huck and Rosebrock [59]. Given a singular diagram $f : D \to X$, we say it is *vertex reduced* if it maps the link of every vertex of D to a path in the link of a vertex in X in which no edge is passed twice in opposite directions. Suppose we have a non vertex reduced diagram $f : D \to X$, with X a strictly systolic angled complex. If we are in the situation where a troublesome link has two edges, the diagram locally looks like in Figure 3.1. We can identify the edges e_1 and e_2 and collapse the corresponding faces to obtain a new diagram. We call this move an *edge reduction* (see Figure 3.1).

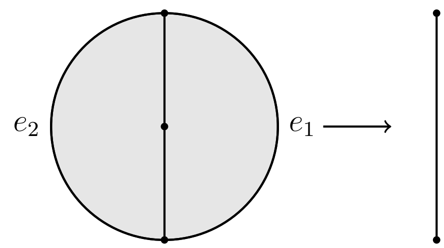

Figure 3.1: Edge reduction.

If the link has more than two edges, we can apply a sequence of diamond moves as in [26, 59], followed by an edge reduction. This is shown in Figure 3.2. Note that, if the diagram is not vertex reduced, there is a vertex v and two triangles in D which are incident to v that are mapped to the same triangle in X. In particular there are two pairs of edges such that each pair is mapped to a single edge. One of these pairs is shown if Figure 3.2. For a detailed exposition on diamond moves see [26].

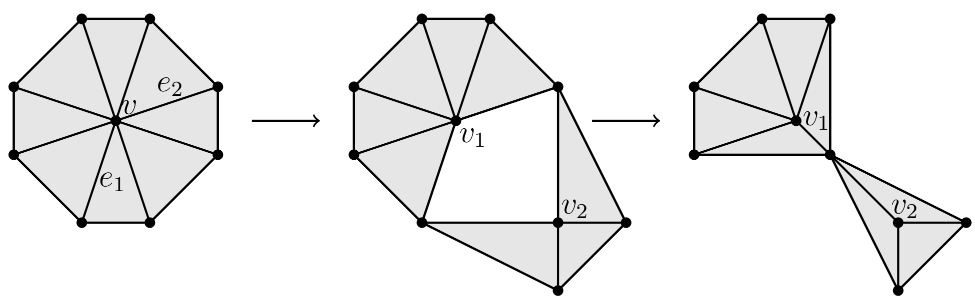

Figure 3.2: Diamond move. Edges e_1 and e_2 are cut along and then glued in a different fashion.

Note that these moves reduce the number of faces of the diagram. Therefore, starting with a nondegenerate singular diagram, we can obtain a nondegenerate and vertex reduced singular diagram with the same boundary. Let $f : D \to X$ be a nondegenerate vertex reduced diagram and v an interior vertex of D. The link of v is a graph and can be decomposed as $\mathrm{lk}_D(v) = \cup_{i=1}^n C_i$, where each C_i is a union of edges, the pairwise intersections of the C_i are empty or contain only vertices, and $f(C_i)$ is a simple cycle in $\mathrm{lk}_X(f(v))$ for every $1 \leq i \leq n$ (see [59, 2.2]). In this case, we say that $f(\mathrm{lk}_D(v))$ admits a *decomposition in simple cycles.* The previous discussion can be summarized in the following lemma.

Lemma 3.1.3. *Let $f : D \to X$ be a nondegenerate singular diagram for a closed edge path γ in a quasi-simplicial complex X. Then it can be modified to obtain a vertex reduced nondegenerate singular diagram for γ. Furthermore, the image of the link of every interior vertex admits a decomposition in simple cycles.*

Lemma 3.1.4. *Let X be a strictly systolic angled complex, v a vertex of X and σ a simple cycle in $\mathrm{lk}_X(v)$. Then $\sum_{c\in\sigma}\omega(c) \geq 2\pi$ or σ is the boundary of a triangulated disk without interior vertices, whose edges map to the simplicial link of v in X.*

Proof. We proceed by induction on the length of σ. There are no cycles with two edges, because X is quasi-simplicial. If it has three edges, then σ is a triangle and it is filled because X is 3-flag, so the claim holds. Suppose σ has more than three edges. If it is 2-full, then it has angular length greater than or equal to 2π. If it is not 2-full, then there exists an edge e in $\mathrm{lk}_X(v)$ that connects two vertices of σ with a common neighbor. This edge subdivides σ in two paths: one of length 2, and another one of length $l(\sigma) - 2$, which we call σ_1 and σ_2 respectively (see Figure 3.3). By the inductive hypothesis, $\sigma_2 \cup e$ either subdivides into triangles or has angular length greater than or equal to 2π. If it subdivides, it induces a subdivision for σ. If it does not, by the triangle inequality we have

$$2\pi \leq \sum_{c\in\sigma_2\cup e}\omega(c) \leq \sum_{c\in\sigma_2\cup\sigma_1}\omega(c) = \sum_{c\in\sigma}\omega(c). \qquad \square$$

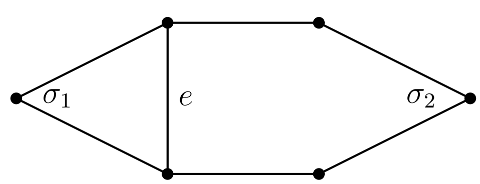

Figure 3.3: Subdividing σ.

Lemma 3.1.5. *Let X be a strictly systolic angled complex and $f : D \to X$ a nondegenerate and vertex reduced singular diagram for a closed edge-path γ. Then there exists a nondegenerate and vertex reduced singular diagram $g : \tilde{D} \to X$ for γ such that the image by g of the links of the interior vertices of $\tilde{D}$ admits a decomposition in simple cycles of angular length greater than or equal to 2π.*

Proof. Let v be an interior vertex of D. By Lemma 3.1.3, the image of its link admits a decomposition in simple cycles. Suppose that one of those simple cycles in the decomposition of $f(\mathrm{lk}_D(v))$, call it σ, has angular length less than 2π. If σ is not the only simple cycle in the decomposition of $f(\mathrm{lk}_D(v))$, then there exist two edges e_1 and e_2 incident to v satisfying $f(e_1) = f(e_2)$. Via a diamond move, we can obtain a new nondegenerate singular diagram $f' : D' \to X$ for γ with the same number of faces and new vertices v_1 and v_2 (see Figure 3.2). Therefore we can assume that σ is the only simple cycle in the decomposition of $f(\mathrm{lk}_{D'}(v_1))$. Since its angular length is less than 2π, by Lemma 3.1.4 it subdivides in triangles in $\mathrm{lk}_X(f(v_1))$. This corresponds with the situation shown in Figure 3.4. Since X is 3-flag, we can modify f by removing the troublesome vertex v_1, as shown in Figure 3.5.

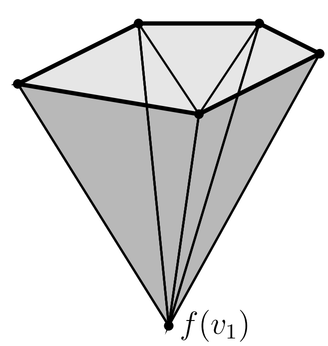

Figure 3.4: The cycle σ subdivides in $\mathrm{lk}_X(f(v_1))$.

After applying this change, we obtain a new nondegenerate singular diagram for γ with fewer faces. Then we can make it vertex reduced by reducing the number of faces once again, and continue with this process. Since the number of faces decreases at each step, the process stops and we obtain a nondegenerate and vertex reduced singular diagram $g : \tilde{D} \to X$ for γ, which

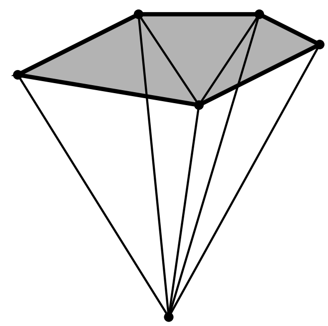

Figure 3.5: Removal of v_1.

satisfies the desired conditions. □

The preceding arguments can be made in terms of minimal diagrams. A diagram for a closed edge-path γ is *minimal* if it has the least amount of triangles. From the previous lemmas it is not hard to deduce the following.

Lemma 3.1.6. *Let X be a strictly systolic angled complex and $f : D \to X$ a minimal singular diagram for a closed edge-path γ. Then f is nondegenerate, vertex reduced and the image by f of the links of the interior vertices of $\tilde{D}$ admits a decomposition in simple cycles of angular length greater than or equal to 2π.*

In Chapter 4 we will work with minimal diagrams instead of modifying diagrams. We believe both approaches are fruitful and so decided to include both of them.

If X is a strictly systolic angled complex and $f : D \to X$ is a singular diagram with f nondegenerate, we can pull back the weights of the corners of X to D. We will apply the combinatorial Gauss–Bonnet theorem to the angled 2-complex D. As the name implies, this theorem will link the combinatorial curvature of a complex with its Euler characteristic in the same fashion as the classical theorem for Riemannian manifolds. We recall first some combinatorial notions of curvature from [88].

Definition 3.1.7. Let L be an angled 2-complex whose cells are simplices. If v is a vertex of L, the curvature of v is defined as

$$\kappa(v) = 2\pi - \pi\chi(\mathrm{lk}_L(v)) - \sum_{c \in v} \omega(c).$$

where the sum is taken over all the corners at v, and χ denotes the Euler characteristic. The curvature of a face (2-simplex) F is

$$\kappa(F) = (\sum_{c \in F} \omega(c)) - \pi,$$

where the sum is taken over all corners in F.

Theorem 3.1.8 (Combinatorial Gauss–Bonnet Theorem [5, 88])**.** *Let L be a finite angled 2-complex. Then*

$$\sum_{F \in faces(L)} \kappa(F) + \sum_{v \in L^{(0)}} \kappa(v) = 2\pi\chi(L).$$

Now we are ready to prove the linear isoperimetric inequality.

Theorem 3.1.9. *Let X be a strictly systolic angled complex. Then there exists a constant $K > 0$ such that*

$$\mathrm{Area}(\gamma) \leq Kl(\gamma),$$

for every closed edge-path γ in X.

Proof. We will find a positive constant K such that for any closed edge-path γ, there exists a singular diagram $g : D \to X$ for γ with $|D| \leq Kl(\gamma)$. Given γ, take a nondegenerate and vertex reduced singular diagram $g : D \to X$ satisfying the conditions of Lemma 3.1.5. Since g is nondegenerate, we can pull back ω to D via g. Since the sum of the internal weights of each face of X is less than π, then $\kappa(F) < 0$ for every face F of D. Furthermore, since the image of ω is finite, $\kappa(F) \leq M < 0$, where M is the maximum of the sums of triples of weights that sum up to less than π, minus π. Note that M is independent from g, D and γ, and strictly negative. The image of the link of each interior vertex admits a decomposition in simple cycles of angular length greater than or equal to 2π. Then $\kappa(v) \leq 0$ if v is an interior vertex of D. Now we apply the combinatorial Gauss–Bonnet theorem to D and we get

$$M|D| \geq \sum_{F \in \mathrm{faces}(D)} \kappa(F) = 2\pi\chi(D) - \sum_{v \in D^{(0)}} \kappa(v) = 2\pi - \sum_{v \in D^{(0)}} \kappa(v),$$

and therefore

$$|D| \leq \frac{1}{-M}(\sum_{v\in D^{(0)}} \kappa(v) - 2\pi) \leq \frac{1}{-M}(\sum_{v\in \partial D^{(0)}} \kappa(v) - 2\pi),$$

where ∂D denotes the boundary of D. Note that the number of vertices in ∂D is less than or equal to $l(\gamma)$. Since the links of the vertices in ∂D have Euler characteristic greater than or equal to 1, and since the weight is nonnegative, their curvature is at most π. Setting $K = \frac{\pi}{-M}$, we obtain $|D| \leq Kl(\gamma)$. □

In the light of Theorem 3.1.9, and by [15, III.2.9], we obtain the following corollaries.

Corollary 3.1.10. *The 1-skeleton $X^{(1)}$ of a strictly systolic angled complex X with its standard geodesic metric is hyperbolic. More generally, if we endow X with a piecewise Euclidean metric with* Shapes(X) *finite, then X is hyperbolic.*

A group Γ which acts properly and cocompactly by simplicial automorphisms on a strictly systolic angled complex, and such that the weight function is Γ-invariant is called a *strictly systolic* group. Note that, similarly as in [57, Theorem 3.1], since the class of locally 2π-large, 3-flag angled complexes is closed under taking full subcomplexes and covers, by [53, Theorem 1.1] finitely presented subgroups of strictly systolic groups are strictly systolic. From Corollary 3.1.10 we obtain the following result.

Corollary 3.1.11. *Strictly systolic groups are hyperbolic. Moreover, all finitely presented subgroup of a strictly systolic group are strictly systolic, and hence, hyperbolic.*

Note that in Corollary 3.1.10, the angles (or weights) of the strictly systolic angled complex X are independent of the metric. However, if we are given a simplicial or quasi-simplicial complex X with a piecewise Euclidean or hyperbolic metric on the 2-skeleton $X^{(2)}$, we can define a weight function for X. If X satisfies a "good enough" link condition, this weight function makes X a strictly systolic angled complex. The following definition is analogous to the notion of metrically systolic complex (see Section 1.4) introduced by Huang and Osajda [57]. It is, in some sense, the metric counterpart to the 7-systolic simplicial complexes of [61].

Definition 3.1.12. A *metrically strictly systolic* complex is a simply connected and 3-flag quasi-simplicial complex X such that its 2-skeleton is equipped either with a piecewise hyperbolic or a piecewise Euclidean metric (with finite shapes) and such that the angular distance induced in the links of the vertices satisfies the weak triangle inequality and the vertex links (with the angular distance) are 2π-large in the hyperbolic case or strictly 2π-large in the Euclidean case, i.e. every 2-full cycle has angular length greater than or equal to 2π (resp. greater than 2π).

Note that the difference between this definition and the one of metrically systolic complex given in Definition 1.4.1 is that, in our case, either the metric is piecewise hyperbolic instead of piecewise Euclidean or, in the Euclidean case, the length of the 2-full cycles is strictly greater than 2π.

Proposition 3.1.13. *Metrically strictly systolic complexes are strictly systolic angled complexes.*

Proof. In the piecewise hyperbolic case, the metric induces a weight function ω in X. The sum of the internal weights of each triangle is less than π, since the metric is piecewise hyperbolic. Then X together with ω is a strictly systolic angled complex.

In the piecewise Euclidean case, since $X^{(2)}$ has finite shapes, there exists $L > 2\pi$ such that every 2-full cycle has angular length greater than or equal to L. Then we can define an appropriate weight in the corners by subtracting a fixed small enough $\delta > 0$ from every angle. $\square$

Corollaries 3.1.10 and 3.1.11 generalize Januszkiewicz and Świątkowski's results on 7-systolic simplicial complexes and groups [61, Theorem 2.1 and Corollary 2.2] (see Section 1.3 for definitions). It follows from their definition that 2-full cycles in a 7-systolic complex have at least seven edges. Therefore, any 7-systolic complex X together with the weight function that assigns $\frac{2}{7}\pi$ to every corner, is a strictly systolic angled complex.

3.2 Application to one-relator groups

We use strictly systolic angled complexes to investigate the geometry of one-relator groups. It is well known that all one-relator groups with torsion are hyperbolic. This is an immediate corollary of Newman's Spelling Theorem. Recall that a one-relator group has torsion if and only if its relation is a proper power (see [64]). On the other hand, the geometry of one-relator groups without torsion is more intricate. Ivanov and Schupp described hyperbolicity in some classes of one-relator groups [60]. More recently Marco Linton showed that one-relator groups with negative immersions are hyperbolic [66].

As seen in Section 1.5, conditions $C'(\frac{1}{6})$ and $C'(\frac{1}{4}) - T(4)$ imply hyperbolicity (see [44, 50]). We introduce a weaker small cancellation hypothesis $C'(\frac{1}{4}) - (T')$, which generalizes both $C'(\frac{1}{6})$ and $C'(\frac{1}{4}) - T(4)$, and show that it suffices to prove hyperbolicity of one-relator groups. We include some examples that illustrate this result and compare it to the classical conditions.

Since all one-relator groups with torsion are hyperbolic, we will consider only one-relator groups without torsion. They are given by presentations $P = \langle \mathcal{A} \mid R \rangle$, where $\mathcal{A}$ is finite and R is a cyclically reduced word which is not a proper power. Note that no proper subword of R is trivial in the presented group Γ (see [86, Theorem 2]). This tells us that 2-cells in the universal cover of the presentation complex do not self-intersect.

In the case of one-relator presentations, the condition $T(4)$ can be restated as follows: the cyclically reduced word R does not contain pieces W_1, W_2, W_3 such that W_1W_2, W_1W_3 and $W_2^{-1}W_3$ are nonempty subwords of R or R^{-1} or any cyclic permutation of them. We will apply Corollary 3.1.11 to prove that $C'(\frac{1}{4})$ together with a much weaker condition than $T(4)$ guarantees hyperbolicity of one-relator groups. Contrary to condition $T(4)$, we allow the existence of pieces W_1, W_2 and W_3 such that W_1W_2, W_1W_3 and $W_2^{-1}W_3$ are nonempty subwords, but we impose a condition on their lengths. Concretely, condition $T(4)$ is replaced by the weaker *Condition* (T').

Condition (T')*:* If there exist pieces W_1, W_2, W_3 of R such that W_1W_2, W_1W_3 and $W_2^{-1}W_3$ are nonempty subwords of R or R^{-1} or any cyclic permutation of them, then $l(W_1) + l(W_2) + l(W_3) < \frac{l(R)}{2}$.

Theorem 3.2.1. *Let Γ be a one-relator group with presentation $P = \langle \mathcal{A} \mid R \rangle$. If P satisfies the metric small cancellation condition $C'(\frac{1}{4})$ and Condition (T'), then Γ is hyperbolic.*

Note that a $C'(\frac{1}{6})$ one-relation presentation automatically satisfies Condition (T'), since

$$l(W_1) + l(W_2) + l(W_3) < \frac{l(R)}{6} + \frac{l(R)}{6} + \frac{l(R)}{6} = \frac{l(R)}{2}$$

Before we proceed with the proof, we illustrate the result with an example.

Example 3.2.2. Consider the presentation

$$P = \langle a, b | a^4 b^{-1} a^{-1} b^{-1} a^4 baba^{-1} ba^{-1} b \rangle.$$

The presented group satisfies the hypotheses of the theorem and, hence, it is hyperbolic. Note that it is neither $C(7)$ (in particular, it is not $C'(\frac{1}{6})$) nor $T(4)$. Also it does not satisfy any hyperbolic weight test [41, 59] and it is not in any of the families classified by Ivanov and Schupp [60].

To prove Theorem 3.2.1, we will construct a strictly systolic angled complex X from P, on which Γ acts properly and cocompactly by simplicial automorphisms, and such that its weight function is Γ-invariant. This construction is inspired in Huang and Osajda's construction for Artin groups [57], but it is adapted to the geometry of one-relator groups.

We start with the construction of the complex X. Let $r = l(R)$. We can assume that $r \geq 4$, since Γ is free (and thus hyperbolic) when $r \leq 3$. Let K_P be the standard 2-complex associated to P. Recall that K_P has one 0-cell, one 1-cell for each generator $a \in \mathcal{A}$ and one 2-cell corresponding to the word R. We denote by $\tilde{K}_P$ its universal cover. Following the terminology of [57], the closed 2-cells of $\tilde{K}_P$ (corresponding to all the lifts of the unique 2-cell of K_P) will be called precells. Observe that, since no proper subword of R is trivial in Γ (see [86, Theorem 2]), precells are embedded in $\tilde{K}_P$. That is, their boundaries have no self-intersections. We triangulate each precell of $\tilde{K}_P$ by adding a central vertex. We will call these new vertices *central*, so as to distinguish them from the *original* vertices of $\tilde{K}_P$.

Now, if two precells intersect, they do so in a disjoint union of vertices and paths. This is because no proper subword of R is trivial in Γ. Notice that each intersection, when it is not a single vertex, amounts to a piece in R. Let C_1 and C_2 be two intersecting precells with corresponding centers c_1 and c_2. For each connected component of their intersection we add an edge between c_1 and c_2. Note that there could be more than one edge between two centers.

Let $v_1, \ldots, v_k$ be the vertices of one component of the intersection. Then for each i we also add triangles with vertices $\{c_1, c_2, v_i\}$ (one of the edges of the boundary of the triangle is the corresponding edge between the centers). We fill the necessary tetrahedra for the complex to be 3-flag (see Figure 3.6).

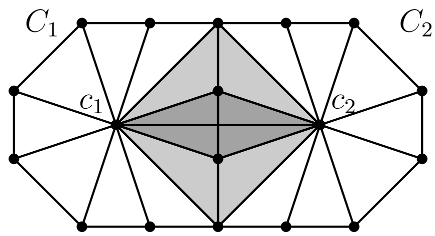

Figure 3.6: Intersection of two precells.

If three precells C_1, C_2 and C_3 intersect, we add triangles with vertices in the three centers (one for each component of $C_1 \cap C_2 \cap C_3$) and the necessary tetrahedra for the resulting complex to be 3-flag. We denote by X the complex that we obtain. Intuitively, we are filling the complex at the original vertices, removing all possible non-negative curvature from the original vertices. In turn, we now have to understand the curvature at the new central vertices. Note that X is quasi-simplicial since the presentation satisfies, in particular, the metric condition $C'(\frac{1}{2})$. It is clear that X is 3-flag by construction. Note that Γ acts simplicially, properly and cocompactly on X since the modifications that we made on $\tilde{K}_P$ are Γ-equivariant.

Lemma 3.2.3. *The complex X is simply connected.*

Proof. First note that X is obtained from (a subdivision of) $\tilde{K}_P$ by adding edges, triangles and tetrahedra. Let e be an edge in X that is not in $\tilde{K}_P$. Then it connects the central vertices of two different precells. By construction, there is a triangle or a tetrahedron by which e homotopes to a path in $\tilde{K}_P$ with the same endpoints. Therefore, since $\tilde{K}_P$ is simply connected, so is X. □

Now we define a weight function ω on X and study under which conditions (X, w) is a strictly systolic angled complex. We will show below that, by construction of X, the links of the original vertices of X do not have 2-full cycles. Therefore we only need to control the weights of the corners at the central vertices and the sum of the internal angles of the triangles. There are three kinds of triangles in X:

1. triangles coming from the subdivision of a precell,
2. triangles with two central vertices,
3. triangles with three central vertices.

Recall that r denotes the length of the relator R. If the triangle is of type (1), then the weight assigned to the angle of the central vertex is $\frac{2\pi}{r}$, and the weights of the other two are 0.

If the triangle is of type (2), the weight of the two central angles is $\frac{l}{r}\pi$, where l is the length of the component of the intersection corresponding to that triangle. The remaining angle equals 0.

If the triangle is of type (3), it corresponds to the intersection of three precells C_1, C_2 and C_3 with centers c_1, c_2 and c_3. Let l_{12} be the length of the component of the intersection between C_1 and C_2. Analogously we define l_{13}, l_{23} and l_{123} (the length of the component of $C_1 \cap C_2 \cap C_3$). Observe that if $l_{123} \neq 0$, then it is equal to the minimum of l_{12}, l_{13} and l_{23}. The weights at the angles at c_1, c_2 and c_3 are $\frac{1}{r}(l_{12}+l_{13}-2l_{123})\pi$, $\frac{1}{r}(l_{12}+l_{23}-2l_{123})\pi$ and $\frac{1}{r}(l_{13}+l_{23}-2l_{123})\pi$ respectively. The sum of the angles is $\frac{1}{r}(2l_{12} + 2l_{13} + 2l_{23} - 6l_{123})\pi$, which is less than π if the presentation is $C'(\frac{1}{4})$ and if, whenever the intersection of the three cells is a vertex, $l_{12} + l_{13} + l_{23} < \frac{r}{2}$, which is Condition (T').

Under these circumstances, all the triangles have inner weight less than π, and the triangle inequality is easily seen to be satisfied. The weight function ω is also easily seen to be Γ-invariant.

Lemma 3.2.4. *The angled complex X with the weight function ω is locally 2π-large.*

Proof. We analyze the links of the vertices in X and their 2-full cycles. There are two types of vertices in X. The original vertices (vertices of $\tilde{K}_P$) and the central vertices. Note that the links of any two vertices of the same kind are isomorphic, so we need to verify only two cases.

First we study the links of central vertices (see Figure 3.7). Let c be a central vertex in a precell C. Its link has two kinds of vertices:

(i) those coming from edges of the triangulation of C,

(ii) those corresponding to edges between c and another central vertex.

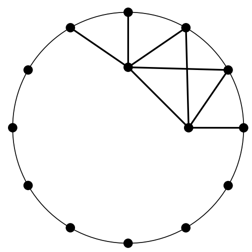

Figure 3.7: Link of a central vertex. Vertices of type (i) are located in the exterior circle, and vertices of type (ii) in the interior.

Let σ be a 2-full cycle in $\mathrm{lk}(c)$. We will show that its angular length is greater than or equal to 2π. We examine the possible cases.

Case 1 If σ only passes through vertices of type (i), then σ is a circle of angular length 2π.

Case 2 Suppose σ only passes through vertices of type (ii). Those vertices correspond to paths in the boundary of the precell. Note that these paths are connected components of the intersections of this precell with other precells. Each path intersects another two paths (because σ is a cycle) and no three paths have common intersection (because σ is 2-full). Therefore their union covers the boundary of the precell. Let $s_1, \dots, s_k$ be those paths. Then the angular length of σ equals

$$\frac{(l(s_1) + l(s_2) - 2l(s_1 \cap s_2)) + \cdots + (l(s_k) + l(s_1) - 2l(s_k \cap s_1))}{r}\pi.$$

By an inclusion exclusion argument this is exactly 2π.

Case 3 Finally, suppose σ passes through vertices of both kinds. Given an orientation for σ, let v_1 be a vertex of type (i) such that the following vertex in σ, say u, is of type (ii). Let v_2 be the next vertex of type (i) that appears following this orientation (note that $v_1 \neq v_2$ since σ is 2-full). The vertex u is adjacent to some type (i) vertices $v_1', \dots, v_l'$ between v_1 and v_2. Denote by $[v_1, v_2]$ the edge-path passing only through vertices of type (i) that connects v_1 with v_2 and which contains $v_1', \dots, v_l'$. It contains all of them because σ

is 2-full. By an argument analogous to that of case 2, the subpath of σ that goes from v_1 to v_2 has angular length greater than or equal to $\frac{2l([v_1,v_2])}{r}\pi$. Inductively, the remaining subpath of σ has angular length greater than or equal to $\frac{2(r-l([v_1,v_2]))}{r}\pi$. Therefore σ has angular length greater than or equal to 2π.

Lastly we analyze the links of the original vertices. By the definition of X, the link of any original vertex in the 2-skeleton of X is obtained from the link of the vertex in the universal cover $\tilde{K}_P$ by subdividing all the edges (adding a new vertex in every edge of the original link), and then adding all the edges between all pairs of new vertices of the link. In particular, the links of the original vertices in $X^{(2)}$ do not have 2-full cycles, and therefore the link condition around these vertices is automatically satisfied. □

Proof of Theorem 3.2.1. By the previous discussion and Lemmas 3.2.3 and 3.2.4, X is a strictly systolic angled complex on which Γ acts properly and cocompactly by simplicial automorphisms, and its weight function is Γ-invariant. Therefore, by Corollary 3.1.11 Γ is hyperbolic. □

Note that, by Corollary 3.1.11, all finitely presented subgroups of the one-relator groups Γ of Theorem 3.2.1 are also hyperbolic. This fact can also be deduced from Theorem 3.2.1 by a well known result of Gersten on finitely presented subgroups of hyperbolic groups in dimension 2 [43]. Recall that one-relator groups without torsion have cohomological dimension 2.

Example 3.2.5. The construction that we introduced in the proof of Theorem 3.2.1 can also be applied to prove hyperbolicity of one-relator groups which do not satisfy the hypotheses of the theorem. One such example is the group presented by $\langle a, t | at^{-1}ata^2t^{-2}a^{-1}t^2 \rangle$. This group appeared in [63], where it was proved to be hyperbolic with very different techniques.

3.3 Condition $\mathcal{T}'$

In this section we define conditions $\mathcal{T}'$ and $\mathcal{T}'_<$. They are defined in terms of the lengths of the pieces and relators incident to interior vertices of reduced diagrams over the presentations. At first glance it may seem that these conditions are difficult to check since the definitions require to analyze all interior vertices of every possible diagram over the presentation P. However we will prove that, for finite presentations, these conditions can be verified by analyzing the directed cycles in a finite weighted graph $\Gamma(P)$ associated to P. In this direction, in Section 3.6 we describe an algorithm which decides whether a given finite presentation P satisfies these conditions. This algorithm has been implemented in the GAP [83] package SmallCancellation [77].

In what follows we consider diagrams $\varphi : M \to K$ to a combinatorial complex K with no interior vertices of degree 2. As in Section 3.1, we define the *geometric link* of a vertex v in a combinatorial complex K as an epsilon sphere about v (which inherits a combinatorial structure), and the *corners* of the 2-cells at v correspond to edges in the link. The endpoints of a corner in M (of a 2-cell f) at v correspond to edges in the diagram incident to v. Given a corner c at an interior vertex v, we denote by $\ell_1(c)$ and $\ell_2(c)$ the lengths of the incident edges and by $\ell_r(c)$ the length of the relator $r \in R$ corresponding to the 2-cell f. Let $d'_F(v) = \sum_{c \ni v} \frac{\ell_1(c)+\ell_2(c)}{\ell_r(c)}$, where the sum is taken over all corners at v.

Definition 3.3.1. We say that a presentation P satisfies the small cancellation condition $\mathcal{T}'$ if for every interior vertex of any reduced diagram over P (with no interior vertices of degree 2), $d'_F(v) \leq d(v) - 2$. Similarly, P satisfies the strict small cancellation condition $\mathcal{T}'_<$ if for every interior vertex of any reduced diagram over P, $d'_F(v) < d(v) - 2$. A group G which admits a presentation P satisfying condition $\mathcal{T}'$ (resp. $\mathcal{T}'_<$) is called a $\mathcal{T}'$-group (resp. $\mathcal{T}'_<$-group).

Condition $\mathcal{T}'$ can be thought of as a "tug of war" between conditions of type C' and T. We allow pieces to be large with respect to the length of the relators, as long as those large pieces do not clash in high numbers at a common vertex. We also allow interior vertices to have high degree if the pieces incident to such a vertex are short. Now we investigate now the first examples of presentations satisfying conditions $\mathcal{T}'$ and $\mathcal{T}'_<$.

Classical metric small cancellation conditions. It is easy to verify that the classical metric small cancellation conditions $C'(\frac{1}{6})$, $C'(\frac{1}{4})-T(4)$ and $C'(\frac{1}{3}) - T(6)$ imply condition $\mathcal{T}'_<$. We will show below that finitely presented $\mathcal{T}'_< - C(3)$-groups are hyperbolic, generalizing the classical result for small cancellation groups (see [15, 50]).

Two-dimensional Artin groups. Recall that an Artin group is two-dimensional if its defining graph Γ satisfies the following condition: for every triangle in the graph Γ with edges labeled by p, q and r we have $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\leq 1$ (see Section 2.2).

We will show that an Artin group is two-dimensional if and only if its standard presentation P_Γ satisfies condition $\mathcal{T}'$. We will also prove below that any group which admits a finite presentation P satisfying conditions $\mathcal{T}'$ and $C'(\frac{1}{2})$ and with all relators of the same length, has quadratic Dehn function and solvable conjugacy problem (see Theorem 3.5.2). These results put together partially recover, with an alternative and simpler proof, similar results for two-dimensional Artin groups recently obtained by Huang and Osajda [57].

Theorem 3.3.2. *An Artin group A_Γ is two-dimensional if and only if its standard presentation P_Γ satisfies condition $\mathcal{T}'$.*

Proof. Let A_Γ be an Artin group and let K be the 2-complex associated to its standard presentation. Note that the 2-cells of K have two distinguished sides in which all edges have the same orientation (see Figure 3.8). If the label of the edge in Γ corresponding to the relator is m, each of these sides has m edges. The terminal vertices of both sides are called initial and final vertices of the relator, according to the orientation of the edges (cf. [57, Section 4.1]).

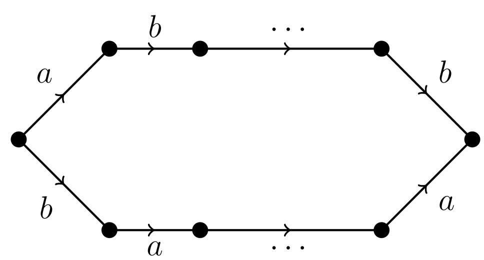

Figure 3.8: A 2-cell of an Artin group.

Let $\varphi\colon M \to K$ be a reduced diagram. We analyze first the interior vertices of degree 3. It is easy to see that vertices of degree 3 only correspond to intersections of faces in the diagram which are mapped to three different relators that form a triangle in the graph Γ. Since the three relators are different, the length of the three pieces involved is 1. If the labels of the edges in the triangle are p, q and r, then the equation for the condition $\mathcal{T}'$ is the following:

$$\frac{1+1}{2p} + \frac{1+1}{2q} + \frac{1+1}{2r} \leq 3 - 2.$$

That is, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$, which is exactly the necessary and sufficient condition for the Artin group to be two-dimensional.

Now we prove that condition $\mathcal{T}'$ is always satisfied in interior vertices of degree greater than or equal to 4 (for any Artin group, not necessarily two-dimensional). Note that such an interior vertex v can be a terminal vertex or it can be inside of one of the sides of the 2-cells containing it. For example, in Figure 3.9, v is a terminal vertex of r_1 and r_4, and it is on one of the sides of r_2 and r_3.

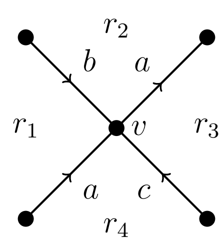

Figure 3.9: An interior vertex of degree 4.

If not all of the 2-cells incident to v are mapped to the same relator, at least four of them will share an edge incident to v with a cell corresponding to a different relator. In Figure 3.9, r_1 shares an edge with r_4, and r_2 with r_3. Since cells corresponding to different relators intersect at paths of length at most 1, at least four of them will have a piece of length 1. Also, the longest piece in a 2-cell with boundary of length $2n$ is $n-1$, and therefore the summands in condition $\mathcal{T}'$ can be at most $\frac{2n-2}{2n}$. In conclusion, if there are 2-cells incident to the vertex which are mapped to different relators, there are at least four summands in the equation for condition $\mathcal{T}'$ which are less than or equal to $\frac{1}{2}$, and every other summand is smaller than 1. Then condition $\mathcal{T}'$ is satisfied.

We analyze now the case where the vertex has degree greater than or equal to 4 and all the 2-cells are mapped to the same relator. Observe that if a 2-cell contains v in one of its sides, the summand corresponding to that 2-cell is at most $\frac{1}{2}$. Therefore there are at most three of such 2-cells, and the rest have to contain v as a terminal vertex. If a 2-cell f contains v as a terminal vertex, then the two adjacent 2-cells to f contain the vertex on a side. This implies that we can reduce ourselves to the cases where v has degree 4, 5 or 6.

We look at the orientation of the edges incident to v, traversing them in clockwise order. If we pass by a 2-cell that has v as a terminal vertex, the orientation of these edges is preserved, and if not, it is reversed. Therefore the number of 2-cells having v on one of its sides is even. Therefore, when v has degree 5 or 6, there are at least four 2-cells that have v on a side.

It only remains to check the case where v has degree 4, two of the 2-cells contain it as a terminal vertex and the other two on a side. This situation is illustrated in Figure 3.10. Let l_1, l_2, l_3, l_4 be the lengths of the pieces involved and let $2n$ be the length of the relator.

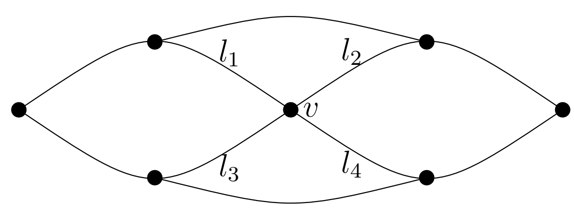

Figure 3.10: A vertex with degree 4 with the 2-cells mapping to the same relator.

Here condition $\mathcal{T}'$ can be rewritten as:

$$\frac{l_1+l_2}{2n}+\frac{l_2+l_4}{2n}+\frac{l_3+l_4}{2n}+\frac{l_1+l_3}{2n}=\frac{2l_1+2l_2+2l_3+2l_4}{2n}\leq 2.$$

Since $l_1+l_2 \leq n$ and $l_3+l_4 \leq n$, condition $\mathcal{T}'$ is satisfied. $\square$

Remark 3.3.1. *Combined with Theorem 3.4.2, Theorem 3.3.2 implies that the standard presentation P_Γ of an Artin group is aspherical (i.e. the group A_Γ is two-dimensional) if and only if P_Γ is diagrammatically reducible (DR, for short). Recall that a presentation P with no proper powers is DR if all* *spherical diagrams over P are reducible. Note that being DR is in general a stronger condition than being aspherical (see [41]).*

We now introduce slightly more general presentations and show that they satisfy condition $\mathcal{T}'$. Let F be a free group on a finite set of generators X. Let $W \subset F$ be a finite subset of words, and let Γ be a finite simple graph with vertex set W and a labeling on the edges by integers $m \geq 2$. We can consider the presentation $P_{\Gamma,F}$ with generators X, and a relation

$$\underbrace{vwvwv\cdots}_{m \text{ factors}} = \underbrace{wvwvw\cdots}_{m \text{ factors}}$$

for every pair of words $v, w \in W$ connected by an edge labeled by m. We denote by $A_{\Gamma,F}$ the group presented by $P_{\Gamma,F}$. This is obviously a generalization of the definition of an Artin group. The following result generalizes Theorem 3.3.2 for the groups $A_{\Gamma,F}$ under certain restrictions on the words and the labeling of the edges.

Theorem 3.3.3. *The presentation $P_{\Gamma,F}$ satisfies $\mathcal{T}'$ provided that all the edges in Γ are labeled by the same integer $m \geq 3$, and the words in W have all the same length, are cyclically reduced and do not share letters (i.e. every generator appears at most in one of the words of W).*

Proof. Let n be the length of the words in W. Let $\varphi\colon M \to K$ be a reduced diagram and let v be an interior vertex of degree at least 3. Let $k = d(v)$ and let $f_1, \ldots, f_k$ be the 2-cells of M incident to v, numbered clockwise.

Note that, since the words in W do not share letters, there is a subset $D(f_i)$ of *distinguished* vertices of the boundary of f_i which is characterized by the following property: $D(f_i)$ splits the word written in the boundary of f_i into words which belong either to W or to W^{-1} (here W^{-1} denotes the set of the inverses of the words in W). Note that a vertex may be distinguished in certain 2-cell f_i but not in a neighbor face.

Let x_i (resp. y_i) be the (possibly empty) word read counterclockwise (resp. clockwise) in the boundary of f_i, starting at v and ending at the first occurrence of a vertex which belongs to $D(f_i)$ (see Figure 3.11). Note that $|x_i| + |y_i| \equiv 0 \pmod{n}$. Here $|x|$ denotes the length of the word x. The *mismatch between f_i and f_{i+1}* is defined as $|y_{i+1}| - |x_i| \in \mathbb{Z}/n\mathbb{Z}$. If the

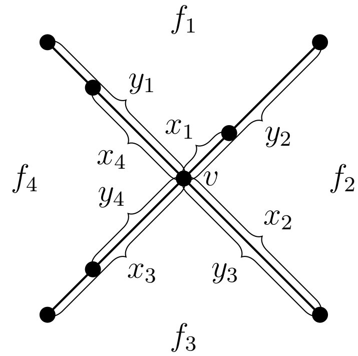

Figure 3.11: Mismatches around an interior vertex.

mismatch is nonzero we say that there is a *proper mismatch* between f_i and f_{i+1}. Let s be the number of proper mismatches around v.

If there are no proper mismatches, then the proof that condition $\mathcal{T}'$ is satisfied is analogous to that of Theorem 3.3.2. If there is a proper mismatch between two faces, their intersection has length less than n. Note that the length of a piece is at most $n(m-1)$, and the length of the relators is $2nm$. Therefore, if $s \geq 2$, condition $\mathcal{T}'$ is satisfied. Therefore, we only need to show that $s \neq 1$.

Suppose that there is exactly one proper mismatch, say between f_1 and f_k. Then the following holds.

$$\begin{aligned} |y_1| &\not\equiv |x_k| \pmod n, \\ |y_{i+1}| &\equiv |x_i| \pmod n \quad \text{for } 1 \leq i \leq k-1, \\ |x_i| &\equiv -|y_i| \pmod n \quad \text{for } 1 \leq i \leq k. \end{aligned}$$

It follows that $|x_1| \not\equiv (-1)^k|x_1| \pmod n$. This is a contradiction if k is even.

Now we study the case where k is odd and $s = 1$. It is easy to see that if one of the x_i or y_i is empty, then there are at least two proper mismatches, so we can assume that no x_i or y_i is empty. Since the words in W do not share letters, there exists $w \in W$ such that for each $1 \leq i \leq k$, the word $y_i^{-1}x_i$ equals w or w^{-1}. Again, we can assume that the unique proper mismatch is between f_1 and f_k. If there exists $1 \leq i \leq k-1$ such that $y_i^{-1}x_i = y_{i+1}^{-1}x_{i+1}$, since $x_i = y_{i+1}$, we deduce that $y_i^{-1}x_{i+1} = w^2$ or $y_i^{-1}x_{i+1} = w^{-2}$. This is a contradiction because w is cyclically reduced. Then $y_i^{-1}x_i = (y_{i+1}^{-1}x_{i+1})^{-1}$

for every $1 \leq i \leq k$. However this cannot happen since k is odd. Therefore $s \geq 2$. $\square$

One-relator groups. In Section 3.2 we introduced a small cancellation condition (T') to study hyperbolicity of one-relator groups. Condition $\mathcal{T}'_<$ generalizes condition (T') to any presentation and Theorem 3.4.4 below provides an alternative, simpler and more general proof of Theorem 3.2.1 for one-relator groups.

Example 3.3.4. The following one-relator presentation does not satisfy conditions $C(6)$ nor $T(4)$, but it is $\mathcal{T}'$. It also does not fall under the hypothesis of Theorem 3.2.1 since it is $C'(\frac{1}{2})$, but not $C'(\frac{1}{4})$.

$$\langle a, b \mid a^3b^4a^3b^4(b^4a^3b^4a^3)^{-1}\rangle.$$

Cyclic presentations. The claims in the following examples can be verified using the GAP [83] package SmallCancellation [77].

Example 3.3.5. The following cyclic presentation of a superperfect group satisfies condition $\mathcal{T}'_<$ but is not $C(6)$ nor $T(4)$.

$$\langle x_0, x_1, x_2, x_3, x_4 \mid x_{i+4}^{-1}x_{i+1}^{-1}x_i^{-1}(x_{i+4}x_{i+1})^2 \text{ for } i = 0, \ldots, 4\rangle$$

Example 3.3.6. The following cyclic presentation of a superperfect group satisfies condition $\mathcal{T}' - C'(\frac{1}{2})$

$$\langle x_0, \ldots, x_6 \mid x_{i+1}x_i^{-1}x_{i+6}x_{i+1}^{-1}x_ix_{i+6}^{-1}x_{i+2}^{-1} \text{ for } i = 0, \ldots, 6\rangle$$

Then by Theorem 3.5.2 this group has a quadratic Dehn function. However the group is not $\mathcal{T}'_<$, nor $C(6)$, $C(4) - T(4)$ or $C(3) - T(6)$.

3.4 Non-positive curvature, diagrammatic reducibility and hyperbolicity

We first define some basic notions on combinatorial curvature analogous to the ones defines in Section 3.1, but in the more general context of combinatorial 2-complexes. Given a combinatorial 2-complex K, we can assign a real number

$\omega(c)$ to the corners, which we think of as angles. This assignment is a *weight function* for the complex. A finite combinatorial 2-complex together with such a weight function is called an angled complex (see [41, 88]).

Let K be an angled complex. If v is a vertex of K, its curvature is defined as

$$\kappa(v) = 2\pi - \pi\chi(\mathrm{lk}_v) - \sum_{c \ni v} \omega(c).$$

Here $\chi(\mathrm{lk}_K(v))$ denotes the Euler characteristic of the link of v, and the sum is taken over all corners at v. The curvature of a face f is defined as

$$\kappa(f) = 2\pi - \pi\ell(\partial f) + \sum_{c \in f} \omega(c),$$

where the sum is taken over all the corners in f and $\ell(\partial f)$ is the number of edges in the boundary of f. As before, we can state a combinatorial Gauss–Bonnet theorem. [5, 88].

Theorem 3.4.1 (Combinatorial Gauss–Bonnet Theorem, [5, 88])**.** *Let K be an angled 2-complex. Then*

$$\sum_{f \in faces(K)} \kappa(f) + \sum_{v \in K^{(0)}} \kappa(v) = 2\pi\chi(K).$$

Assignment of weight functions

Let P be a presentation satisfying condition $\mathcal{T}'$ or $\mathcal{T}'_<$. Given a reduced diagram $f\colon M \to K_P$ over P, we define the following weight function in M. The weight of a corner c at an interior vertex v is $\omega(c) = \pi - \frac{\ell_1(c)+\ell_2(c)}{\ell_r(c)}\pi$ (recall that there are no interior vertices of degree 2). The weight of a corner c at a vertex $v \in \partial M$ of degree 2 is $\omega(c) = \pi$. If c is a corner at a vertex $v \in \partial M$ of degree greater than 2, we define $\ell_1(c)$ and $\ell_2(c)$ similarly as we did with interior vertices (the lengths of the incident edges obtained if we remove the vertices of degree 2) and $\omega(c) = \pi - \frac{\ell_1(c)+\ell_2(c)}{\ell_r(c)}\pi$.

With this assignment, the curvature of the faces of M is 0, and the curvature of the interior vertices is non-positive if P satisfies condition $\mathcal{T}'$, and strictly negative if P satisfies condition $\mathcal{T}'_<$.

We will show that presentations satisfying condition $\mathcal{T}'$ and without proper powers are DR and that finitely presented $\mathcal{T}'_< - C(3)$-groups are hyperbolic.

Given a 2-complex M, we denote by $\mathcal{V}(M), \mathcal{E}(M)$ and $\mathcal{F}(M)$ the number of vertices, edges and faces of M respectively.

Theorem 3.4.2. *If a presentation P satisfies condition $\mathcal{T}'$ and has no proper powers, then it is DR.*

Proof. Since P has no proper powers, our notion of reduced spherical diagram over P coincides with that of [41]. Therefore, in order to prove that P is DR, we only have to verify that there are no reduced spherical diagrams over P. Suppose $\varphi\colon M \to K_P$ is a reduced spherical diagram. We have the following identities:

$$\mathcal{E}(M) = \frac{1}{2}\sum_{v\in M^{(0)}} d(v),$$
$$\mathcal{F}(M) = \frac{1}{2}\sum_{v\in M^{(0)}} d'_F(v).$$

The first one is clear, since every edge is incident to two vertices. The second one is deduced from the fact that in the right hand side of the second equality we are summing two times the length of each relator, divided by the length of each relator. That is, we are summing 2 for each face. Then

$$2 = \mathcal{V}(M) - \mathcal{E}(M) + \mathcal{F}(M) = \mathcal{V}(M) - \frac{1}{2}\sum_{v\in M^{(0)}} d(v) + \frac{1}{2}\sum_{v\in M^{(0)}} d'_F(v) \leq 0,$$

where the last inequality holds because all the vertices in the sphere are interior vertices. This is a contradiction, and therefore P is diagrammatically reducible. $\square$

Equations over groups

A system of equations over a group G with unknowns $x_1, x_2, \dots, x_n$ is a set $\{w_j(x_1, x_2, \dots, x_n)\}_j$ of words in $G * F(x_1, \dots, x_n)$. Here $F(x_1, \dots, x_n)$ is the free group with basis $\{x_1, x_2, \dots, x_n\}$. The letters of w_j which lie in G are the *coefficients* of w_j. The (non-necessarily reduced) word r_j in the alphabet $\{x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}\}$ obtained by deleting the coefficients of w_j is the *shape* of w_j, and the word r_j considered as an element of the free group

$F(x_1, x_2, \dots, x_n)$ is called the *content* of w_j. We say that the system has a solution in an overgroup of G if there exits a group H of which G is a subgroup and elements $h_1, h_2, \dots, h_n$ in H such that

$$w_j(h_1, h_2, \dots, h_n) = 1 \in H$$

for every j. The Kervaire–Laudenbach Conjecture states that for any group G, a unique equation w with a unique unknown x has a solution in an overgroup of G if w is non-singular, which means that the total exponent of x is non-zero. The so called Kervaire–Laudenbach–Howie Conjecture generalizes this to an arbitrary finite number n of unknowns and a non-singular system of m equations (in this case, non-singular means that the rank of the $m \times n$ matrix of total exponents is equal to m).

Let S be a system of equations $w_1, w_2, \dots, w_m$ over a group G. Let P be the presentation $\langle x_1, x_2, \dots, x_n | r_1, r_2, \dots, r_m \rangle$ whose generators are the unknowns of S and its relators are the shapes of the equations w_j. A well known result by Gersten [41] states that if P is DR, then S has a solution in an overgroup of G. In other words, for any group H, any system of equations modeled by the presentation P has a solution in an overgroup of H. A presentation with this property is said to be *Kervaire*. The converse of this result is false. The presentation $P = \langle t | ttt^{-1} \rangle$ is not DR, but it is Kervaire: any equation $atbtct^{-1}$ modeled by P over any coefficient group H has a solution in an overgroup of H [56].

From the results above we deduce the following.

Proposition 3.4.3. *Let $k, l, m \in \mathbb{N}$ and w_1, w_2, w_3 be cyclically reduced words of the same length in the unknowns x_i, y_j, z_s respectively. Then for any $p \geq 2$, the presentation*

$$P = \langle x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_m \ | (w_1w_2)^p(w_2w_1)^{-p}, (w_2w_3)^p(w_3w_2)^{-p}, \\ (w_3w_1)^p(w_1w_3)^{-p} \rangle$$

is DR. Therefore for any group G, any system of equations modeled by P has a solution in an overgroup of G.

Proof. The presentation P corresponds to a presentation of type $P_{\Gamma,F}$ where Γ is a triangle with vertices w_1, w_2 and w_3 and whose three edges are labeled by $2p$. Now the result follows from Theorems 3.3.3 and 3.4.2 and from Gersten's result mentioned above. $\square$

Hyperbolicity

We now show that the strict metric condition $\mathcal{T}'_<$ implies hyperbolicity. We do so by proving that the Dehn function is linear. This is a strengthening of Theorem 3.2.1.

Theorem 3.4.4. *Let G be a group which admits a finite presentation satisfying conditions $\mathcal{T}'_<$ and $C(3)$. Then G is hyperbolic.*

Proof. We show that a finite presentation P satisfying conditions $\mathcal{T}'_<$ and $C(3)$ has a linear isoperimetric inequality. Note that it suffices to consider non-singular disk diagrams.

Let $\varphi\colon M \to K_P$ be a non-singular reduced disk diagram. We assign weights to the corners in M as we did before. Then, by the Combinatorial Gauss–Bonnet Theorem,

$$2\pi = \sum_{v \in M^{(0)}} \kappa(v) + \sum_{f \in \text{faces}(M)} \kappa(f) = \sum_{v \in M^{(0)}} \kappa(v).$$

Since P satisfies $\mathcal{T}'_<$, then $\kappa(v) < 0$ for every interior vertex v, and since P is finite, by Corollary 3.6.3 there is a constant $N < 0$, which is independent of the diagram, such that $\kappa(v) \leq N$ for every interior vertex v. Also, for every boundary vertex v, it holds that $\kappa(v) < \pi$ since the weights in M are non-negative. Then

$$\begin{aligned} 2\pi &\leq \mathcal{V}^\circ(M)N + \sum_{v \in (\partial M)^{(0)})} \kappa(v) \\ &\leq \mathcal{V}^\circ(M)N + \mathcal{V}(\partial M)\pi \\ &= \mathcal{V}(M)N + \ell(\partial M)(\pi - N), \end{aligned}$$

where $\mathcal{V}^\circ(M)$ denotes the number of interior vertices of M, and $\ell(\partial M)$ is the length of the boundary. The last equality holds because M is non-singular. Then,

$$-\mathcal{V}(M)N \leq \ell(\partial M)(\pi - N) - 2\pi$$

and therefore

$$\mathcal{V}(M) \leq \ell(\partial M)\frac{\pi - N}{-N} + \frac{2\pi}{N}.$$

Now, since P satisfies condition $C(3)$, the number of faces can be linearly bounded by the number of vertices in the diagram. Consequently, the number of faces in the diagram is linearly bounded by the length of its boundary. $\square$

3.5 Quadratic Dehn function and conjugacy problem

In this section we will show that a finitely presented group which admits a presentation P satisfying conditions $\mathcal{T}'$ and $C'(\frac{1}{2})$ and such that all relators of P have the same length r, has a quadratic Dehn function and solvable conjugacy problem.

Let $\varphi\colon M \to K_P$ be a diagram over P. The *boundary layer* L of M consists of every vertex in the boundary of M, every edge incident to a vertex in the boundary, and every open face with a vertex in the boundary. Note that L is usually not a combinatorial complex. Let $M_1 = M \backslash L$ be the complement of the boundary layer. Note that M_1 is a subcomplex of M. The following lemma will be used to prove the main result of this section.

Lemma 3.5.1. *Let P be a presentation satisfying conditions $\mathcal{T}' - C'(\frac{1}{2})$ and such that all its relators have length r, and let $\varphi\colon M \to K_P$ be an annular or disk diagram over P. Then*

$$\mathcal{V}(\partial M_1) \leq \mathcal{V}(\partial M) - r\chi(M).$$

Proof. We had previously removed interior vertices of degree 2 from the diagrams. We subdivide the boundary of M_1 reintroducing the vertices of degree 2, and still denote this diagram by M.

For the vertices of M of degree greater than 2, we assign weights to the corners as before, and in vertices of degree 2, both weights are equal to π. With this assignment, every face has curvature 0 and all interior vertices have non-positive curvature, since P satisfies $\mathcal{T}'$. Note that $\kappa(v) = 0$ for interior vertices of degree 2.

In what follows, we can assume without loss of generality that M is non-singular, since we are going to bound the length of the boundary of M_1 in terms of the length of the boundary of M. Since P is $C'(\frac{1}{2})$ we may assume that each boundary 2-cell f has at least two edges which are not on the boundary of M, for otherwise we can remove f decreasing the length of the boundary and without changing M_1. In particular this reduction allows us to assume that $M_1 \neq \emptyset$.

We consider the complex B constructed by taking the disjoint union of the 0-cells, 1-cells and 2-cells (now closed) of the boundary layer of M and identifying the boundaries of the closed 2-cells but only in the vertices and edges of the boundary layer of M (see Figure 3.12).

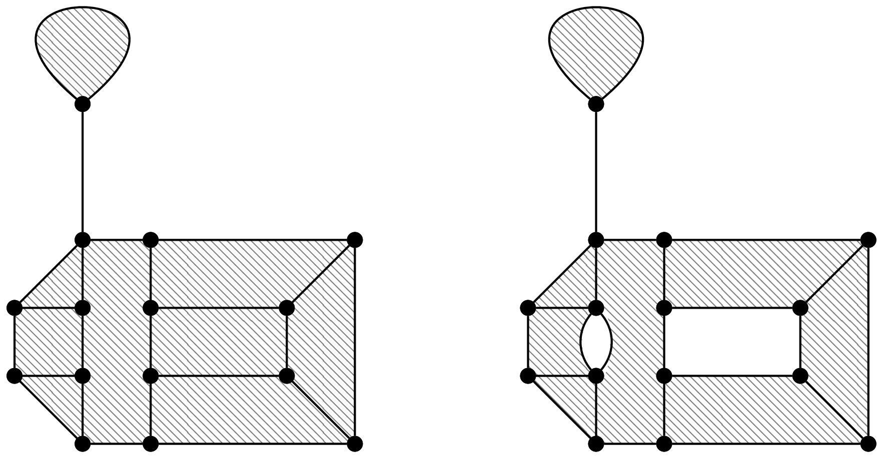

Figure 3.12: At the left the complex M. At the right the complex B constructed from M in the proof of Lemma 3.5.1.

We omit vertices of degree 2 in the cell structure of B. Note that $\ell(\partial B) = \ell(\partial M) + \ell(\partial M_1)$. If M is a disk, B is a planar and connected combinatorial complex, so its Euler characteristic is less than or equal to $1 = \chi(M)$. If M is an annulus, B may have more than one connected component, but none of them would be a disk, since they all have a disconnected complement. Therefore its Euler characteristic is less than or equal to $0 = \chi(M)$.

We separate its vertices into two sets: V_1 will denote the set of vertices of B that are in the boundary of M, and V_2 the set of remaining vertices of B. Since P satisfies condition $\mathcal{T}'$, by Gauss–Bonnet we have

$$2\pi\chi(M) \leq \sum_{v\in V_1} \kappa(v).$$

Also by Gauss–Bonnet, we have

$$\sum_{v\in B^{(0)}} \kappa(v) = 2\pi\chi(B).$$

Therefore

$$\sum_{v\in V_2}\kappa(v)\leq 0.$$

Now since each boundary 2-cell has at least two edges which are not on the boundary of M we have

$$\mathcal{V}_1+\sum_{v\in V_2}\sum_{c\ni v}1=\mathcal{V}(B)+\mathcal{E}(B)-\ell(\partial B)=\mathcal{V}_2+\sum_{v\in V_1}\sum_{c\ni v}1-\mathcal{V}^\circ(B).$$

Putting everything together

$$\begin{aligned}
2\pi\chi(M) &\leq \sum_{v\in V_1}\kappa(v)-\sum_{v\in V_2}\kappa(v)\\
&=\sum_{v\in V_1}\left(\pi-\sum_{c\ni v}\left(\pi-\frac{\ell_1(c)+\ell_2(c)}{r}\pi\right)\right)\\
&\quad-\sum_{v\in V_2}\left(\pi-\sum_{c\ni v}\left(\pi-\frac{\ell_1(c)+\ell_2(c)}{r}\pi\right)\right)\\
&\leq \sum_{v\in V_1}\sum_{c\ni v}\frac{\ell_1(c)+\ell_2(c)}{r}\pi-\sum_{v\in V_2}\sum_{c\ni v}\frac{\ell_1(c)+\ell_2(c)}{r}\pi\\
&=\frac{2\mathcal{V}(\partial M)-2\mathcal{V}(\partial M_1)}{r}\pi.
\end{aligned}$$

It follows that

$$\mathcal{V}(\partial M_1)\leq\mathcal{V}(\partial M)-r\chi(M).$$

$\square$

Theorem 3.5.2. *Let P be a presentation satisfying conditions $\mathcal{T}'-C'(\frac{1}{2})$ and such that all its relators have length r, then P has a quadratic Dehn function. Moreover, if P is finite the group G presented by P has solvable conjugacy problem.*

Proof of the quadratic Dehn function. Let $\varphi\colon M\to K_P$ be a reduced disk diagram. We show that $\mathcal{V}(M)\leq\frac{1}{r}\mathcal{V}(\partial M)^2$ by induction in the number of interior vertices of M. We have

$$\begin{aligned}\mathcal{V}(M) &= \mathcal{V}(M_1) + \mathcal{V}(\partial M) \\ &\leq \frac{1}{r}\mathcal{V}(\partial M_1)^2 + \mathcal{V}(\partial M) \\ &\leq \frac{1}{r}(\mathcal{V}(\partial M) - r)^2 + \mathcal{V}(\partial M) \\ &= \frac{1}{r}\mathcal{V}(\partial M)^2 - 2\mathcal{V}(\partial M) + r + \mathcal{V}(\partial M) \\ &\leq \frac{1}{r}\mathcal{V}(\partial M)^2.\end{aligned}$$

The first inequality follows by induction, and the second one follows from Lemma 3.5.1. Finally, since P satisfies $C'(\frac{1}{2})$, each face of M has at least three sides. Then we can bound the number of faces of M by the number of vertices of M, obtaining the desired quadratic isoperimetric inequality. □

It follows that the finitely generated groups which admit presentations satisfying the hypotheses of Theorem 3.5.2 have solvable word problem. This will be used to prove that they also have solvable conjugacy problem.

We attack the conjugacy problem following the strategy of [68, Section V.7]).

Remark 3.5.1. *Let $P = \langle X \mid R\rangle$ be a finite presentation of a group G with solvable word problem. Suppose that all the relators have the same length r. Let w_1 and w_2 be words in the free group $F(X)$. We write $w_1 \sim w_2$ if there exists a word b in $F(X)$ with $|b| < r$ such that $bw_1b^{-1}w_2^{-1} = 1$ in G. Here $|b|$ denotes the length of the word b. Since the word problem is solvable, the relation $\sim$ is decidable. Now let u and v be cyclically reduced words in $F(X)$ and let $d = |u| + |v|$. Take $W = \{w \in F(X), |w| \leq d\}$. Note that W is finite since X is finite. Note also that the set W depends on the lengths of u and v and that $u, v \in W$. We write $u \overline{\sim} v$ if there exist words $w_1, \dots, w_k$ in W such that $u \sim w_1 \sim \dots \sim w_k \sim v$. Equivalently, $\overline{\sim}$ is the transitive closure in W of the relation $\sim$. Note that this relation is also decidable since W is finite. In order to prove that the conjugacy problem is solvable it suffices to prove that if two words $u, v \in F(X)$ are conjugate in G, then $u \overline{\sim} v$.*

We will also use the following result of Schupp [68, Section V.7]. Let A be an annular diagram and L its boundary layer. The diagram $A_1 = A \backslash L$

(the complement of L in A) may be disconnected, but it has at most one annular component. A simply connected component of A_1 is called a *gap*. Let $K_1, \ldots, K_n$ be the gaps. Then $H = A \backslash (L \cup \bigcup_{i=1}^{n} K_i)$ is the annular component of A_1, assuming there is any. Let σ and τ be the outer and inner boundaries of H. A pair (D_1, D_2) of faces (not necessarily distinct) in A is called a *boundary linking pair* if $\sigma \cap \partial D_1 \neq \emptyset$, $\partial D_1 \cap \partial D_2 \neq \emptyset$, and $\partial D_2 \cap \tau \neq \emptyset$.

Lemma 3.5.3 (Schupp). *Let A be an annular diagram having at least one region, and let H be the diagram obtained by removing its boundary layer and its gaps. If there are no boundary linking pairs, H is an annular diagram.*

Proof of the conjugacy problem. Take cyclically reduced words $u, v \in F(X)$ and suppose that they are conjugate in G. Let $d = |u| + |v|$. By Remark 3.5.1, we only have to prove that $u \overset{\sim}{\sim} v$. Let A be an annular diagram with u and v^{-1} as inner and outer boundaries. Construct the diagrams $A = H_0, H_1, \ldots, H_k$, where H_{i+1} is obtained from H_i by removing its boundary layer and its gaps, and let H_k be the first of such diagrams with a linking pair. By Lemma 3.5.1, $\ell(\partial H_{i+1}) \leq \ell(\partial H_i)$ for each $0 \leq i \leq k-1$. Therefore, $\ell(\partial H_i) \leq d$ for every i (and so the boundary labels of ∂H_i are in the set W).

Let σ_i and τ_i be the outer and inner boundaries of H_i respectively. Let S_i be the subdiagram of M consisting of σ_i, σ_{i+1} and all the cells of M between these two paths. Define T_i in the same manner with respect to τ_i and τ_{i+1}. It is clear that any boundary face of S_i intersects both boundaries of S_i. So there is a path γ_i from σ_i to σ_{i+1} with a label of length less than or equal to r. Let s_i and s_{i+1}^{-1} be the labels of σ_i and σ_{i+1} starting at a given vertex. Then $s_i \sim s_{i+1}$. Analogously, we have $t_i \sim t_{i+1}$ where t_i^{-1} and t_{i+1} are the labels of T_i.

The last annulus H_k has a boundary linking pair (D_1, D_2). We have vertices $v_0 \in \sigma_k \cap \partial D_1$, $v_1 \in \partial D_1 \cap \partial D_2$, and $v_2 \in \partial D_2 \cap \tau_k$. Therefore there are paths β_1 and β_2 from v_0 to v_1 and from v_1 to v_2 labeled by words b_1 and b_2 of length smaller than or equal to $\frac{r}{2}$. Let $\beta = \beta_1\beta_2$, then its label is a word of length less than r. Let s be the word read in the outer boundary of H_k starting at v_0, and t^{-1} the word read in the inner boundary of H_k starting at v_2. We have that $sb_1b_2t^{-1}b_2^{-1}b_1^{-1} = 1$ in G. Then $s \sim t$.

Since s_0 and t_0 are cyclic permutations of u and v respectively, and s and

t are cyclic permutations of s_k and t_k respectively, we have

$$u \overline{\approx} s_0 \sim s_1 \sim \ldots \sim s_k \overline{\approx} s \sim t \overline{\approx} t_k \sim \ldots \sim t_0 \overline{\approx} v.$$

□

A slight modification in the proof of Lemma 3.5.1 allows one to obtain a lower bound on the length of the words which represent the trivial element in the group G, even if the relators have different lengths. This result is in the spirit of Greendlinger's lemma (Lemma 1.5.6).

Proposition 3.5.4. *Let $P = \langle X \mid R \rangle$ be a presentation of a group G, satisfying conditions $\mathcal{T}' - C'(\frac{1}{2})$. Let $r_{\min}$ be the length of the shortest relator. Then any nontrivial word W in the free group generated by X representing the trivial element in G has length at least $r_{\min}$. In particular, if P has a relator of length greater than or equal to 2, then G is nontrivial.*

Proof. Let M be a reduced disk diagram. We follow the same steps as in the proof of Lemma 3.5.1 and we get that

$$2\pi\chi(M) \leq \sum_{v\in V_1}\sum_{c\ni v}\frac{\ell_1(c)+\ell_2(c)}{l_r(c)}\pi - \sum_{v\in V_2}\sum_{c\ni v}\frac{\ell_1(c)+\ell_2(c)}{l_r(c)}\pi.$$

In particular since the terms $\frac{\ell_i(c)}{\ell_r(c)}$ in the first sum which do not correspond to edges in the boundary cancel with terms in the second sum, we have $r_{\min}\chi(M) \leq V(\partial M)$. Therefore, since M is a disk, $r_{\min} \leq V(\partial M)$, which implies that words representing the trivial element have length at least $r_{\min}$.

For the second statement, by removing all the relators of length 1 along with the corresponding generators, we can assume that each relator of P has length at least 2. Note that condition $C'(\frac{1}{2})$ guarantees that each of these generators can appear in only one relator. □

In Theorem 3.5.2 we proved the existence of quadratic Dehn functions and solvability of the conjugacy problem for presentations satisfying conditions $\mathcal{T}' - C'(\frac{1}{2})$, provided the relators have the same length. We believe that the result is still valid without the assumption on the lengths of the relators. We discuss now a strategy to prove solvability of the word problem for a wider class of groups. Given a presentation $P = \langle X \mid R \rangle$ of a group G, our aim is to obtain a new presentation P' of a group H such that G embeds in H, and such that P' satisfies conditions $\mathcal{T}' - C'(\frac{1}{2})$ and the relators have the same length. By Theorem 3.5.2, this would imply that H, and therefore G, has solvable word problem. To do so, we will choose a positive integer n_x for some $x \in X$ and replace every occurrence of x in the relators by x^{n_x}. If the element x in the group G has infinite order, this corresponds to adding an n_x-th root, or equivalently, to taking the amalgamated product of G with $\mathbb{Z}$ along the subgroup $n_x\mathbb{Z}$. If we make these replacements for a finite number of $x \in X$, we obtain a new presentation P' of an overgroup H of G. Of course, it is not always possible to choose the n_x so that all the relators in the new presentation have the same length and even if this is possible, the presentation P' obtained may not satisfy conditions $\mathcal{T}' - C'(\frac{1}{2})$ (even if P does). The following example illustrates this technique.

Example 3.5.5. Consider the following presentation

$$P = \langle a, b, c, s, t \mid tats^{-1}b^{-1}s^{-1}, tbts^{-1}c^{-2}s^{-1}, tc^2ts^{-1}a^{-1}s^{-1} \rangle.$$

Note that the relators do not have the same length. This presentation does not satisfy conditions $C(5)$, $T(4)$ nor $\mathcal{T}'$. Now, it is easy to see that a and b have infinite order in the group G presented by P, and by choosing $n_a = 2$ and $n_b = 2$, we obtain the following presentation

$$P' = \langle a, b, c, s, t \mid ta^2ts^{-1}b^{-2}s^{-1}, tb^2ts^{-1}c^{-2}s^{-1}, tc^2ts^{-1}a^{-2}s^{-1} \rangle.$$

Now all the relators have the same length. One can verify that P' satisfies conditions $\mathcal{T}' - C'(\frac{1}{2})$ (although it does not satisfy conditions $C(5)$ nor $T(4)$). This implies that G has solvable word problem.

3.6 Computability of condition $\mathcal{T}'$

In this section we give an algorithm to verify whether a finite presentation $P = \langle X \mid R \rangle$ satisfies condition $\mathcal{T}'$. Note that a priori it is not clear that such an algorithm exists, since the definition involves checking a condition for every possible diagram over P. The algorithm described here has been implemented in the GAP[83] package SmallCancellation [77].

We describe a weighted directed graph $\Gamma(P)$. The vertices of this graph are the tuples (r, p, q) such that

- $r \in R^*$,
- p and q are pieces, and
- we can write $r = qsp$ without cancellations.

There is an edge $(r, p, q) \to (r', p', q')$ if

- $p' = q^{-1}$, and
- $r' \neq r^{-1}$.

The weight of this edge is $1 - \frac{|p|+|q|}{|r|}$ (by simplicity, we divide by π the weights that we considered in Section 3.4). The weight of a cycle is the sum of the weights of its edges.

Given a diagram $\varphi \colon M \to K_P$, we can fix an orientation in M as explained in Section 1.5. The corners in the diagram inherit the orientations of the corresponding faces. Note that if c is a corner at an interior vertex $v(c)$, then $(r(c), w_1(c), w_2(c))$ is a vertex in $\Gamma(P)$. Here $r(c)$ denotes the relator read in the boundary of the face, starting from the vertex $v(c)$ and following the orientation of the face, $w_1(c)$ and $w_2(c)$ are the subwords written in the edges of the oriented corner (the first edge being the one oriented towards $v(c)$). This remark and the following proposition make clear why this graph is meaningful. Essentially, this graph codifies the cycles that appear in interior vertices of reduced diagrams over P.

Proposition 3.6.1. *(i) Let v be an interior vertex in a reduced diagram $\varphi \colon M \to K_P$. Then there is a directed cycle γ in $\Gamma(P)$ of length at least 3 and weight $d(v) - d'_F(v)$.*

(ii) Let γ be a directed cycle in $\Gamma(P)$ of length at least 3 and weight w. Then there is a reduced diagram over P and an interior vertex v such that $d(v) - d'_F(v) = w$.

Proof. We first prove (i). Let v be an interior vertex in a reduced diagram $\varphi \colon M \to K_P$. Let $c_1, \ldots, c_n$ be the corners around v, numbered clockwise.

Then $w_1(c_{i+1}) = w_2(c_i)^{-1}$ (indices are modulo n). Since the diagram is reduced we have $r(c_{i+1})^{-1} \neq r(c_i)^{-1}$ and therefore there is an edge

$$(r(c_i), w_1(c_i), w_2(c_i)) \xrightarrow{e_i} (r(c_{i+1}), w_1(c_{i+1}), w_2(c_{i+1}))$$

in $\Gamma(P)$ with weight $1 - \frac{\ell_1(c_i)+\ell_2(c_i)}{\ell_r(c_i)}$. Then the cycle $\gamma = (e_1, \ldots, e_n)$ has weight $d(v) - d'_F(v)$.

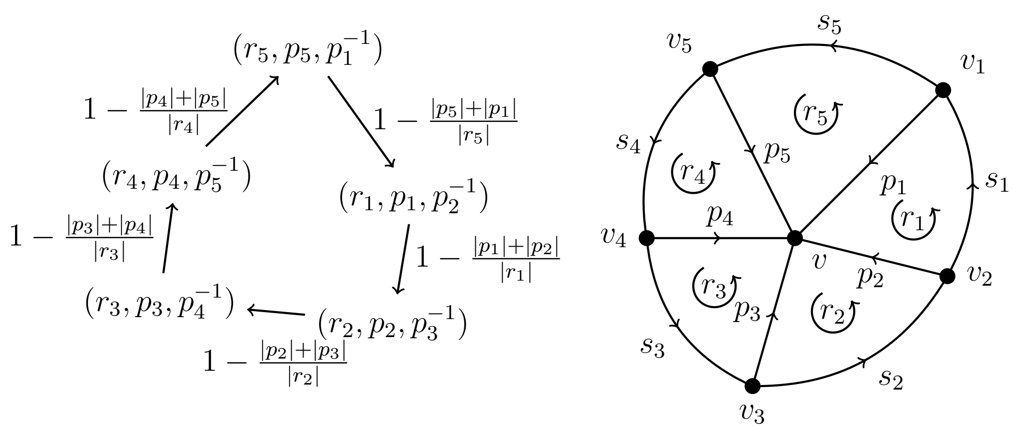

Figure 3.13: On the left a cycle γ in $\Gamma(P)$, on the right the corresponding diagram constructed in the proof of part (ii) of Proposition 3.6.1.

We now prove (ii). Let $n \geq 3$ and let γ be a cycle in $\Gamma(P)$ of length n. By the first condition for the edges of $\Gamma(P)$, the vertices of γ can be named (r_1, p_1, p_2^{-1}), (r_2, p_2, p_3^{-1}), $\ldots$, (r_n, p_n, p_1^{-1}). For each i we consider the word s_i such that $r_i = p_{i+1}^{-1} s_i p_i$ without cancellations. We construct a disk diagram Δ with $n+1$ vertices, $2n$ edges and n faces as follows. The vertices of Δ will be denoted by $v, v_1, \ldots, v_n$. For each i the diagram has an edge $v_i \xrightarrow{e_i} v$ which reads p_i and an edge $v_{i+1} \xrightarrow{\alpha_i} v_i$ which reads s_i. For each i there is a face f_i attached with boundary $(e_{i+1}^{-1}, \alpha_i, e_i)$ which reads r_i (starting at v) (see Figure 3.13). By the second condition for an edge in $\Gamma(P)$, the diagram is reduced. Note that by construction, $d(v) - d'_F(v)$ is the weight of γ. $\square$

Corollary 3.6.2. *A presentation P satisfies condition $\mathcal{T}'$ if and only if each directed cycle in $\Gamma(P)$ of length at least 3 has weight greater than or equal to 2.*

A presentation P satisfies condition $\mathcal{T}'_<$ if and only if each directed cycle in $\Gamma(P)$ of length at least 3 has weight greater than 2.

Note that Corollary 3.6.2 gives an algorithm to check if a finite presentation satisfies $\mathcal{T}'$, for it is possible to use Dijkstra's algorithm to find the least weight of a directed cycle of length at least k in a directed graph with positive edge weights. This can be done by constructing an auxiliary graph having $(k+1)$ vertices for each vertex in the original graph. For more details on this see the implementation in the GAP package SmallCancellation[77].

From Proposition 3.6.1 we deduce the following result, which is used in the proof of Theorem 3.4.4.

Corollary 3.6.3. *If a finite presentation P satisfies condition $\mathcal{T}'_<$ there is a constant $N < 0$ such that $\kappa(v) \leq N$ for every diagram Δ and every interior vertex $v \in \Delta$.*

Proof. Since the weights are positive, we can take N to be $-\pi$ times the minimum weight of a simple directed cycle of length at least 3 in $\Gamma(P)$. Note that, since the graph $\Gamma(P)$ is finite, there is a finite number of such cycles. □

The following examples of groups which do not satisfy $\mathcal{T}'$ are consistent with our conjecture that $\mathcal{T}' - C'(\frac{1}{2})$ implies a quadratic isoperimetric inequality even if the presentation has relators of different lengths.

Example 3.6.4. From [6] we know the Baumslag–Solitar group $BS(p, q)$ has exponential Dehn function if $|p| \neq |q|$. Therefore, by Theorem 3.5.2 the groups $BS(n, n+1)$ do not satisfy $\mathcal{T}'$. It can be seen that the minimum of $d(v) - d'_F(v)$ for v an interior vertex in a diagram for the usual presentation of $BS(n, n+1)$ is $2 - \frac{1}{2n+3}$, which tends to 2 as $n \to \infty$. Note that $BS(n, n)$ satisfies $\mathcal{T}' - C'(\frac{1}{2})$ so by Theorem 3.5.2 one can verify the well-known fact that these groups have quadratic Dehn function.

Example 3.6.5. In [6, Lemma 11] a family of groups $M_{c,d}$ is considered and it is proved that the Dehn function of $M_{c,d}$ has order n^{c+d}. We have that $M_{1,1}$ (which is a RAAG) satisfies $\mathcal{T}' - C'(\frac{1}{2})$. Some GAP computations suggest that the minimum of $d(v) - d'_F(v)$ for these groups is $\frac{23}{20}$ for any $(c, d) \neq (1, 1)$.

Example 3.6.6. In [6] it is proved that the Dehn function of the group $E = \langle b, s, t \mid s^{-1}bs = b^2, t^{-1}bt = b\rangle$ is at least 2^n. This group does not satisfy $\mathcal{T}'$ (the minimum of $d(v) - d'_F(v)$ is $\frac{8}{5}$).

As these examples suggest, it would be interesting to know more about what the minimum of $d(v) - d'_F(v)$ says about a presentation.

3.7 Systolic angled complexes and $\mathcal{T}'$

In Section 3.1 we introduced (strictly) systolic angled complexes. Then we defined a metric small cancelation condition (T') that together with condition $C'(\frac{1}{4})$ assured that one-relator groups were strictly systolic. Conditions $\mathcal{T}'$ and $\mathcal{T}'_<$ were introduced in an attempt to generalize and study conditions $C'(\frac{1}{4}) - (T')$ in a combinatorial fashion. It turns out that we could obtain stronger and more general properties in this way. Even though the approach in the second half of this chapter has been combinatorial, there is a relationship with angled systolic complexes. More precisely we can state the following theorem.

Theorem 3.7.1. *Let P be a finite presentation satisfying conditions $C'(\frac{1}{2}) - \mathcal{T}'$ (resp. $\mathcal{T}'_<$). Then the group presented by P acts geometrically by simplicial automorphisms on a systolic angled complex (resp. strictly systolic angled complex).*

The proof is exactly the same as the proof of Theorem 3.2.1. The only ingredient of the proof that is missing is the fact that 2-cells are embedded in the presentation complex $\tilde{K}_P$. That is, their boundaries have no self-intersections. This follows if no proper subword of a relator is trivial in the presented group (which holds for one-relator groups [86, Theorem 2]). Fortunately we have proved this in Proposition 3.5.4.

Theorem 3.7.1 can be thought of as an angled version of Wise's result that states that $C(6)$ groups are systolic [87].

Chapter 4

Systolicity-by-function and two-dimensional Artin groups

As mentioned in Section 2.3, in this chapter we extend previous results by Cumplido, Martin and Vaskou [31] on parabolic subgroups of large-type Artin groups to a broader family of two-dimensional Artin groups. We prove that an arbitrary intersection of parabolic subgroups of a $(2, 2)$-free two-dimensional Artin group is itself a parabolic subgroup. An Artin group is $(2, 2)$-*free* if its defining graph does not have two consecutive edges labeled by 2. As a consequence of this result, we solve the conjugacy stability problem for this family by applying an algorithm introduced by Cumplido [29]. All of this is accomplished by considering systolic-by-function complexes, which generalize systolic complexes. Systolic-by-function complexes have a more flexible structure than systolic complexes since we allow the edges to have different lengths. At the same time, their geometry is rigid enough to satisfy an analogue of the Cartan–Hadamard theorem and other geometric properties similar to those of systolic complexes. The results of this chapter can be found in [8].

4.1 The Artin complex

In [31] Cumplido, Martin and Vaskou used a geometric approach to solve the problem of intersection of parabolic subgroups for Artin groups of large-type (i.e. those with $m_{st} \geq 3$ for all $s, t \in S$). They introduced a simplicial complex associated to an Artin group, called the Artin complex, on which the Artin group acts cocompactly and without inversions. This complex was also previously defined in [20] under the name of Deligne complex (now the term Deligne complex is commonly reserved for the modified Deligne complex introduced in [20]).

In this section we recall the construction of the Artin complex. We follow the description and notation from [31]. The definitions and notations related to complexes of groups are those of [15, Chapter II.12].

Let A_Σ be an Artin group with generator set Σ (with $|\Sigma| \geq 2$). Take K a simplex of dimension $|\Sigma|-1$ and define a simplex of groups over K. First, give the simplex K a trivial local group. Simplices of codimension 1 are in one-to-one correspondence with elements $\sigma_i \in \Sigma$, and are denoted by Δ_{σ_i}. The simplex Δ_{σ_i} is given the local group $\langle \sigma_i \rangle$. Now every simplex of codimension k is in one-to-one correspondence with a subset of Σ of cardinality k. Given $\Sigma' \subset \Sigma$ with $|\Sigma'| = k$, its corresponding face can be written uniquely as

$$\Delta_{\Sigma'} = \cap_{\sigma_i \in \Sigma'} \Delta_{\sigma_i}.$$

The simplex $\Delta_{\Sigma'}$ is given the local group $A_{\Sigma'}$.

Given an inclusion $\Delta_{\Sigma''} \subset \Delta_{\Sigma'}$ there is a natural inclusion $\psi_{\Sigma'\Sigma''} : A_{\Sigma'} \to A_{\Sigma''}$. Let $\mathcal{P}$ be the poset of standard parabolic subgroups of A_Σ with the order given by the natural inclusions. Since every standard parabolic subgroup is itself an Artin group [65], there is a simple morphism $\varphi : G(\mathcal{P}) \to A_\Sigma$, given by inclusion, from the complex of groups to A_Σ.

Definition 4.1.1. The *Artin complex* associated to A_Σ is the development $X_\Sigma := D_K(\mathcal{P}, \varphi)$ of $\mathcal{P}$ over K along φ ([15, Theorem II.12.18]).

In the proof of [15, Theorem II.12.18], an explicit description of X_Σ is given. The simplicial complex X_Σ can be defined as

$$X_\Sigma := A_\Sigma \times K/\sim,$$

where $(g, x) \sim (g', x')$ if and only if $x = x'$ and $g^{-1}g'$ is in the local group of the smallest simplex of K containing x.

The action of A_Σ in X_Σ is by simplicial isomorphisms, without inversions and cocompact, with strict fundamental domain K. Any simplex Δ of X_Σ is in the orbit of exactly one $\Delta_{\Sigma'} \subset K$ for some $\Sigma' \subset \Sigma$. In that case Δ is said to be *of type* Σ'. We now recall some results from [31] about the complex X_Σ.

Lemma 4.1.2 ([31], Lemma 4)**.** *Let A_Σ be an Artin group and let X_Σ be its Artin complex. Then X_Σ in connected. Additionally, if $|\Sigma| \geq 3$, then X_Σ is simply connected.*

Lemma 4.1.3 ([31], Lemma 6)**.** *Let A_Σ be an Artin group with Artin complex X_Σ. The link of a simplex of type Σ' is isomorphic to the Artin complex $X_{\Sigma'}$ associated to the Artin group $A_{\Sigma'}$.*

Lemma 4.1.4 ([31], Lemma 9)**.** *Let A_Σ be an Artin group with $\Sigma = \{\sigma_x, \sigma_y\}$. Then any cycle in X_Σ has at least $2m_{xy}$ edges, and it is a tree if $m_{xy} = \infty$.*

Remark 4.1.1. *In [31], they show the previous result for $m_{xy} \in \{3, 4, \dots, \infty\}$, since they work with large-type Artin groups. However, the result also holds for the case $m_{xy} = 2$, and the proof is the same as in the other cases.*

Now we state the connection between the Artin complex and the intersection of parabolic subgroups. This result gives a novel approach to the problem of intersection of parabolic subgroups.

Theorem 4.1.5 ([31], Theorem 11, Remark 15, Corollary 16)**.** *Let A_Σ be an Artin group and X_Σ its Artin complex. If any time an element of A_Σ fixes two vertices of X_Σ it fixes pointwise a combinatorial path joining them, then an arbitrary intersection of parabolic subgroups of A_Σ is a parabolic subgroup of A_Σ.*

With this theorem in mind, the question can now be answered in a completely geometric way. In order to show that large-type Artin groups satisfy the conditions of the theorem, Cumplido, Martin and Vaskou proved that their Artin complexes are systolic in the sense of [61]. Then they used the fact that if a group G acts without inversions on a systolic complex and fixes two vertices, then it fixes pointwise every combinatorial geodesic between them ([31, Lemma 14]).

We want to generalize Cumplido, Martin and Vaskou's result to a broader class of two-dimensional Artin groups. This will be accomplished by considering a geometric structure more flexible than systolicity. This flexibility allows us to include a broader family of examples, while maintaining a rigid enough geometry.

4.2 Systolic-by-function complexes

In this section we define systolic-by-function complexes, which are a generalization of systolic complexes. We prove some basic properties and a local-to-global theorem analogous to the Cartan–Hadamard theorem. In Section 4.3 we will make use of this geometric structure to prove the path fixing condition required by Theorem 4.1.5.

Definition 4.2.1. A *length function* for a simplicial complex X is a function $l : \text{edges}(X) \to [0, \frac{1}{2}]$ that assigns a real number between 0 and $\frac{1}{2}$ to each edge of X, satisfying the two following conditions:

- the sum of the lengths of the three edges of any triangle is less than or equal to 1;
- the triangle inequality holds. That is, given three edges e_0, e_1, e_2 that form a triangle, $l(e_i) \leq l(e_{i+1}) + l(e_{i+2})$ (indices modulo 3).

A simplicial complex together with a length function is called a *length complex*.

A *cycle* in a (length) complex X is a subcomplex σ homeomorphic to S^1. We denote by $|\sigma|$ the number of edges in σ. The length of σ is the sum of the lengths of its edges, and we denote it by $l(\sigma)$. A *path* in X is a subcomplex γ homeomorphic to $[0, 1]$. We define $|\gamma|$ and $l(\gamma)$ analogously.

We recall some definitions from Section 1.3. A subcomplex K of a simplicial complex X is *full* if any simplex of X spanned by a set of vertices in K is a simplex of K. A *diagonal* in a cycle σ in a simplicial complex X is an edge of X connecting two nonconsecutive vertices of σ. Thus, a cycle is full if and only if it has no diagonals and does not span a simplex. A simplicial complex X is *flag* if every set of vertices pairwise connected by edges spans a simplex of X.

Definition 4.2.2. A length complex X is *large* if it is flag and if every full cycle has length greater than or equal to 2. It is *locally large* if the link of every vertex is large.

It is clear from the definitions that a large length complex is locally large. This is because, since the complex is flag, the links of its vertices are flag and full cycles in the links are full cycles in the complex. The rest of this section is devoted to showing that the converse holds when X is simply connected. This is a local-to-global theorem analogous to the classical result for systolic complexes [61].

Definition 4.2.3. A length complex X is *systolic-by-function* if it is connected, simply connected and locally large.

Remark 4.2.1. *A simplicial complex is systolic if and only if it is systolic-by-function with constant length function $l \equiv \frac{1}{3}$. In general, a simplicial complex is k-systolic if and only if it is systolic-by-function with constant length function $l \equiv \frac{2}{k}$.*

Theorem 4.2.4. *Let X be a systolic-by-function length complex. Then X is large.*

In order to prove this theorem, we will have to study the structure of diagrams over a systolic-by-function complex. A *diagram* Δ in X is a simplicial map $\varphi : M \to X$. If M is a simplicial structure of a 2-dimensional disk, we say that Δ is a *disk diagram.* A simplicial map is called *nondegenerate* if it is injective in every simplex.

Lemma 4.2.5 ([61], Lemma 1.6)**.** *Let X be a simplicial complex, and σ a homotopically trivial cycle in X. Then there exists a nondegenerate disk diagram $\varphi : D \to X$, which maps the boundary of D isomorphically onto σ.*

Such a diagram is called a *filling diagram* for σ. In a simply connected length complex, the previous lemma implies that every cycle has a filling diagram. To understand these diagrams, we will recall some basic notions of combinatorial curvature. These definitions are analogous to the ones used in Chapter 3, but in the context of length complexes.

Let X be a 2-dimensional length complex. If v is a vertex of X, its curvature is defined as

$$\kappa(v) = 2 - \chi(\mathrm{lk}_X(v)) - \sum_{e \in \mathrm{lk}_X(v)} l(e).$$

Here $\chi(\mathrm{lk}_X(v))$ denotes the Euler characteristic of the link of v. We define the curvature of a 2-simplex f of X as

$$\kappa(f) = \left(\sum_{e \in \partial f} l(e)\right) - 1,$$

where ∂f is the boundary of f and the sum is over its three edges. Note that the curvature of a face is always non-positive.

Theorem 4.2.6 (Combinatorial Gauss–Bonnet Theorem for Length Complexes)**.** *Let X be a 2-dimensional length complex. Then*

$$\sum_{f \in faces(X)} \kappa(f) + \sum_{v \in vertices(X)} \kappa(v) = 2\chi(X).$$

Note that the above formulas are not exactly equal to those of Chapter 3. The size of an angle can be thought of as the length of the side opposite to it, multiplied by π. We omit the factor of π for simplicity.

Definition 4.2.7. Let σ be a cycle in a simplicial complex X. A filling diagram $\varphi : D \to X$ for σ is *minimal* if D has the least amount of 2-simplices among all filling diagrams for σ.

Observe that if $\varphi : D \to X$ is a minimal filling diagram for a cycle σ, it is nondegenerate: if an edge e were mapped to a vertex, we could take the two triangles containing e, delete the interior of their union and glue the remaining four edges, thus obtaining a filling diagram for σ with fewer 2-simplices. Given a nondegenerate diagram $\varphi : M \to X$, where X is a length complex, we can pull back the length function to M, so that M is a length complex itself.

Lemma 4.2.8. *Let X be a large length complex and σ a cycle in X of length less than 2. Then there exists a filling diagram $\varphi : D \to X$ for σ such that D has no interior vertices.*

Proof. We proceed by induction on $|\sigma|$. If $|\sigma| = 3$, then the result follows by flagness. Now suppose $|\sigma| > 3$. Since X is large, σ cannot be full. Then σ has a diagonal e that connects two nonconsecutive vertices of σ. This edge subdivides σ in two paths, both with less than $|\sigma| - 1$ edges. We call them

σ_1 and σ_2. Attaching e to both σ_1 and σ_2, we get two cycles with fewer edges than σ. By the triangle inequality $l(\sigma_i \cup e) \leq l(\sigma)$ for $i = 1, 2$. By inductive hypothesis, there exist filling diagrams without interior vertices for both cycles. Gluing these two diagrams along the two edges mapped to e we obtain the desired diagram for σ. □

Lemma 4.2.9. *Let σ be a homotopically trivial cycle in a locally large length complex X. Then for any minimal filling diagram $\varphi : D \to X$ for σ, D is locally large when considered with the pullback length.*

Proof. Let $\varphi : D \to X$ be a minimal filling diagram for σ. Suppose there is an interior vertex v of D such that $\mathrm{lk}_D(v)$ is not large. Since D is simplicial, $\mathrm{lk}_D(v)$ is a full cycle in D (that has length less than 2). Consider $\varphi(\mathrm{lk}_D(v))$ as a cycle in $\mathrm{lk}_X(\varphi(v))$. Since X is locally large, $\mathrm{lk}_X(\varphi(v))$ is a large length complex. Thus, by Lemma 4.2.8, there is a filling diagram $\psi : D' \to \mathrm{lk}_X(\varphi(v)) \subset X$ for $\varphi(\mathrm{lk}_D(v))$ with no interior vertices. There are $|\mathrm{lk}_D(v)|$ closed 2-simplices in D that contain v. Since ψ is a filling diagram for $\varphi(\mathrm{lk}_D(v))$ and D' has no interior vertices, the number of 2-simplices in D' is $|\varphi(\mathrm{lk}_D(v))| - 2 < |\mathrm{lk}_D(v)|$. Therefore, if we replace the set of closed 2-simplices of D that contain v by this new diagram, we obtain a filling diagram for σ with fewer 2-simplices, which is a contradiction. Hence D is locally large. □

Remark 4.2.2. *If $\varphi : M \to X$ is a nondegenerate disk diagram, then being locally large is equivalent to $\kappa(v) \leq 0$ for every interior vertex v of M. This is because, for an interior vertex v, $\kappa(v) = 2 - \sum_{e \in \mathrm{lk}_M(v)} l(e)$.*

Let $\varphi : D \to X$ be a disk diagram. The *boundary layer* L of D consists of every vertex in the boundary of D, every edge incident to a vertex in the boundary, and every open 2-simplex whose closure has a vertex in the boundary. Here we consider open 2-simplices because we do not want edges that are not incident to a vertex in the boundary to be part of the boundary layer. Note that L is usually not a simplicial complex. If D has at least two interior vertices, and no edge connecting nonconsecutive vertices of the boundary, we define the following complex. Consider the simplicial complex A constructed by taking the disjoint union of the vertices, edges and 2-simplices (now closed) of the boundary layer of D and identifying the boundaries of the closed 2-simplices but only in the vertices and edges of the boundary layer of D (see Figure 4.1). Since D has more than one interior vertex and no edge connecting nonconsecutive vertices of the boundary, A is an annulus without interior vertices. We call A the *boundary complex* of D. It has two boundary components $\partial_1 A$ and $\partial_2 A$, the first of which is isomorphic to ∂D. If D is a length complex, then A is a length complex with the induced length. Note that if D has exactly two interior vertices, its boundary complex A is not a simplicial complex, since it has a double edge. However, it is easy to see that all the definitions and results can be adapted to this case.

Note that if the disk had only one interior vertex, its boundary complex would be the disk itself. This is why that case is excluded from the previous definition.

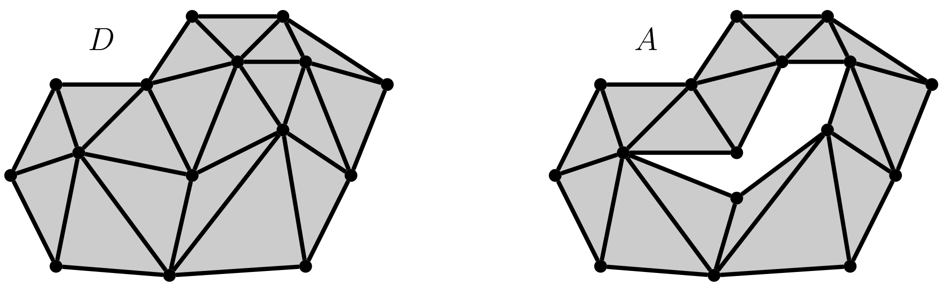

Figure 4.1: A disk D and its boundary complex A

Lemma 4.2.10. *Let $\varphi : D \to X$ be a minimal filling diagram for a cycle σ in a locally large length complex X, where D has at least two interior vertices, and no edge connecting nonconsecutive vertices of the boundary. Let A be the boundary complex of D. Then:*

$$l(\partial_1 A) \geq l(\partial_2 A) + 2.$$

Proof. We apply Gauss–Bonnet to D and A to obtain (after simplifying the notation of the indices of the sums)

$$2 = \sum_{f \in D} \kappa(f) + \sum_{v \in D} \kappa(v) \leq \sum_{f \in A} \kappa(f) + \sum_{v \in \partial D} \kappa(v) = \sum_{f \in A} \kappa(f) + \sum_{v \in \partial_1 A} \kappa(v),$$

$$0 = \sum_{f \in A} \kappa(f) + \sum_{v \in \partial_1 A \cup \partial_2 A} \kappa(v).$$

The first and last equalities hold because the Euler characteristic of a disk and an annulus are 1 and 0 respectively. The first inequality is due to the fact that the curvature of faces and interior vertices is always non-positive.

By taking the double of the first expression and subtracting the second expression we get

$$\begin{aligned} 4 &\leq 2\left(\sum_{f\in A}\kappa(f)+\sum_{v\in\partial_1 A}\kappa(v)\right)-\sum_{f\in A}\kappa(f)-\sum_{v\in\partial_1 A\cup\partial_2 A}\kappa(v) \\ &= \sum_{f\in A}\kappa(f)+\sum_{v\in\partial_1 A}\kappa(v)-\sum_{v\in\partial_2 A}\kappa(v). \end{aligned} \tag{4.1}$$

Observe that the Euler characteristic of the link of a vertex in the boundary is equal to 1. We note by F_1 and F_2 the sets of 2-simplices of A having one edge in $\partial_1 A$ and $\partial_2 A$ respectively. For a face f in F_1 or F_2 we denote its three sides by e_1^f, e_2^f and e_3^f, where e_1^f is the one lying in the corresponding boundary component. We also denote their respective lengths by l_1^f, l_2^f and l_3^f. Note that the cardinality of F_i is $|\partial_i A|$ for $i = 1, 2$. From this we have

$$\begin{aligned} 4 &\leq \sum_{f\in A}\left(\left(\sum_{e\in\partial f} l(e)\right)-1\right)+\sum_{v\in\partial_1 A}\left(1-\sum_{e\in Lk_A(v)} l(e)\right) \\ &\quad -\sum_{v\in\partial_2 A}\left(1-\sum_{e\in Lk_A(v)} l(e)\right) \\ &= \sum_{f\in F_1}\left(\left(\sum_{e\in\partial f} l(e)\right)-1+1+l_1^f-l_2^f-l_3^f\right) \\ &\quad +\sum_{f\in F_2}\left(\left(\sum_{e\in\partial f} l(e)\right)-1-1-l_1^f+l_2^f+l_3^f\right) \\ &= \sum_{f\in F_1} 2l_1^f+\sum_{f\in F_2}\left(-2+2l_2^f+2l_3^f\right) \\ &\leq \sum_{f\in F_1} 2l_1^f+\sum_{f\in F_2} -2l_1^f \\ &= 2l(\partial_1 A)-2l(\partial_2 A). \end{aligned}$$

The first equality is a rearrangement of the terms using the new notation and the remark about the cardinalities of the F_i. The last inequality holds because the sum of the lengths of the sides of any 2-simplex is less than or equal to 1. Dividing both sides by 2, we obtain the desired inequality $l(\partial_1 A) \geq l(\partial_2 A) + 2$. □

We now have all the necessary ingredients to prove Theorem 4.2.4.

Proof of Theorem 4.2.4. We have to show that every full cycle in X has length greater than or equal to 2, and that X is flag. Let σ be a full cycle in X. Since X is simply connected, by Lemma 4.2.5 there is a minimal filling diagram for σ, say $\varphi : D \to X$. We know that φ is nondegenerate because it is minimal. Hence by Lemma 4.2.9, D is locally large. Since σ is full, there are no edges in D connecting nonconsecutive vertices of its boundary, and D has at least one interior vertex. If D has only one interior vertex v we have

$$0 \geq \kappa(v) = 2 - \sum_{e \in \mathrm{lk}_D(v)} l(e) = 2 - l(\sigma).$$

Therefore $l(\sigma) \geq 2$. If D has more than one interior vertex, then we are under the hypotheses of Lemma 4.2.10, and $l(\sigma) = l(\partial_1 A) \geq 2$.

Now we show that X is flag. We are going to see that it suffices to show that every cycle with three edges spans a 2-simplex in X. Indeed, suppose we have vertices $v_1, \ldots, v_n$ that are pairwise connected. If every triangle is filled, we have the 1-skeleton of an $(n-1)$-simplex in $\mathrm{lk}_X(v_1)$. Since the links of the vertices are flag, $v_1, \ldots, v_n$ must span an n-simplex in X.

Take a cycle σ with three edges, and let $\varphi : D \to X$ be a minimal filling diagram for σ. If D has more than one interior vertex, then by Lemma 4.2.10, $l(\sigma) \geq 2$, which is impossible. If D has exactly one interior vertex v, then just as before

$$0 \geq \kappa(v) = 2 - \sum_{e \in \mathrm{lk}_D(v)} l(e) = 2 - l(\sigma).$$

Once again, this would imply that $l(\sigma) \geq 2$. So the only possibility is that D has no interior vertices. Hence, σ spans a 2-simplex in X. □

4.3 Parabolic subgroups

We now define a length function for the Artin complex of a given $(2,2)$-free two-dimensional Artin group, and show that it is a systolic-by-function length complex. Then we make use of its geometric structure to prove the following theorem. The definition of geodesic used in the theorem will be introduced once we have defined the length function for the Artin complex.

Theorem 4.3.1. *Let A_Σ be a $(2,2)$-free two-dimensional Artin group with $|\Sigma| \geq 3$ and X_Σ its Artin complex. Let u and v be vertices of X_Σ. Then there exists a path joining u and v such that if an element of A_Σ fixes u and v, it fixes this path pointwise.*

As an immediate consequence, the results of Theorem 4.1.5 hold for all $(2,2)$-free two-dimensional Artin groups (the cases with less than 3 generators were established in [30, 31]). In particular, we derive the main result of this chapter.

Theorem 4.3.2. *Let A_Σ be a $(2,2)$-free two-dimensional Artin group. Then the intersection of an arbitrary family of parabolic subgroups is a parabolic subgroup.*

We start by proving a characterization of $(2,2)$-free two-dimensional Artin groups.

Proposition 4.3.3. *A two-dimensional Artin group A_Γ is $(2,2)$-free if and only if there exist numbers $m'_{xy} \in \{2,3,4,6\}$ with $m'_{xy} \leq m_{xy}$ and $m'_{xy} = m'_{yx}$ for every $x, y \in V(\Gamma)$, such that $\frac{1}{m'_{xy}} + \frac{1}{m'_{yz}} + \frac{1}{m'_{zx}} \leq 1$ and $\frac{1}{m'_{xy}} \leq \frac{1}{m'_{yz}} + \frac{1}{m'_{zx}}$ for every $x, y, z \in V(\Gamma)$.*

Proof. It is clear that if such m'_{xy} exist, then A_Σ is $(2,2)$-free. Now suppose that Γ does not have two consecutive edges labeled by 2. Then we can define the m'_{xy} in the following way:

- if $m_{xy} = 2$, then $m'_{xy} = 2$;
- if $m_{xy} = 3$, then $m'_{xy} = 3$;
- if $m_{xy} > 3$ and the edge is not adjacent to an edge labeled by 2, then $m'_{xy} = 3$;
- if m_{xy} forms a triangle with a 2 and a 3, then $m'_{xy} = 6$;
- in any other case, $m'_{xy} = 4$.

It is easy to see that such labeling is well defined since A_Σ is a $(2,2)$-free two-dimensional Artin group, and that it satisfies the required conditions. □

Let A_Σ be a $(2,2)$-free two-dimensional Artin group. We define a length function for X_Σ, $l : \mathrm{edges}(X_\Sigma) \to [0, \frac{1}{2}]$ as follows. Edges in X_Σ are simplices of codimension $|\Sigma| - 2$, so they correspond to subsets of Σ that are missing two elements. We define the length of an edge of type $\Sigma \backslash \{\sigma_x, \sigma_y\}$ to be $\frac{1}{m'_{xy}}$, where the m'_{xy} are the ones in Proposition 4.3.3. Since A_Σ is $(2,2)$-free and two-dimensional, the sum of the lengths of the three edges of every triangle is less than or equal to 1, and the triangle inequality holds, so l is well defined.

Theorem 4.3.4. *Let A_Σ be a $(2,2)$-free two-dimensional Artin group with $|\Sigma| \geq 3$. Then X_Σ with the length function defined as above is systolic-by-function.*

Proof. The proof proceeds by induction on $|\Sigma|$ by using our local-to-global Theorem 4.2.4.

If $|\Sigma| = 3$, by Lemma 3.2.3 X_Σ is connected and simply connected. Let v be a vertex of X_Σ. Lemma 4.1.3 says that $\mathrm{lk}_{X_\Sigma}(v)$ is isomorphic to the Artin complex $X_{\Sigma'}$ associated to the Artin group $A_{\Sigma'}$, where $\Sigma' \subset \Sigma$ with $|\Sigma'| = 2$. The complex $X_{\Sigma'}$ is a graph and it inherits the length function from X_Σ. If $X_{\Sigma'}$ is a tree, then it is clearly a large length complex. If it is not a tree, then by Proposition 4.3.3, the length of its edges is greater than or equal to $\frac{1}{m}$, where m is the label in $A_{\Sigma'}$. By Lemma 4.1.4 all cycles in $X_{\Sigma'}$ have at least $2m$ edges, so it is a large length complex. Thus X_Σ is systolic-by-function.

Now assume that $|\Sigma| > 3$ and that the claim holds for every $(2,2)$-free two-dimensional Artin group with fewer generators. Once again, we know X_Σ is connected and simply connected from Lemma 3.2.3. Applying Lemma 4.1.3 we get that the link of every vertex is systolic-by-function. Hence, by Theorem 4.2.4, the link of every vertex is large. Therefore X_Σ is systolic-by-function. □

With the length function defined as above, one can give X_Σ a metric such that it is metrically systolic in the sense of [57] (see Section 1.4). Concretely, if an edge $e \in X_\Sigma$ has $l(e) = \frac{1}{k}$, then the length of e in the metric is $\sin(\frac{\pi}{k})$. More generally, if X is a systolic-by-function complex with length function l, where the image of l is finite and l is positive, then X can be made into a metrically systolic complex. Just as in the Artin complex, the length of an edge e in the metric is $\sin(\pi l(e))$. This assignment works because of the law of the sines. We will not use metric systolicity in this chapter, but this fact may be of interest for other applications.

We define the *distance* between two vertices $u, v \in X_\Sigma$ as

$$d(u,v) = \min\{l(\gamma) \mid \gamma \text{ is a path connecting } u \text{ and } v\}.$$

Note that this minimum is attained, because X_Σ is connected and the image of l is a finite subset of $[0, \frac{1}{2}]$. We say that a path γ between u and v is a *geodesic* if it is of minimum length.

We are now ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Let u and v be vertices of X_Σ. We want to show that for every pair of vertices $u, v \in X_\Sigma^{(0)}$ there exists a path joining u and v such that if an element $g \in A_\Sigma$ fixes both u and v, it fixes pointwise this path. Suppose it is not the case. Take vertices u and v such that there is no such path and such that $d(u,v)$ is minimal among such pairs. Let γ be a geodesic between them (it exists because X_Σ is connected), and $g \in A_\Sigma$ an element that fixes u and v, but not γ. Then g maps γ to another geodesic γ' between u and v. Since $d(u,v)$ is minimal, the union of γ and γ' determines a cycle in X_Σ. We will show that we can fill the cycle with a minimal filling diagram and find a shortcut (i.e. a path shorter than γ) between u and v, contradicting the fact that γ is a geodesic.

Let $\varphi : D \to X_\Sigma$ be a minimal filling diagram for the concatenation of γ and γ' (it exists because X_Σ is simply connected). We label the vertices of γ in D as $u = v_0, v_1, \ldots, v_{|\gamma|-1}, v_{|\gamma|} = v$, and the vertices of γ' in D as $u = v'_0, v'_1, \ldots, v'_{|\gamma'|-1}, v'_{|\gamma'|} = v$. Note that

1. we may assume, without loss of generality, that there is no edge between nonconsecutive vertices of γ, or between nonconsecutive vertices of γ', because γ is a geodesic, and

2. there is no edge between v_i and v'_i for $1 \leq i \leq |\gamma| - 1$, because vertices of the same type are not connected by an edge in X_Σ.

However, there may be an edge between v_i and v'_j if $i \neq j$. Take the rightmost of these edges. We call it e and assume that it connects v_k with

v'_{k+r} (see Figure 4.2). Let $\tilde{D}$ be the disk delimited by e, and let $\tilde{\gamma}$ and $\tilde{\gamma'}$ be the paths connecting v_k and v along the boundary of $\tilde{D}$, where $\tilde{\gamma'}$ contains e. If there is no such edge e, we consider $\tilde{D} = D$, $\tilde{\gamma} = \gamma$ and $\tilde{\gamma'} = \gamma'$, and continue in the same way. It is clear that $\tilde{\gamma}$ is a geodesic. Then $l(e) \geq d(v_k, v_{k+r})$. We also have that $l(e) \leq d(v_k, v_{k+r})$. To see so, we project both e and the path in $\tilde{\gamma}$ between v_k and v_{k+r} to the fundamental domain K, and apply the triangle inequality $r-1$ times. Hence, $l(\tilde{\gamma'}) = l(\tilde{\gamma})$. From the previous observations, $\tilde{D}$ has no edges connecting nonconsecutive boundary vertices. Therefore, it has at least one interior vertex. As in the proof of Theorem 4.2.4 we get that $l(\partial\tilde{D}) = l(\tilde{\gamma}) + l(\tilde{\gamma'}) \geq 2$.

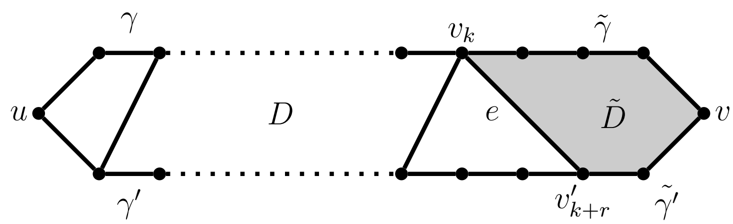

Figure 4.2: Disk D, the rightmost edge e and disk $\tilde{D}$

If either v_k or v have degree greater than 3, then by Lemma 4.2.10 we could find a path in $\tilde{D}$ connecting v_k and v shorter than $\tilde{\gamma}$. That would contradict the fact that γ is a geodesic. So we can assume that both v_k and v have degree 3 (it is clear that they cannot have degree 2). Let e_1 and e_2 be the interior edges incident to v_k and v respectively. If either of them has length less than $\frac{1}{2}$, then either by Lemma 4.2.10, in case $\tilde{D}$ has more than one interior vertex, or by the inequality $l(\partial\tilde{D}) \geq 2$ if $\tilde{D}$ has exactly one interior vertex, we can find a shortcut and get a contradiction.

The only situation remaining is when v_k and v have degree 3, and $l(e_1) = l(e_2) = \frac{1}{2}$. In that case we can find a path σ in $\tilde{D}$ starting at v_k and ending with e_2, with $l(\sigma) \leq l(\tilde{\gamma})$. If the inequality is strict the proof is finished. If $l(\sigma) = l(\tilde{\gamma})$, consider the geodesic τ from u to v consisting of concatenating the subpath of γ that goes from u to v_k with $\varphi(\sigma)$. We can assume $\varphi(\sigma)$ is a path in X_Σ, because otherwise we would have found a shortcut. Since this geodesic ends with an edge of length $\frac{1}{2}$ and no triangle has two edges of length $\frac{1}{2}$, applying the same procedure to τ gives us a path between u and v shorter than γ, obtaining the desired contradiction. □

Theorem 4.3.2 follows immediately from Theorems 4.1.5 and 4.3.1. Applying this Theorem 4.3.2 together with previous results we get following corollary.

Corollary 4.3.5. *Let A_Σ be an Artin group with at most three generators. Then the intersection of an arbitrary family of parabolic subgroups of A_Σ is a parabolic subgroup of A_Σ.*

Proof. Such an Artin group is either spherical type, right-angled or $(2,2)$-free two-dimensional. Therefore, either by [30] for the spherical case; by [38] for the right-angled case; or by Theorem 4.3.2 for the $(2,2)$-free two-dimensional case, we get the desired result. □

Combining recent work of Möller, Paris and Varghese [69] with Theorem 4.3.2, we get the a partial result for all two-dimensional Artin groups.

Corollary 4.3.6. *Let A_Γ be a two-dimensional Artin group. Let P_1 be a parabolic subgroup corresponding to a complete subgraph of Γ, and let P_2 be an arbitrary parabolic subgroup. Then $P_1 \cap P_2$ is a parabolic subgroup of A_Γ.*

Unfortunately, a result analogous to Theorem 4.3.1 does not hold for all Artin groups. Hence, this method of finding a fixed path in the Artin complex does not work as a unifying approach to the problem of intersection of parabolic subgroups. We give two examples where it fails. One of them is not $(2,2)$-free and the other one is not two-dimensional.

Example 4.3.7. Consider first the example in the left of Figure 4.3. The label n can be taken to be ∞, in which case the corresponding Artin group is two-dimensional, but still not $(2,2)$-free. In any case, the element $\sigma_x\sigma_z$ fixes all vertices of type $\langle\sigma_x,\sigma_z\rangle$. However it does not fix any vertex of type $\langle\sigma_x,\sigma_y\rangle$ or $\langle\sigma_y,\sigma_z\rangle$. Hence it does not fix any path between vertices of type $\langle\sigma_x,\sigma_z\rangle$.

Now lets look at the example on the right. This Artin group is $(2,2)$-free, but not two-dimensional. The element $\sigma_a\sigma_b$ fixes vertices $\langle\sigma_a,\sigma_b\rangle$ and $\sigma_c\sigma_a\sigma_b\sigma_c\langle\sigma_a,\sigma_b\rangle$, but does not fix any path joining them.

In both examples it can be easily checked by hand that the proposed elements work. Nonetheless, there is a geometric intuition behind them. When the Artin group is spherical, the corresponding Coxeter group is finite. The associated Coxeter complex (which is defined in the same fashion as the Artin complex, but considering the Coxeter group) is a sphere. If vertices in the Artin complex were to be fixed by an automorphism, then the corresponding elements in the Coxeter complex should also be fixed. What we are doing in these examples is picking opposite poles in the Coxeter complex and rotating along the axis that connects them.

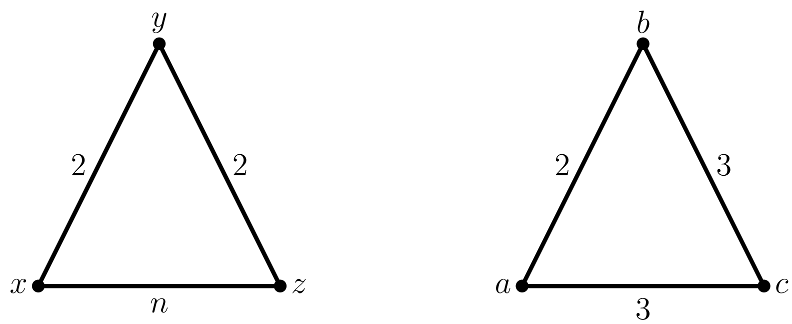

Figure 4.3: Graphs of the failing examples.

We finish this chapter by applying our results and the algorithm introduced in [29, Algorithm 4] to solve the conjugacy stability problem for $(2,2)$-free two-dimensional Artin groups. A subgroup H of a group G is *conjugacy stable* if, for every pair $h, h' \in H$ such that there exists $g \in G$ with $g^{-1}hg = h'$, there is $\tilde{h} \in H$ such that $\tilde{h}^{-1}h\tilde{h} = h'$. The conjugacy stability problem consists in deciding which of the parabolic subgroups of an Artin group are conjugacy stable. We follow the notation and definitions of [29].

Theorem 4.3.8 ([29], Theorem A)**.** *Let A_Σ be a standardisable Artin group satisfying the ribbon property, and such that every element in A_Σ admits a parabolic closure. Then, there is an algorithm that decides if a parabolic subgroup P of A_Σ is conjugacy stable or not.*

It is known by results of Godelle [47] that two-dimensional Artin groups are standardisable and satisfy the ribbon property. Also, by Theorem 4.3.1 and Theorem 4.1.5, any element in a $(2,2)$-free two-dimensional Artin group has a parabolic closure. Therefore, $(2,2)$-free two-dimensional Artin groups satisfy the hypothesis of Theorem 4.3.8. By examining the aforementioned algorithm in the $(2,2)$-free two-dimensional case we obtain the following classification theorem.

Theorem 4.3.9. *Let A_Γ be a $(2,2)$-free two-dimensional Artin group and A_{Γ_X} a standard parabolic subgroup. Then A_{Γ_X} is not conjugacy stable if and only if there exist vertices x, y in Γ_X that are connected by an odd-labeled path in Γ, but are not connected by an odd-labeled path in Γ_X.*

Proof. We need to understand [29, Algorithm 4] in the case of a $(2,2)$-free two-dimensional Artin group A_Γ and a standard parabolic subgroup A_{Γ_X}. Since A_Γ is two-dimensional, the only spherical-type parabolic subgroups are dihedral Artin groups of type $I_2(m)$. Thus, the algorithm reduces to checking if there exist vertices x, y in Γ_X that are connected by an odd-labeled path in Γ, but are not connected by an odd-labeled path in Γ_X. This is exactly the criterion we wanted to prove. $\square$

For a more detailed proof of this fact, see [31, Theorem C]. Their proof is for large-type Artin groups, but it also works in the $(2,2)$-free two-dimensional case.

Chapter 5

Parabolics inside parabolics

In this chapter we prove Theorem 2.3.1. That is, we show that a parabolic subgroup of an Artin group which is contained in another parabolic subgroup is a parabolic subgroup of said parabolic subgroup. The results presented are joint work with Luis Paris and can be found in [11].

Theorem 2.3.1 was proved by Rolfsen in [76] and by Fenn, Rolfsen and Zhu in [40] for braid groups, by Paris in [73] and by Godelle in [45] for Artin groups of spherical type, by Godelle in [46] for Artin groups of FC type, by Godelle in [47] for two-dimensional Artin groups and by Haettel in [51] for some Euclidean type Artin groups. Our proof is independent from these works and it is valid for all Artin groups. Notice that results proved for all Artin groups are quite uncommon in the literature, so our theorem is in some sense a rarity.

We follow the notation introduced in Chapter 2. Let $X \subset V(\Gamma)$. In order to achieve our goal we construct a set-retraction $\pi_X : A_\Gamma \to A_X$ to the inclusion map $A_X \hookrightarrow A_\Gamma$ (see Proposition 5.1.3). This map is defined directly on the words that represent the elements of A, but it is not a homomorphism, although its restriction to the so-called colored subgroup is a homomorphism. The construction of this map is interesting by itself and it can be considered as an important result of the chapter. However, we underline that this construction is implicit in the proof of Theorem 1.2 of Charney and Paris [22] and our contribution consists in making it explicit.

5.1 The proofs

Let X be a subset of $V(\Gamma)$. We set $S_X = \{s_x \mid x \in X\}$ and we denote by W_X the subgroup of W_Γ generated by S_X. We know by Bourbaki [12] that W_X is naturally isomorphic to W_{Γ_X}, hence, as for Artin groups, we will not differentiate W_X from W_{Γ_X}. The subgroup W_X is called a *standard parabolic subgroup* of W_Γ and a subgroup conjugate to W_X is called a *parabolic subgroup* of W_Γ.

We denote by $\theta : A_\Gamma \to W_\Gamma$ the natural epimorphism which sends σ_x to s_x for all $x \in V(\Gamma)$. The kernel of θ is denoted by CA_Γ and it is called the *colored Artin group* of Γ. The epimorphism θ has a natural set-section $\iota : W_\Gamma \to A_\Gamma$ defined as follows. For $w \in W_\Gamma$ the word length of w with respect to S is denoted by $\ell_S(w)$, and an expression $w = s_{x_1} s_{x_2} \cdots s_{x_p}$ is called *reduced* if $p = \ell_S(w)$. Let $w \in W_\Gamma$. We choose a reduced expression $w = s_{x_1} s_{x_2} \cdots s_{x_p}$ and we set $\iota(w) = \sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_p}$. By Tits [85] this definition does not depend on the choice of the reduced expression. Notice that ι is not a homomorphism, but, if $u, v \in W_\Gamma$ are such that $\ell_S(uv) = \ell_S(u) + \ell_S(v)$, then $\iota(uv) = \iota(u)\,\iota(v)$. We clearly have $\theta \circ \iota = \mathrm{id}$.

For $X \subset V(\Gamma)$ we set $\mathrm{CA}_X = \mathrm{CA}_\Gamma \cap A_X$. Since the inclusion map from Γ_X to Γ induces isomorphisms $W_{\Gamma_X} \to W_X$ and $A_{\Gamma_X} \to A_X$, the isomorphism $A_{\Gamma_X} \to A_X$ restricts to an isomorphism $\mathrm{CA}_{\Gamma_X} \to \mathrm{CA}_X$. So, as for W_X and A_X, we will not differentiate CA_X from CA_{Γ_X}.

The following lemma arises from the exercises of Chapter 4 of Bourbaki [12] (see also Davis [34, Section 4.3]) and it is widely used in the study of Coxeter groups.

Lemma 5.1.1 (Bourbaki [12]). *Let $X, Y \subset V(\Gamma)$ and let $w \in W_\Gamma$.*

(1) There exists a unique element of minimal length in the double-coset $W_X\, w\, W_Y$.

(2) Let w_0 be the element of minimal length in $W_X\, w\, W_Y$. For each $v \in W_X\, w\, W_Y$ there exist $u_1 \in W_X$ and $u_2 \in W_Y$ such that $v = u_1 w_0 u_2$ and $\ell_S(v) = \ell_S(u_1) + \ell_S(w_0) + \ell_S(u_2)$.

(3) Let w_0 be the element of minimal length in $W_X\, w\, W_Y$. For each $u_1 \in W_X$ we have $\ell_S(u_1 w_0) = \ell_S(u_1) + \ell_S(w_0)$, and for each $u_2 \in W_Y$ we have

$$\ell_S(w_0u_2) = \ell_S(w_0) + \ell_S(u_2).$$

Let $X, Y \subset V(\Gamma)$ and $w_0 \in W_\Gamma$. We say that w_0 is (X,Y)-*minimal* if it is of minimal length in the double-coset $W_X\, w_0\, W_Y$.

The first ingredient in the proof of Theorem 2.3.1 is the following. It essentially states that the conclusion of Theorem 2.3.1 holds if we are under the same hypothesis, but over the Coxeter group.

Lemma 5.1.2. *Let $X, Y \subset V(\Gamma)$ and $w \in W_\Gamma$ such that $wW_Yw^{-1} \subset W_X$. Then there exist $Y' \subset X$ and $\alpha \in A_X$ such that $\iota(w)\, A_Y\, \iota(w)^{-1} = \alpha A_{Y'} \alpha^{-1}$. In particular, $\iota(w)\, A_Y\, \iota(w)^{-1} \subset A_X$.*

Proof. Let w_0 be the element of minimal length in the double-coset $W_X\, w\, W_Y$. By Lemma 5.1.1 there exist $u_1 \in W_X$ and $u_2 \in W_Y$ such that $w = u_1w_0u_2$ and $\ell_S(w) = \ell_S(u_1) + \ell_S(w_0) + \ell_S(u_2)$. Since $wW_Yw^{-1} \subset W_X$, $u_1 \in W_X$ and $u_2 \in W_Y$, we have $w_0W_Yw_0^{-1} \subset W_X$.

Let $y \in Y$, and let $\psi(y) = w_0s_yw_0^{-1} \in W_X$. We have that $w_0s_y = \psi(y)\, w_0$. Furthermore, by Lemma 5.1.1 (3), we have $\ell_S(w_0) + 1 = \ell_S(w_0s_y) = \ell_S(\psi(y)\, w_0) = \ell_S(\psi(y)) + \ell_S(w_0)$, and hence $\ell_S(\psi(y)) = 1$. So, there exists $f(y) \in X$ such that $w_0s_yw_0^{-1} = \psi(y) = s_{f(y)}$. Note that the above defined map $f : Y \to X$ is injective since conjugation by w_0 is an automorphism. We set $Y' = f(Y) \subset X$.

Let $y \in Y$. We have $w_0s_y = s_{f(y)}w_0$ and $\ell_S(w_0s_y) = \ell_S(s_{f(y)}w_0) = \ell_S(w_0) + 1$, hence

$$\iota(w_0)\, \sigma_y = \iota(w_0)\, \iota(s_y) = \iota(w_0s_y) = \iota(s_{f(y)}w_0) = \iota(s_{f(y)})\, \iota(w_0) = \sigma_{f(y)}\, \iota(w_0)\, .$$

This implies that $\iota(w_0)\, \Sigma_Y\, \iota(w_0)^{-1} = \Sigma_{Y'}$, thus $\iota(w_0)\, A_Y\, \iota(w_0)^{-1} = A_{Y'}$.

We set $\alpha = \iota(u_1) \in A_X$. Then, since $\iota(u_2) \in A_Y$,

$$\begin{aligned} &\iota(w)\, A_Y\, \iota(w)^{-1} = \iota(u_1)\, \iota(w_0)\, \iota(u_2)\, A_Y\, \iota(u_2)^{-1}\iota(w_0)^{-1}\iota(u_1)^{-1} = \\ &\iota(u_1)\, \iota(w_0)\, A_Y\, \iota(w_0)^{-1}\iota(u_1)^{-1} = \iota(u_1)\, A_{Y'}\, \iota(u_1)^{-1} = \alpha A_{Y'} \alpha^{-1}\, . \end{aligned}$$

□

We now turn to construct a set-retraction of the inclusion map from A_X into A_Γ, that is, a map $\pi_X : A_\Gamma \to A_X$ which satisfies $\pi_X(\alpha) = \alpha$ for all $\alpha \in A_X$. This map will be used to prove Lemma 5.1.4 which is the second and last ingredient in the proof of Theorem 2.3.1. Note that the main ideas of the proof of Proposition 5.1.3 come from the proof of Theorem 1.2 of Charney and Paris [22].

Recall that $(\Sigma \sqcup \Sigma^{-1})^*$ denotes the free monoid freely generated by $\Sigma \sqcup \Sigma^{-1}$, that is, the set of words over the alphabet $\Sigma \sqcup \Sigma^{-1}$. Let $X \subset V(\Gamma)$. Let $\hat{\alpha} = \sigma_{z_1}^{\varepsilon_1} \sigma_{z_2}^{\varepsilon_2} \cdots \sigma_{z_p}^{\varepsilon_p} \in (\Sigma \sqcup \Sigma^{-1})^*$. We set $u_0 = 1 \in W_\Gamma$ and, for $i \in \{1, \dots, p\}$, we set $u_i = s_{z_1} s_{z_2} \cdots s_{z_i} \in W_\Gamma$. We write each u_i in the form $u_i = v_i w_i$ where $v_i \in W_X$ and w_i is $(X, \emptyset)$-minimal. Let $i \in \{1, \dots, p\}$. We set $t_i = w_{i-1} s_{z_i} w_{i-1}^{-1}$ if $\varepsilon_i = 1$ and $t_i = w_i s_{z_i} w_i^{-1}$ if $\varepsilon_i = -1$. If $t_i \notin S_X$, then we set $\tau_i = 1$. Suppose that $t_i \in S_X$, and let $x_i \in X$ such that $t_i = s_{x_i}$. Then we set $\tau_i = \sigma_{x_i}^{\varepsilon_i}$. Finally, we set

$$\hat{\pi}_X(\hat{\alpha}) = \tau_1 \tau_2 \cdots \tau_p \in (\Sigma_X \sqcup \Sigma_X^{-1})^* \,.$$

Proposition 5.1.3. *Let $X \subset V(\Gamma)$.*

(1) Let $\hat{\alpha}, \hat{\beta} \in (\Sigma \sqcup \Sigma^{-1})^*$. If $\hat{\alpha}$ and $\hat{\beta}$ represent the same element of A_Γ, then $\hat{\pi}_X(\hat{\alpha})$ and $\hat{\pi}_X(\hat{\beta})$ represent the same element of A_X. In other words, the map $\hat{\pi}_X : (\Sigma \sqcup \Sigma^{-1})^* \to (\Sigma_X \sqcup \Sigma_X^{-1})^*$ induces a set-map $\pi_X : A_\Gamma \to A_X$.

(2) We have $\pi_X(\alpha) = \alpha$ for all $\alpha \in A_X$.

(3) The restriction of π_X to CA_Γ is a homomorphism $\pi_X : \mathrm{CA}_\Gamma \to \mathrm{CA}_X$.

While the definition of $\hat{\pi}_X$ may seem ad hoc at first, it will become clear in Section 5.2, where we introduce the Salvetti complex and prove Proposition 5.1.3. Now, thanks to Proposition 5.1.3 we can prove the second ingredient of the proof of Theorem 2.3.1.

Lemma 5.1.4. *Let $X \subset V(\Gamma)$, $\alpha \in A_X$ and $\beta \in \mathrm{CA}_\Gamma$. If $\beta\alpha\beta^{-1} \in A_X$, then $\beta\alpha\beta^{-1} = \pi_X(\beta)\, \alpha\, \pi_X(\beta)^{-1}$.*

Proof. We assume that $\beta\alpha\beta^{-1} \in A_X$. We choose a word $\sigma_{z_1}^{\varepsilon_1} \sigma_{z_2}^{\varepsilon_2} \cdots \sigma_{z_p}^{\varepsilon_p} \in (\Sigma \sqcup \Sigma^{-1})^*$ which represents β and a word $\sigma_{x_1}^{\mu_1} \sigma_{x_2}^{\mu_2} \cdots \sigma_{x_q}^{\mu_q} \in (\Sigma_X \sqcup \Sigma_X^{-1})^*$ which represents α. We start with the definition of $\pi_X(\beta\alpha\beta^{-1})$ which uses the representative word $\sigma_{z_1}^{\varepsilon_1} \cdots \sigma_{z_p}^{\varepsilon_p} \sigma_{x_1}^{\mu_1} \cdots \sigma_{x_q}^{\mu_q} \sigma_{z_p}^{-\varepsilon_p} \cdots \sigma_{z_1}^{-\varepsilon_1}$. We set $u_{0,1} = 1$ and, for $i \in \{1, \dots, p\}$, we set $u_{i,1} = s_{z_1} s_{z_2} \cdots s_{z_i}$. We write each $u_{i,1}$ in the form $u_{i,1} = v_{i,1} w_{i,1}$ where $v_{i,1} \in W_X$ and $w_{i,1}$ is $(X, \emptyset)$-minimal. Let $i \in \{1, \dots, p\}$. We set $t_{i,1} = w_{i-1,1} s_{z_i} w_{i-1,1}^{-1}$ if $\varepsilon_i = 1$, and $t_{i,1} = w_{i,1} s_{z_i} w_{i,1}^{-1}$ if $\varepsilon_i = -1$. We set $\tau_{i,1} = 1$ if $t_{i,1} \not\in S_X$, and $\tau_{i,1} = \sigma_{x_{i,1}}^{\varepsilon_i}$ if $t_{i,1} \in S_X$, where $x_{i,1}$ is the element of X such that $t_{i,1} = s_{x_{i,1}}$. We set $u_{0,2} = \theta(\beta)$ and, for $i \in \{1, \dots, q\}$, we set $u_{i,2} = \theta(\beta)\, s_{x_1} s_{x_2} \cdots s_{x_i}$. We write each $u_{i,2}$ in the form $u_{i,2} = v_{i,2} w_{i,2}$, where $v_{i,2} \in W_X$ and $w_{i,2}$ is $(X, \emptyset)$-minimal. Let $i \in \{1, \dots, q\}$. We set $t_{i,2} = w_{i-1,2} s_{x_i} w_{i-1,2}^{-1}$ if $\mu_i = 1$, and $t_{i,2} = w_{i,2} s_{x_i} w_{i,2}^{-1}$ if $\mu_i = -1$. We set $\tau_{i,2} = 1$ if $t_{i,2} \not\in S_X$, and $\tau_{i,2} = \sigma_{x_{i,2}}^{\mu_i}$ if $t_{i,2} \in S_X$, where $x_{i,2}$ is the element of X such that $t_{i,2} = s_{x_{i,2}}$. We set $u_{p+1,3} = \theta(\beta)\, \theta(\alpha)$ and, for $i \in \{1, \dots, p\}$, we set $u_{i,3} = \theta(\beta)\, \theta(\alpha)\, s_{z_p} s_{z_{p-1}} \cdots s_{z_i}$. We write each $u_{i,3}$ in the form $u_{i,3} = v_{i,3} w_{i,3}$, where $v_{i,3} \in W_X$ and $w_{i,3}$ is $(X, \emptyset)$-minimal. Let $i \in \{1, \dots, p\}$. We set $t_{i,3} = w_{i+1,3} s_{z_i} w_{i+1,3}^{-1}$ if $\varepsilon_i = -1$, and $t_{i,3} = w_{i,3} s_{z_i} w_{i,3}^{-1}$ if $\varepsilon_i = 1$. We set $\tau_{i,3} = 1$ if $t_{i,3} \not\in S_X$, and $\tau_{i,3} = \sigma_{x_{i,3}}^{-\varepsilon_i}$ if $t_{i,3} \in S_X$, where $x_{i,3}$ is the element of X such that $t_{i,3} = s_{x_{i,3}}$. Then, by definition,

$$\pi_X(\beta \alpha \beta^{-1}) = \tau_{1,1} \tau_{2,1} \cdots \tau_{p,1} \tau_{1,2} \tau_{2,2} \cdots \tau_{q,2} \tau_{p,3} \cdots \tau_{2,3} \tau_{1,3}\,.$$

We also have $\pi_X(\beta \alpha \beta^{-1}) = \beta \alpha \beta^{-1}$, since $\beta \alpha \beta^{-1} \in A_X$.

We have $\tau_{1,1} \tau_{2,1} \cdots \tau_{p,1} = \pi_X(\beta)$ by definition. Let $i \in \{0, 1, \dots, q\}$. We have $\theta(\beta) = 1$ since $\beta \in \mathrm{CA}$, hence $u_{i,2} = s_{x_1} s_{x_2} \cdots s_{x_i} \in W_X$. It follows that $v_{i,2} = u_{i,2}$ and $w_{i,2} = 1$. Let $i \in \{1, \dots, q\}$. Then $t_{i,2} = s_{x_i} \in S_X$ and $\tau_{i,2} = \sigma_{x_i}^{\mu_i}$. So,

$$\tau_{1,2} \tau_{2,2} \cdots \tau_{q,2} = \sigma_{x_1}^{\mu_1} \sigma_{x_2}^{\mu_2} \cdots \sigma_{x_q}^{\mu_q} = \alpha\,.$$

Let $i \in \{0, 1, \dots, p\}$. We have $1 = \theta(\beta) = s_{z_1} \cdots s_{z_i} s_{z_{i+1}} \cdots s_{z_p}$, hence $s_{z_p} \cdots s_{z_{i+1}} = s_{z_1} \cdots s_{z_i} = u_{i,1}$, and therefore

$$u_{i,3} = \theta(\beta)\, \theta(\alpha)\, s_{z_p} \cdots s_{z_i} = \theta(\alpha)\, u_{i-1,1} = \theta(\alpha)\, v_{i-1,1} w_{i-1,1}\,.$$

Since $\theta(\alpha) \in W_X$, it follows that $v_{i,3} = \theta(\alpha)\, v_{i-1,1}$ and $w_{i,3} = w_{i-1,1}$. Let $i \in \{1, \dots, p\}$. If $\varepsilon_i = 1$, then

$$t_{i,3} = w_{i,3} s_{z_i} w_{i,3}^{-1} = w_{i-1,1} s_{z_i} w_{i-1,1}^{-1} = t_{i,1}\,.$$

Similarly, if $\varepsilon_i = -1$, then

$$t_{i,3} = w_{i+1,3} s_{z_i} w_{i+1,3}^{-1} = w_{i,1} s_{z_i} w_{i,1}^{-1} = t_{i,1}\,.$$

In both cases it follows that $\tau_{i,3} = \tau_{i,1}^{-1}$. So,

$$\tau_{p,3} \cdots \tau_{2,3} \tau_{1,3} = \tau_{p,1}^{-1} \cdots \tau_{2,1}^{-1} \tau_{1,1}^{-1} = \pi_X(\beta)^{-1}\,.$$

Finally,
$$\beta\alpha\beta^{-1} = \pi_X(\beta\alpha\beta^{-1}) = \pi_X(\beta)\,\alpha\,\pi_X(\beta)^{-1}\,.$$
□

Proof of Theorem 2.3.1. Let $X, Y \subset V(\Gamma)$ and $\alpha \in A$ such that $\alpha A_Y \alpha^{-1} \subset A_X$. Let $w = \theta(\alpha)$. We have $wW_Yw^{-1} \subset W_X$, hence, by Lemma 5.1.2, there exist $Y' \subset X$ and $\beta_2 \in A_X$ such that $\iota(w)\,A_Y\,\iota(w)^{-1} = \beta_2 A_{Y'} \beta_2^{-1}$. Let $\beta_1 = \alpha\,\iota(w)^{-1}$. Then
$$\alpha A_Y \alpha^{-1} = \alpha\,\iota(w)^{-1}\,\iota(w)\,A_Y\,\iota(w)^{-1}\,\iota(w)\,\alpha^{-1} = \beta_1\beta_2 A_{Y'}\beta_2^{-1}\beta_1^{-1}\,.$$
We have $\beta_1 \in \mathrm{CA}$, since $\theta(\beta_1) = ww^{-1} = 1$. Now, $\beta_2 A_{Y'} \beta_2^{-1} \subset A_X$ and $\beta_1(\beta_2 A_{Y'}\beta_2^{-1})\beta_1^{-1} \subset A_X$, hence, by Lemma 5.1.4,
$$\alpha A_Y \alpha^{-1} = \beta_1(\beta_2 A_{Y'}\beta_2^{-1})\beta_1^{-1} = \pi_X(\beta_1)\,(\beta_2 A_{Y'}\beta_2^{-1})\,\pi_X(\beta_1)^{-1}\,.$$
So, if $\gamma = \pi_X(\beta_1)\,\beta_2$, then $\gamma \in A_X$ and $\alpha A_Y \alpha^{-1} = \gamma A_{Y'} \gamma^{-1}$. □

5.2 The Salvetti complex

In this section we recall a geometric construction associated to an Artin group and use it to prove Proposition 5.1.3. The *Salvetti complex* of Γ is a CW-complex $\overline{\mathrm{Sal}}(\Gamma)$ whose 2-skeleton coincides with the 2-complex associated with the standard presentation of A_Γ (see Godelle and Paris [48], Paris [74], Salvetti [78], or Charney and Davis [19] for a precise definition). In particular, $\overline{\mathrm{Sal}}(\Gamma)$ has a unique vertex o_0, and it has one edge $\bar{a}_x$ for each $x \in V(\Gamma)$. We also have an isomorphism $A_\Gamma \to \pi_1(\overline{\mathrm{Sal}}(\Gamma))$ which sends σ_x to the homotopy class of $\bar{a}_x$ for all $x \in V(\Gamma)$. Let $p : \mathrm{Sal}(\Gamma) \to \overline{\mathrm{Sal}}(\Gamma)$ be the regular covering associated with $\theta : A_\Gamma \to W_\Gamma$. Note that $\mathrm{Sal}(\Gamma)$ has fundamental group CA_Γ. The set of vertices of $\mathrm{Sal}(\Gamma)$ is a set $\{o(u) \mid u \in W_\Gamma\}$ in one-to-one correspondence with W_Γ and the set of edges is a set $\{a_x(u) \mid x \in V(\Gamma)\,,\ u \in W_\Gamma\}$ in one-to-one correspondence with $V(\Gamma) \times W_\Gamma$. An edge $a_x(u)$ connects $o(u)$ with $o(us_x)$, and it is assumed to be oriented from $o(u)$ to $o(us_x)$. We have $p(o(u)) = o_0$ for all $u \in W_\Gamma$ and $p(a_x(u)) = \bar{a}_x$ for all $(x, u) \in V(\Gamma) \times W_\Gamma$. We have an action of W_Γ on $\mathrm{Sal}(\Gamma)$ by deck transformations, and $\mathrm{Sal}(\Gamma)/W = \overline{\mathrm{Sal}}(\Gamma)$. This action is defined on the vertices and edges as follows:
$$v\,o(u) = o(vu)\,,\ v\,a_x(u) = a_x(vu)\,.$$

Let $X \subset V(\Gamma)$. We have an embedding $\bar\nu_X : \overline{\mathrm{Sal}}(\Gamma_X) \to \overline{\mathrm{Sal}}(\Gamma)$ which sends $\bar a_x$ to $\bar a_x$ for all $x \in X$ and which induces the natural embedding of A_X into A_Γ. We also have an embedding $\nu_X : \mathrm{Sal}(\Gamma_X) \to \mathrm{Sal}(\Gamma)$ which sends $o(u)$ to $o(u)$ for all $u \in W_X$, which sends $a_x(u)$ to $a_x(u)$ for all $(x, u) \in X \times W_X$, and which induces the natural embedding of CA_X into CA_Γ. These two embeddings are linked with the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Sal}(\Gamma_X) \arrow[r, "\nu_X"] \arrow[d, "p"] & \mathrm{Sal}(\Gamma) \arrow[d, "p"] \\
\overline{\mathrm{Sal}}(\Gamma_X) \arrow[r, "\bar\nu_X"] & \overline{\mathrm{Sal}}(\Gamma)
\end{tikzcd}$$

We know by Godelle and Paris [48, Theorem 2.2] that the embedding $\nu_X : \mathrm{Sal}(\Gamma_X) \to \mathrm{Sal}(\Gamma)$ admits a continuous retraction $\rho_X : \mathrm{Sal}(\Gamma) \to \mathrm{Sal}(\Gamma_X)$. This retraction is cellular in the sense that it sends the k-skeleton of $\mathrm{Sal}(\Gamma)$ to the k-skeleton of $\mathrm{Sal}(\Gamma_X)$ for all $k \ge 0$. The following explicit description of ρ_X on the 0 and 1-skeletons of $\mathrm{Sal}(\Gamma)$ is proved by Charney and Paris in [22, Lemma 2.6]. Let $u \in W_\Gamma$ and $z \in V(\Gamma)$. We write u in the form $u = vw$ where $v \in W_X$ and w is $(X, \emptyset)$-minimal.

- $\rho_X(o(u)) = o(v)$.
- If $w s_z w^{-1} \notin S_X$, then $\rho_X(a_z(u)) = o(v)$.
- Suppose that $w s_z w^{-1} \in S_X$. Let $x \in X$ such that $w s_z w^{-1} = s_x$. Then $\rho_X(a_z(u)) = a_x(v)$.

In what follows we compose paths from left to right. Let $\hat\alpha = \sigma_{z_1}^{\varepsilon_1} \sigma_{z_2}^{\varepsilon_2} \cdots \sigma_{z_p}^{\varepsilon_p}$ be an element of $(\Sigma \sqcup \Sigma^{-1})^*$. Let

$$\bar\gamma(\hat\alpha) = \bar a_{z_1}^{\varepsilon_1} \bar a_{z_2}^{\varepsilon_2} \cdots \bar a_{z_p}^{\varepsilon_p}\,.$$

We see that, if α is the element of A_Γ represented by $\hat\alpha$, then α, regarded as an element of $\pi_1(\overline{\mathrm{Sal}}(\Gamma)) = A_\Gamma$, is represented by the loop $\bar\gamma(\hat\alpha)$. Let $\gamma(\hat\alpha)$ be the lift of $\bar\gamma(\hat\alpha)$ in $\mathrm{Sal}(\Gamma)$ starting at $o(1)$. We set $u_0 = 1 \in W_\Gamma$ and, for $i \in \{1, \dots, p\}$, we set $u_i = s_{z_1} s_{z_2} \cdots s_{z_i} \in W_\Gamma$. For $i \in \{1, \dots, p\}$ we set $a_i = a_{z_i}(u_{i-1})$ if $\varepsilon_i = 1$, and $a_i = a_{z_i}(u_i)$ if $\varepsilon_i = -1$. Then

$$\gamma(\hat\alpha) = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_p^{\varepsilon_p}\,.$$

Let $\gamma_X(\hat{\alpha}) = \rho_X(\gamma(\hat{\alpha}))$. We write each u_i in the form $u_i = v_i w_i$ where $v_i \in W_X$ and w_i is $(X, \emptyset)$-minimal. Let $i \in \{1, \dots, p\}$. We set $t_i = w_{i-1} s_{z_i} w_{i-1}^{-1}$ if $\varepsilon_i = 1$, and $t_i = w_i s_{z_i} w_i^{-1}$ if $\varepsilon_i = -1$. If $t_i \notin S_X$, then, as shown by Charney and Paris in [22, Lemma 2.6], $v_i = v_{i-1}$. In that case we denote by b_i the constant path at $o(v_{i-1}) = o(v_i)$. Suppose that $t_i \in S_X$. Let $x_i \in X$ such that $t_i = s_{x_i}$. We set $b_i = a_{x_i}(v_{i-1})$ if $\varepsilon_i = 1$, and $b_i = a_{x_i}(v_i)^{-1}$ if $\varepsilon_i = -1$. It follows from the description of the map ρ_X on the 0 and 1-skeletons given above that

$$\gamma_X(\hat{\alpha}) = b_1 b_2 \cdots b_p \,.$$

Let $\bar{\gamma}_X(\hat{\alpha}) = p(\gamma_X(\hat{\alpha}))$. Let $i \in \{1, \dots, p\}$. If $t_i \notin S_X$, then we denote by $\bar{b}_i$ the constant loop in $\overline{\mathrm{Sal}}(\Gamma_X)$ based at o_0. Suppose $t_i \in S_X$. Let $x_i \in X$ such that $t_i = s_{x_i}$ as before. We set $\bar{b}_i = \bar{a}_{x_i}$ if $\varepsilon_i = 1$, and $\bar{b}_i = \bar{a}_{x_i}^{-1}$ if $\varepsilon_i = -1$. Then

$$\bar{\gamma}_X(\hat{\alpha}) = \bar{b}_1 \bar{b}_2 \cdots \bar{b}_p \,.$$

Let $\alpha' \in A_X = \pi_1(\overline{\mathrm{Sal}}(\Gamma_X))$ be the element represented by the loop $\bar{\gamma}_X(\hat{\alpha})$. Then we easily see that α' is exactly the element of A_X represented by the word $\hat{\pi}_X(\hat{\alpha}) \in (\Sigma_X \sqcup \Sigma_X^{-1})^*$.

Proof of Proposition 5.1.3. Proof of Part (1). Let $\hat{\alpha}, \hat{\beta} \in (\Sigma \sqcup \Sigma^{-1})^*$ be two words that represent the same element of A_Γ. Then $\bar{\gamma}(\hat{\alpha})$ and $\bar{\gamma}(\hat{\beta})$ represent the same element of $A_\Gamma = \pi_1(\overline{\mathrm{Sal}}(\Gamma))$, hence $\bar{\gamma}(\hat{\alpha})$ and $\bar{\gamma}(\hat{\beta})$ are homotopic loops. Since $p : \mathrm{Sal}(\Gamma) \to \overline{\mathrm{Sal}}(\Gamma)$ is a covering map, $\gamma(\hat{\alpha})$ and $\gamma(\hat{\beta})$ are homotopic relative to the extremities. Since ρ_X is continuous, it follows that $\gamma_X(\hat{\alpha})$ and $\gamma_X(\hat{\beta})$ are also homotopic relative to the extremities. Again, the map $p : \mathrm{Sal}(\Gamma_X) \to \overline{\mathrm{Sal}}(\Gamma_X)$ is continuous, hence $\bar{\gamma}_X(\hat{\alpha})$ and $\bar{\gamma}_X(\hat{\beta})$ are homotopic loops, and therefore they represent the same element of $A_X = \pi_1(\overline{\mathrm{Sal}}(\Gamma_X))$. We conclude that $\hat{\pi}_X(\hat{\alpha})$ and $\hat{\pi}_X(\hat{\beta})$ represent the same element of A_X.

Proof of Part (2). Let $\alpha \in A_X$. We choose a word $\hat{\alpha} = \sigma_{x_1}^{\varepsilon_1} \sigma_{x_2}^{\varepsilon_2} \cdots \sigma_{x_p}^{\varepsilon_p} \in (\Sigma_X \sqcup \Sigma_X^{-1})^*$ which represents α. Following the above definition, we set $u_0 = 1$ and, for $i \in \{1, \dots, p\}$, we set $u_i = s_{x_1} s_{x_2} \cdots s_{x_i}$. We write each u_i in the form $u_i = v_i w_i$ where $v_i \in W_X$ and w_i is $(X, \emptyset)$-minimal. Note that $u_i \in W_X$, hence $v_i = u_i$ and $w_i = 1$. Let $i \in \{1, \dots, p\}$. We set $t_i = w_{i-1} s_{x_i} w_{i-1}^{-1}$ if $\varepsilon_i = 1$, and $t_i = w_i s_{x_i} w_i^{-1}$ if $\varepsilon_i = -1$. In both cases we have $t_i = s_{x_i}$, and so $\tau_i = \sigma_{x_i}^{\varepsilon_i}$. So,

$$\hat{\pi}_X(\hat{\alpha}) = \tau_1 \tau_2 \cdots \tau_p = \sigma_{x_1}^{\varepsilon_1} \sigma_{x_2}^{\varepsilon_2} \cdots \sigma_{x_p}^{\varepsilon_p} = \hat{\alpha} \,,$$

hence $\pi_X(\alpha) = \alpha$.

Proof of Part (3). Observe that the restriction of π_X to CA_Γ coincides with the homomorphism $\rho_{X,*} : \mathrm{CA}_\Gamma = \pi_1(\mathrm{Sal}(\Gamma)) \to \pi_1(\mathrm{Sal}(\Gamma_X)) = \mathrm{CA}_X$ induced by the map $\rho_X : \mathrm{Sal}(\Gamma) \to \mathrm{Sal}(\Gamma_X)$. To see this, note that ρ_X does to edge paths in $\mathrm{Sal}(\Gamma)$ what $\hat{\pi}_X$ does to elements in $(\Sigma \sqcup \Sigma^{-1})^*$ (where the ε appearing in the definition of $\hat{\pi}_X$ reflect the orientation of the edges in $\mathrm{Sal}(\Gamma)$). Hence the restriction of π_X to CA_Γ is a homomorphism $\pi_X : \mathrm{CA}_\Gamma \to \mathrm{CA}_X$. $\square$

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