Numerical blow-up for the p-laplacian equation with a source

RAÚL FERREIRA
ARTURO DE PABLO
MAYTE PÉREZ-LLANOS
Departamento de Matemáticas, U. Carlos III de Madrid
28911 Leganés, Madrid

Abstract
We study numerical approximations of nonnegative solutions of the p-laplacian equation with a nonlinear source,

\[
\begin{align*}
    u_t &= (|u_x|^{p-2}u_x)_x + |u|^{q-2}u, & (x, t) &\in (-L, L) \times (0, T), \\
    u(-L, t) &= u(L, t) = 0, & t &\in [0, T), \\
    u(x, 0) &= \varphi(x) > 0, & x &\in (-L, L),
\end{align*}
\]

where \( p > 2, \ q > 2 \) and \( L > 0 \) are parameters. We describe in terms of \( p, q \) and \( L \) when solutions of a semidiscretization in space exist globally in time and when they blow up in a finite time. We also find the blow-up rates and the blow-up sets by means of the discrete self-similar profiles.

Keywords: numerical blow-up, parabolic p-laplacian, eigenvalue problem.
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1 Introduction

In this paper we deal with numerical approximations of the following problem:

\[
\begin{align*}
    u_t &= (|u_x|^{p-2}u_x)_x + |u|^{q-2}u, & (x, t) &\in (-L, L) \times (0, T), \\
    u(-L, t) &= u(L, t) = 0, & t &\in [0, T), \\
    u(x, 0) &= \varphi(x) > 0, & x &\in (-L, L),
\end{align*}
\]

(1.1)
where $p > 2$, $q > 2$ and $L > 0$ are parameters.

We assume that $\varphi$ is smooth, positive in $(-L, L)$ and compatible with the boundary conditions. Also for simplicity, we assume that $\varphi$ is symmetric and decreasing in $[0, L]$; these symmetry properties will be preserved by our numerical scheme and make computations easier.

Problem (1.1) can be thought of as a model for nonlinear heat propagation in a reactive medium (reaction given by the power $|u|^{q-2}u$). Under this point of view $u$ stands for the temperature, [20]. This problem also appears in the study of non-Newtonian fluids and nonlinear filtration theory [31]. In this context, the quantity $p$ is a characteristic of the medium: media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics; if $p = 2$ they are Newtonian fluids.

The solution of (1.1) may only exist for a finite period of time, see [24]. The time $T$ is the maximal existence time for the solution, which may be finite or infinite. If $T < \infty$ then $u$ becomes unbounded in finite time and we say that it blows up. If $T = \infty$ we say that the solution is global. The blow-up phenomena have deserved a great deal of attention in recent years, see for example the book [28] and the surveys [10, 20].

Since the solution $u$ may develop a singularity in finite time, it is an interesting question to ask what can be said about the numerical approximations of this kind of problems. Our aim in this paper is to analyze whether a numerical semidiscretization in space of (1.1) has a similar behaviour than the original problem. The case $p = 2$ is the well know semilinear heat equation and the numerical approximations have been deeply studied, see for example, [1, 3, 5, 22].

The continuous problem. Let us summarize what is known for the continuous problem (1.1), see for instance [29, 16, 18, 17, 11, 6, 25, 15] and the references therein.

i) If $\varphi(x)$ is a nontrivial and nonnegative continuous function, then there exists a unique nonnegative weak solution $u \in C^{1+\alpha, \frac{1}{2}}([-L, L] \times (0, T))$ for some $0 < \alpha < 1$.

ii) $q = p$ is the blow-up critical exponent, that is, if $q < p$ problem (1.1) has a unique global solution for all nonnegative initial values, and if $q > p$ there are both global and blowing-up solutions.

In the critical case $q = p$ the existence of blowing-up solutions depends on the length of the interval: the solution blows up if and only if $L$ is large.
More precisely, if \( \lambda_1(L) < 1 \) there are no nontrivial global weak solutions, and if \( \lambda_1(L) \geq 1 \) all weak solutions are global, where \( \lambda_1(L) \) is the first eigenvalue of the nonlinear eigenvalue problem

\[
\begin{align*}
- (|\psi_x|^{p-2} \psi_x)_x &= \lambda |\psi|^{p-2} \psi, & -L < x < L, \\
\psi(-L) &= \psi(L) = 0.
\end{align*}
\]

(1.2)

iii) The blow-up rate for the blowing-up solutions is given by

\[ \|u(\cdot, t)\|_\infty \sim (T - t)^{-\frac{1}{q-2}}. \]

iv) Concerning the blow-up set, we have single-point blow-up, \( B(u) = \{0\} \) if \( q > p \), while for \( q = p \) the regional or global blow-up depends on the length of the interval. More precisely, it holds

\[
B(u) = \begin{cases}
[-L, L] & \text{if } L_0 < L \leq L_1, \quad \text{(global blow-up)} \\
[-L_1, L_1] & \text{if } L > L_1, \quad \text{(regional blow-up)}
\end{cases}
\]

where \( L_0 \) is the length of the interval for which the first eigenvalue of problem (1.2) is equal to one, and \( L_1 \) is the maximal length of existence for the positive (in \((-L_1, L_1)\)), self-similar even profiles. This last assertion follows from the results of [17] and [8].

The numerical scheme. Now we introduce the numerical scheme. We discretize using piecewise linear finite elements with mass lumping in a uniform mesh for the space variable. We denote with \( U(t) = (u_{-N}(t), \cdots, u_N(t)) \) the value of the numerical approximation at nodes \( x_i = ih \) \((h = L/N)\) and at time \( t \). Then the vector \( U(t) \) verifies the following equation

\[
\begin{align*}
MU' &= h^{-p} D_p U + MU^{q-1}, \\
u_{-N} &= u_N = 0, \\
U(0) &= \varphi^I,
\end{align*}
\]

(1.3)

where

\[
D_p U = D_+ |D_- U|^{p-2} D_- U,
\]

\( D_+ \) and \( D_- \) being the stiffness matrices, \( M \) the mass matrix obtained with lumping, and \( \varphi^I \) is the Lagrange interpolation of the initial datum \( \varphi \).

We remark that the operator \( D_p \) is not given by a matrix. This fact makes the analysis of (1.3) different to the semilinear heat equation \((p = 2)\).
Writing this equation explicitly we obtain the following ODE system,

\[
\begin{align*}
    u_{-N} &= 0, \\
    u'_k(t) &= h^{-p} \mathcal{D}_p u_k(t) + u_k^{q-1}(t), \quad -N + 1 \leq k \leq N - 1, \\
    u_N &= 0, \\
    u_k(0) &= \varphi(x_k),
\end{align*}
\]

where \( \mathcal{D}_p u_k \), the \( k \)th component of \( \mathcal{D}_p U \), is given by

\[
\mathcal{D}_p u_k = |u_{k-1} - u_k|^{p-2} (u_{k-1} - u_k) - |u_k - u_{k+1}|^{p-2} (u_k - u_{k+1}).
\]

**Remark 1.1** Other methods can also be applied to study blow-up problems, like the moving-mesh method used in [5, 23, 7, 12]. This method requires some scale invariance of the problem. Though the quantitative results are commonly better when the mesh is refined properly, we do not know of any rigorous result of convergence in this direction.

**Main Results.** First of all we state a convergence result that says that the above method converges in sets of the form \([-L, L] \times [0, T - \tau]\), for every \( 0 < \tau < T \). Let us observe that, due to the singularity developed by the solution at time \( t = T \), we cannot expect that the convergence result extends up to \( T \).

**Theorem 1.1** Let \( u \in C^{1+\alpha,\frac{1}{2}}([-L, L] \times [0, T - \tau]) \), \( 0 < \alpha < 1 \), be a nonnegative solution of (1.1) and \( u_h \) the numerical approximation, considered as the linear interpolation of the components of the vector \( U \) given by (1.3). Then there exists a constant \( C \), that depends on the norm of \( u \) in the previous space and \( \tau \), such that for every \( h \) small enough it holds

\[
\begin{align*}
    \max_{0 \leq t \leq T-\tau} \|u(\cdot, t) - u_h(\cdot, t)\|_{L^2([-L, L])} &\leq Ch^\alpha, \\
    \max_{0 \leq t \leq T-\tau} \|u_x(\cdot, t) - (u_h)_x(\cdot, t)\|_{L^2([-L, L])} &\leq Ch^{2\alpha/p}, \\
    \max_{0 \leq t \leq T-\tau} \|u(\cdot, t) - u_h(\cdot, t)\|_{L^\infty([-L, L])} &\leq Ch^{2\alpha/p}.
\end{align*}
\]

Moreover, if \( u(\cdot, t) \in W^{2,2}([-L, L]) \) we can take \( \alpha = 1 \) in the above estimates.

Next, we show that whenever the solution \( u \) of the continuous problem (1.1) blows up, so does the solution \( U \) of the discrete problem (1.3).
Proposition 1.1 Let \( u \) be a blowing-up solution of (1.1). Then the solution \( U \) of (1.3) also blows up for every \( h \) small enough.

This fact enables us to state the conditions for which numerical blow-up takes place. To accomplish that we introduce the discrete eigenvalue problem corresponding to (1.2)

\[
\begin{aligned}
\mathcal{D}_p \psi + \lambda |\psi|^{p-2}\psi &= 0, \\
\psi(-L) &= \psi(L) = 0.
\end{aligned}
\]

Denote by \( \lambda_1(L, h) \) the first eigenvalue to this problem.

Theorem 1.2 Let us consider the Dirichlet problem (1.4). We have:

i) If \( q < p \) every solution is global.

ii) If \( q = p \), the solution blows up in finite time provided that \( \lambda_1(L, h) < 1 \), whereas if \( \lambda_1(L, h) \geq 1 \) every solution of (1.4) is global.

iii) If \( q > p \) there are solutions that blow up in finite time.

The conditions for the presence of blowing-up solutions are analogous to the ones given for the continuous problem. Moreover, in the case \( q = p \) we have the convergence of the discrete eigenvalues to the continuous one.

Theorem 1.3 Let \( \lambda_1(L) \) and \( \lambda_1(L, h) \) be the first eigenvalues of problems (1.2) and (1.5), respectively. We have

\[
\lim_{h \to 0} \lambda_1(L, h) = \lambda_1(L).
\]

Observe that in this case \( q = p \), as \( \lambda_1(L) \leq \lambda_1(L, h) \) we have that if the numerical approximation blows up so must do the continuous solution. If \( q > p \) it is not known if there exist global solution approximated by blowing-up solutions.

We now turn our attention to the asymptotic behavior of the numerical approximation near the blow-up time. That is, the blow-up rate, the blow-up set and the convergence of the blow-up time. For the case \( q > p \) the reaction is stronger than the diffusion and this makes the analysis easier. For the case \( q = p \) we construct a suitable family of self-similar solutions

\[
V(t) = (T_h - t)^{-1/(p-2)} W,
\]
where the profile \( W = (\omega_k) \) verifies

\[
\begin{align*}
0 &= 2h^{-p}|\omega_1 - \omega_0|^{p-2}(\omega_1 - \omega_0) + |\omega_0|^{p-2}\omega_0 - \frac{1}{p-2}\omega_0, \\
0 &= h^{-p} \mathcal{D}_p \omega_k + |\omega_k|^{p-2}\omega_k - \frac{1}{p-2}\omega_k, \quad k > 0.
\end{align*}
\]

We note that for any \( \omega_0 > 0 \) there exists a unique solution. Moreover, this solution converges to the corresponding continuous self-similar profile. This fact and the discrete version of the Intersection-Comparison Theory (see Lemma 2.4 below) give us the following results.

**Theorem 1.4** Let \( q \geq p \) and \( U \) be a blowing-up solution of (1.3) with blow-up time \( T_h \). Then there exist two positive constants independent of \( h \) such that

\[
C_1(T_h - t)^{-\frac{1}{q-2}} \leq \|U(t)\|_\infty \leq C_2(T_h - t)^{-\frac{1}{q-2}}.
\]

Moreover, if \( q > p \) we have

\[
\lim_{t \to T_h} (T_h - t)^{-\frac{1}{q-2}}\|U(t)\|_\infty = (q - 2)^{-\frac{1}{q-2}}.
\]

**Theorem 1.5** Let \( q \geq p \) and \( \varphi \) be an initial datum for (1.1) such that \( u \) blows up. If we call \( T \) and \( T_h \) the blow-up times for \( u \) and \( U \) respectively, we have

\[
\lim_{h \to 0} T_h = T.
\]

Concerning the blow-up sets for the numerical approximation, that is the set of nodes \( x_k \) such that \( u_k(t) \to \infty \) as \( t \to T \), we prove

**Theorem 1.6** Let \( U \) be a blowing-up solution of (1.3).

i) If \( q > p \) the blow-up set is given by a finite number of nodes, \( B(U) = [-Kh, Kh] \), where \( K \equiv K(p, q) \) is given in (5.20).

ii) If \( q = p \) the blow-up set is the whole interval \( B(U) = [-L, L] \).

We observe that in the case \( q > p \) the blow-up set of the numerical solution can be larger than a single point. Nevertheless, since \( K \) does not depend on \( h \), we have that

\[
B(U) \to \{0\} = B(u), \quad \text{as } h \searrow 0.
\]

On the other hand, when \( q = p \) we have global blow-up for every \( h \). Therefore we find that regional blow-up is not possible for a numerical scheme with a
fixed mesh. However we can recover the expected regional blow-up for the numerical scheme (see [13]) by means of the self-similar variables

\[ Y(s) = (T_h - t)^{1/(q-2)}U(t), \quad (T_h - t) = e^{-s}. \]  

(1.7)

We prove that the rescaled function \( Y(s) \) converges to a self-similar profile \( W \) as \( s \to \infty \), where \( W \) is a solution to the problem (1.6) which satisfies \( \omega_N = 0 \). On the other hand, \( W \) tends to zero outside \([0, L_1]\) as \( h \to 0 \). This means that, for \( t \sim T_h \),

\[ u_k(t) \sim \omega_k(h)(T_h - t)^{-\alpha}, \]

where \( \omega_k(h) \to 0 \) for the nodes lying in the interval \([L_1, L]\). It is in this sense that we obtain the expected regional blow-up if \( L > L_1 \).

In the sequel the letter \( C \) will denote a generic constant, which can be different in different occurrences. We also use small letter \( x_k \) to denote the components of a vector \( X \).

## 2 Properties of the numerical scheme

In this section we collect some preliminary results for our numerical method. First, we state a symmetry property for the numerical problem (1.4). We call a vector symmetric if it verifies \( u_{-k} = u_k \).

**Lemma 2.1** Let \( U(0) \) be a symmetric vector; then the solution \( U(t) \) to problem (1.3) is also symmetric for all \( t \in (0, T_h) \).

**Proof.** The result follows easily from reflection and uniqueness. \( \square \)

**Remark 2.1** Since \( u_k(0) = \varphi(x_k) \) and \( \varphi \) is symmetric, then \( U(0) \) (and therefore \( U(t) \)) is symmetric. So we can restrict ourselves to the half interval \([0, L]\), reducing the size of the system of ODEs to be solved numerically.

**Lemma 2.2** Let \( U \) be a solution of (1.4) with \( u_k(0) > u_{k+1}(0) \), for every \( 0 \leq k \leq N - 1 \). Then \( u_k(t) > u_{k+1}(t) \), for every \( 0 \leq k \leq N - 1 \).

**Proof.** We argue by contradiction. Let us assume that there exists a first time \( t_0 \) and two consecutive nodes where the conclusion of the lemma
fails; let us call them \( j, j + 1 \). So we are assuming that \( u_{j+1}(t_0) = u_j(t_0) \).
From the equations (1.4) we get
\[
0 \geq u'_j(t_0) - u'_{j+1}(t_0) = \frac{1}{h^p} \left\{ |u_{j-1}(t_0) - u_j(t_0)|^{p-2}(u_{j-1}(t_0) - u_j(t_0)) \\
+ |u_{j+1}(t_0) - u_{j+2}(t_0)|^{p-2}(u_{j+1}(t_0) - u_{j+2}(t_0)) \right\} \geq 0.
\]
We conclude that \( u_{j-1}(t_0) = u_j(t_0) = u_{j+1}(t_0) = u_{j+2}(t_0) \). Using the same reasoning at all nodes, we get \( u_j(t_0) = u_N(t_0) = 0 \). This is a contradiction with the uniqueness of the backward problem for \( t < t_0 \), since \( U(0) > 0 \).

**Remark 2.2** The previous result ensures that the maximum of \( U(t) \) is attained at the central node \( x_0 = 0 \).

Now we want to establish a comparison lemma. To do this we need the following definition,

**Definition 2.1** We call \( \overline{U} \) a supersolution (resp. \( \underline{U} \) a subsolution) if it satisfies (1.3) with upper (resp. lower) inequalities instead of equalities.

**Lemma 2.3** Let \( \overline{U} \) and \( \underline{U} \) be a supersolution and a subsolution respectively, then
\[
\overline{U}(t) \geq U(t) \geq \underline{U}(t).
\]

**Proof.** By an approximation procedure we restrict ourselves to consider strict inequalities for the supersolution. Let us prove \( \overline{U}(t) > U(t) \). We argue again by contradiction. Let us assume that there exists a first time \( t_0 \) and a node \( j \) such that \( \overline{u}_j(t_0) = u_j(t_0) = a \); then for \(-N + 1 < j < N - 1\) we have
\[
0 \geq \overline{u}'_j(t_0) - u'_j(t_0) \\
> \frac{1}{h^p} \left\{ |\overline{u}_{j-1}(t_0) - a|^{p-2} (\overline{u}_{j-1}(t_0) - a) - |u_{j-1}(t_0) - a|^{p-2} (u_{j-1}(t_0) - a) \right\} \\
+ \frac{1}{h^p} \left\{ |a - u_{j+1}(t_0)|^{p-2} (a - u_{j+1}(t_0)) - |a - \overline{u}_{j+1}(t_0)|^{p-2} (a - \overline{u}_{j+1}(t_0)) \right\} \\
\geq 0,
\]
a contradiction. The inequality \( U(t) \geq \underline{U}(t) \) is handled in a similar way. □

Next we include a result concerning the number of intersections between two solutions. This is a discrete version of the well known Sturm Comparison Theorem, see for instance [28].

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Lemma 2.4 Let $U_h$ and $V_h$ be two solutions of the system of equations

$$u'_k(t) = h^{-p} \mathcal{D}_p u_k(t) + u^{q-1}_k(t), \quad J_1 \leq k \leq J_2.$$ 

Suppose that at some time $t^*$ they have a unique intersection in $[x_{J_1}, x_{J_2}]$, i.e., for some $k_0$, $u_j(t^*) \geq v_j(t^*)$ for every $j \leq k_0$ and $u_j(t^*) < v_j(t^*)$ for $j > k_0$. Assume also that $u_{J_1}(t) > v_{J_1}(t)$ and $u_{J_2}(t) < v_{J_2}(t)$ for every $t \geq t^*$. Then, the number of intersections for $t > t^*$ is at most one.

**Proof.** Denote by $E(t) = V_h(t) - U_h(t)$. Assume by contradiction that there exists a first time $t_0$ such that $e_k(t_0) = 0$, for $k \neq k_0'$, where $k_0'$ is the node of the existing intersection at time $t_0$. Suppose $k < k_0'$ (for the situation $k > k_0'$ next reasoning is the same with a change of sign). We have that

$$0 \leq e'_k(t_0) = h^{-p} (\mathcal{D}_p v_k(t_0) - \mathcal{D}_p u_k(t_0)) + v^{q-1}_k(t_0) - u^{q-1}_k(t_0) \leq 0,$$

which repeated for each node implies that $v_j(t_0) = u_j(t_0)$, $J_1 \leq j \leq k_0'$. This is a contradiction. \hfill $\square$

Remark 2.3 Notice that if the number of intersections is greater than one, the above argument gives that this number does not increase.

Let us end this section by enclosing the proof of our convergence result. We need some preliminary technical results, see [4, 21].

Lemma 2.5 i) If $f \in C^{1+\alpha}([-L, L])$, $0 < \alpha < 1$, then

$$\|f - f^I\|_\infty = O(h^{1+\alpha}), \quad \|(f - f^I)'\|_\infty = O(h^\alpha).$$

ii) If $f \in W^{2,s}([-L, L])$, $s \geq 1$, then

$$\|f - f^I\|_{W^{1,s}([-L,L])} \leq C h \|f\|_{W^{2,s}([-L,L])}.$$ 

Lemma 2.6 Let $p \in (1, \infty)$ be fixed. There exist positive constants such that for all $x$, $y \in \mathbb{R}$ it holds

$$|x|^{p-2}x - |y|^{p-2}y \leq C_1 |x - y|(|x| + |y|)^{p-2},$$

$$|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq C_2 |x - y|^2(|x| + |y|)^{p-2}.$$
Lemma 2.7  Let $u$ be the solution to (1.1), $u^h$ its numerical approximation and assume that both of them are bounded in $[0,t_0]$. Then

$$\int_0^{t_0} \int_{-L}^L (u_t)^2 \, dx \, dt \leq C \quad \text{and} \quad \int_0^{t_0} \int_{-L}^L (u^h_t)^2 \, dx \, dt \leq C.$$ 

**Proof.** Using $u_t$ as a test function in the weak formulation of (1.1) we have

$$\int_0^{t_0} \int_{-L}^L (u_t)^2 \, dx \, dt = -\frac{1}{p} \int_0^{t_0} \int_{-L}^L (|u_x|^p)_t \, dx \, dt + \frac{1}{q} \int_0^{t_0} \int_{-L}^L (u^q)_t \, dx \, dt \leq C(\|u(t_0)\|_{\infty}, \|u_0\|_{\infty}).$$

The second bound is shown in a similar way. \hfill \Box

**Proof of Theorem 1.1.** Let us define the error function as $\varepsilon(x,t) = u(x,t) - u^h(x,t)$, where $u$ is the solution of the continuous problem (1.1), and $u^h$ its numerical approximation.

Let $$t_0 = \max\{t \in [0, T - \tau] : \max_{x \in [-L,L]} |\varepsilon(x,t)| \leq 1\},$$
to ensure neither of the solutions blow up in $[0,t_0]$. We will prove at the end that $t_0 = T - \tau$ for every $h$ small enough.

Let us consider $V = W_0^{1,p}([-L,L])$ and $V_h$ the finite dimensional subspace of $V$, consisting on the approximation of $V$ by piecewise linear finite elements. We consider then the variational formulation for both, the continuous problem and the discrete problem. If we subtract the identities obtained, taking into account that $\int_{-L}^L |f| \, dx = \int_{-L}^L |f| \, dx + O(h^2)$, we arrive at

$$\int_{-L}^L u_t \Psi \, dx + \int_{-L}^L \nabla_p u \Psi_x \, dx - \int_{-L}^L u^h_t v^h \, dx - \int_{-L}^L \nabla_p u^h (v^h)_x \, dx$$

$$= \int_{-L}^L u^{g-1} \Psi \, dx - \int_{-L}^L (u^h)^{g-1} v^h \, dx + O(h^2), \tag{2.8}$$

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for every $\Psi \in V$ and $v^h \in V_h$, where $\nabla_p u = |u_x|^{p-2} u_x$. We now take as test functions $\Psi = v^h = u^l - u^h \in V_h \subset V$. Thus (2.8) gives

$$
\int_{-L}^L \varepsilon_t \varepsilon dx + \int_{-L}^L (\nabla_p u - \nabla_p u^h) \varepsilon_x dx dt \leq C \left\{ \int_{-L}^L |\varepsilon_t| |u - u^l| dx \right. \\
+ \int_{-L}^L |u^{q-1} - (u^h)^{q-1}| |\varepsilon| dx + \int_{-L}^L |u^{q-1} - (u^h)^{q-1}| |u - u^l| dx \\
+ \int_{-L}^L |\nabla_p u - \nabla_p u^h| |(u - u^l)| dx + h^2 \right\} = J_1 + J_2 + J_3 + J_4 + Ch^2.
$$

Acting with $J_4$ as in [4], using Lemma 2.6, we get

$$(\nabla_p u - \nabla_p u^h)(u - u^l)_x \leq C(|u_x| + |u^h_x|)^{p-2} |u - u^l)_x|

\leq C(|u_x| + |u^h_x|)^{p-2} \left( \delta |x|^2 + C\delta^{-1} |(u - u^l)_x|^2 \right)$$

for every $\delta > 0$. Observe that Lemma 2.6 also implies

$$(\nabla_p u - \nabla_p u^h) \varepsilon_x \geq C_1 \varepsilon^2_x (|u_x| + |u^h_x|)^{p-2}.$$

Therefore, choosing $\delta = C_1/2C$ we can absorb the first term in $J_4$ into the left hand side. Now the Mean Value Theorem, the fact that $u$ and $u^h$ are bounded up to $t_0$ and Lemma 2.7 imply

$$J_1 \leq C \left( \int_{-L}^L |u - u^l|^2 \right)^{1/2}, \quad J_2 \leq C \int_{-L}^L \varepsilon^2, \quad J_3 \leq C \int_{-L}^L |u - u^l|.$$

Summing up, and taking account the inequality

$$\varepsilon^2_x (|u_x| + |u^h_x|)^{p-2} \geq C \varepsilon^2_x (|u_x| + \varepsilon_x)^{p-2} \geq \varepsilon^p_x,$$

we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{-L}^L \varepsilon^2 dx \leq \frac{1}{2} \frac{d}{dt} \int_{-L}^L \varepsilon^2 + \int_{-L}^L |\varepsilon_x|^p \leq C \int_{-L}^L \varepsilon^2 \\
+ C \left\{ \int_{-L}^L |u - u^l| + \left( \int_{-L}^L |u - u^l|^2 \right)^{1/2} \right\}^2 + \int_{-L}^L |(u - u^l)_x|^2 \right\}.
$$

(2.9)

Integrating this inequality in $[0, t]$ for any $0 < t < t_0$, and using the error estimate for the interpolation given above, we get

$$\int_{-L}^L \varepsilon^2(t) dx \leq C e^{Ct_0} h^{2\alpha},$$

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for some $0 < \alpha < 1$. We remark that if we assume $u \in W^{2,p}$ we can take $\alpha = 1$ in the above estimate. Considering again (2.9) we also obtain

$$\int_{-L}^{L} |\varepsilon_x|^p \leq C h^{2\alpha}.$$  

Let us observe that thanks to the inclusion $W^{1,p}_0([-L, L]) \subset L^{\infty}([-L, L])$ we obtain the corresponding error bound in $L^{\infty}$ norm. Finally, it is easy to see that we can take $t_0 = T - \tau$ for every $h$ small enough, and the proof is finished. 

\section{Blow-up for the numerical scheme}

In this section we begin by characterizing when problem (1.4) has solutions with blow-up. To do this we consider the energy functional for the continuous problem (1.1) given by

$$H(v) = \frac{1}{p} \int_{-L}^{L} |v_x|^p \, dx - \frac{1}{q} \int_{-L}^{L} |v|^q \, dx,$$  

$v \in W^{1,p}_0([-L, L]).$ 

It is nonincreasing along the orbits of our problem, i.e. $\frac{d}{dt} H(u(t)) \leq 0$. It is also known (see [24, 25, 26]) that if $q \geq p$ and $\varphi \in W^{1,p}_0([-L, L]) \cap L^{\infty}([-L, L])$ satisfies $H(\varphi) < 0$, then the corresponding solution of problem (1.1) blows up in finite time.

By means of next lemma, whose proof follows the same ideas as in [9], we will show that whenever $u$ blows up so does $U$.

\textbf{Lemma 3.1} If $u$ blows up in finite time $T$ then

$$\lim_{t \to T} H(u(t)) = -\infty.$$  

\textbf{Proof of Proposition 1.1.} Let us define $H_h$, the discrete analogous of $H,$ as follows

$$H_h(t) \equiv H_h(U(t)) = \frac{1}{p} \langle -\nabla_p U, U \rangle(t) - \frac{1}{q} \langle |U|^{q-2} U, U \rangle(t).$$  

It is also nonincreasing along the orbits $H'_h(t) = -\langle U', U'' \rangle(t) \leq 0$.

By Theorem 1.1 it is easy to check the convergence $\|H_h(t) - H(t)\|_p \to 0$ as $h \to 0$, which added to Lemma 3.1 allows us to conclude that if $u$ blows
up in finite time, then $H_h(t_0) < 0$ for some $t_0$ and every $h$ small enough. Therefore, $H_h(t) < 0$ for every $t \geq t_0$.

Let us introduce the following functional

$$J(t) \equiv J(U(t)) = \frac{1}{2} \langle U, U(t) \rangle.$$  

We have

$$J'(t) = \langle U, U'(t) \rangle = -pH_h(t) + (1 - p/q) \langle U^{q/2}, U^{q/2}(t) \rangle.$$  

Since an additional term appears in the expression of the first derivative of $J$ if $q > p$, this case needs a different treatment. In fact, only one of the terms will be needed in each case in order to obtain the blow-up of $U$. For the case $q > p$ we have

$$J'(t) \geq C \langle U^{q/2}, U^{q/2}(t) \rangle \geq C \langle U, U \rangle^{q/2}(t) = C J(t)^{q/2}.$$  

Since $q > 2$, this means that $J$ blows up in finite time and so does $U$. For the case $q = p$ we have that the first two derivatives of the operator $J$ are nonnegative. Indeed, $J'(t) = -pH_h(t) > 0$ and $J''(t) = -pH_h'(t) = p(U', U')(t) > 0$.

On the other hand, we have the following inequality

$$(J'(t))^2 = \langle U, U' \rangle^2(t) \leq \langle U, U \rangle \langle U', U' \rangle(t) = \frac{2}{p} J(t) J''(t).$$  

Hence the function $J(t)^{1-\frac{2}{p}}$ is decreasing and concave. So it vanishes for some finite time, which means that $J(t)$ blows up in finite time. Therefore, $U$ also blows up in finite time \hfill \square

**Remark 3.1** In the first part of the proof we were just assuming that $q > \max(p, 2)$ and thus the result makes sense even for $1 < p \leq 2$.

Once we have proved the result above we are able to establish for the discrete problem blow-up conditions that are completely analogous to the ones given in Section 1 for the continuous case.

**Proof of Theorem 1.2.** The proof is given by comparison with a blowing-up subsolution or with a global supersolution.

i) If $q < p$ we consider the first eigenfunction $\psi$ of the discrete $p$-laplacian operator (problem (1.5)) in the interval $[-L - 1, L + 1]$. It is easy to see that,
for $k$ large enough we have that $k\psi$ is a global stationary supersolution of (1.3) and $\varphi(x) \leq k\psi(x)$. The comparison principle ends the first part of the theorem.

\textit{ii}) Let $\psi(x) > 0$ be the first eigenfunction to problem (1.5) with $\|\psi\|_{\infty} = 1$. Then, if $\lambda_1(L, h) < 1$ we have, for any $k > 0$,

$$H_h(0) = \frac{1}{p} \langle -\mathcal{D}_p (k\psi), k\psi \rangle - \frac{1}{p} \langle |k\psi|^{p-2}k\psi, k\psi \rangle = k^p \frac{\lambda_1 - 1}{p} \langle |\psi|^{p-2}\psi, \psi \rangle < 0.$$  

Therefore, the solution of (1.1) with the initial datum $k\psi(x)$ blows up in finite time by Proposition 1.1.

Given any nontrivial initial datum $\varphi(x) > 0$, and choosing $k > 0$ so small in order to have $\varphi(x) \geq k\psi$, we can conclude that the corresponding solution blows up in finite time by the comparison principle.

If on the contrary $\lambda_1(L, h) \geq 1$, we observe that $k\psi$ is a stationary supersolution of (1.1). Therefore, choosing $k$ large enough, and applying again the comparison principle, we obtain that $u$ is global.

\textit{iii}) Proposition 1.1 assures that for $q > p$ blowing-up solutions exist, those whose initial datum verify that $H_h(0) < 0$. \hfill $\square$

To finish this section we prove the convergence of the first eigenvalues.

\textbf{Proof of Theorem 1.3} We follow here the ideas given in [27]. We consider the characterization of the first eigenvalue of the continuous problem (1.2) as the solution of the minimization problem,

$$\lambda_1(L) = \inf_{\psi \in W^{1,p}_0([-L,L])} \left\{ \int_{-L}^L |\nabla \psi|^p dx : \int_{-L}^L |\psi|^p dx = 1 \right\}. \tag{3.10}$$

Also, the corresponding first discrete eigenvalue of problem (1.5) satisfies

$$\lambda_1(L, h) = \inf_{\psi_h \in V_h} \left\{ \int_{-L}^L |\nabla \psi_h|^p dx : \int_{-L}^L |\psi_h|^p dx = 1 \right\}, \tag{3.11}$$

where $V_h$ is the usual approximation of the space $W^{1,p}_0$ by finite elements considered in Section 1.2.

Let $\psi$ be an extremal for (3.10), that is, a solution of (1.2). We prove that there exists a constant $C$ independent of $h$ such that, for every $h$ small enough, it holds

$$|\lambda_1(L) - \lambda_1(L, h)| \leq C \inf_{v \in V_h} \|\nabla (\psi - v)\|_p^p.$$
Since $V_h \subset W_0^{1,p}$ we have the first estimate
\[ \lambda_1(L) \leq \lambda_1(L, h). \] (3.12)

Let us now choose $v \in V_h$ such that
\[ \|\nabla (\psi - v)\|_p^p \leq \inf_{\omega \in V_h} \|\nabla (\psi - \omega)\|_p^p + \epsilon. \]

If $\psi_h$ is an extremal for (3.11), we have
\[ \lambda_1(L, h)^{1/p} = \frac{\|\nabla \psi_h\|_p}{\|v\|_p} \leq \frac{\|\nabla (v - \psi)\|_p + \|\nabla \psi\|_p}{\|v\|_p} = \frac{\|\nabla (v - \psi)\|_p + \lambda_1(L)^{1/p}}{\|v\|_p}. \]

Now we use the inequalities
\[ 1 - \|v\|_p \leq \|v\|_p - 1 \leq \|\nabla \psi\|_p \leq C \|\nabla (v - \psi)\|_p, \]
and the fact that
\[ \lim_{h \to 0} \inf_{v \in V_h} \|v - \psi\|_{W_0^{1,p}} = 0, \]

to obtain that for every $h$ small enough,
\[ \lambda_1(L, h) \leq \left( \frac{\|\nabla (v - \psi)\|_p + \lambda_1(L)^{1/p}}{1 - C \|\nabla (v - \psi)\|_p} \right)^p \leq \lambda_1(L) + C \|\nabla (v - \psi)\|_p. \] (3.13)

From (3.12) and (3.13) the desired result follows. \qed

4 Blow-up rate and blow-up time

We devote this section to the proof of Theorems 1.4 and 1.5 related to the blow-up time. We start with the study of the blow-up rate. Observe that, from (1.4), the central node $u_0$ verifies
\[ u_0^{q-1} \geq u_0' \geq u_0^{q-1} (1 - \frac{2}{h^p} u_0^{q-p}). \] (4.14)

Integrating in time the first inequality we obtain the lower estimate of Theorem 1.4. In fact,
\[ u_0(t) \geq (q - 2)^{-1/(q-2)} (T_h - t)^{-1/(q-2)}. \] (4.15)
If \( q > p \) we can integrate the second inequality in (4.14) to conclude that,

\[
\lim_{t \to T_h} (T_h - t)^{\frac{1}{q-2}} u_0 = (q - 2)^{-1/(q-2)}
\]

proving Theorem 1.4 in the case \( q > p \). Notice that for \( q = p \) and \( h \) small the second inequality in (4.14) is meaningless.

In order to obtain the upper estimate for the case \( q = p \) we argue in a different way. We construct a solution of problem (1.6) which verifies:

**Lemma 4.1** Let \( W_h \) be the solution of (1.6). If the initial data \( w_0 \) is large enough then there exists a node \( x_J \) such that \( w_J < 0 \) and \( w_k > w_{k+1} \) for \( 0 < k < J \). Moreover, as \( h \) goes to zero \( W_h \) tends to the solution of the problem

\[
\begin{cases}
|z'|^{p-2}z' + |z|^{p-2}z - \frac{1}{p-2}z = 0, & x > 0, \\
z(0) = w_0, \\
z'(0) = 0,
\end{cases}
\]

in its positivity set.

This problem has a unique weak solution with compact support for a precise value of \( z(0) \), see [17]. Also, for \( z(0) \) large there exists a unique classical solution which crosses the horizontal axis at a finite point \( x_* \), and when \( z(0) \) increases the point \( x_* \) decreases; if \( z(0) \to \infty \) then \( x_* \to L_0 \), where \( L_0 \) is given in Section 1. See also [15].

Therefore, for any bounded initial datum \( \varphi \) there exists an initial condition \( z(0) \) large enough such that \( z(0) > T_h^{\frac{1}{p-2}} \varphi(0) \) and intersecting \( \varphi \) only once, at some point \( x < L \).

Assuming the previous lemma, we observe that since \( W_h \) converge towards the profile \( z \), we have for \( h \) sufficiently small,

\[
\omega_j \geq T_h^{\frac{1}{p-2}} \varphi(x_j) \quad \forall j \leq k_0 \quad \text{and} \quad \omega_j < T_h^{\frac{1}{p-2}} \varphi(x_j) \quad \forall j > k_0,
\]

for some unique \( 0 \leq k_0 \leq N \). Therefore, considering the solutions of (1.4) given by \( U \) and \( Y = (T_h - t)^{-1/(q-2)}W \), we are in the hypothesis of Lemma 2.4 with \( t^* = 0 \), therefore the number of intersections of \( U \) and \( Y \) is at most one for every \( t > 0 \). On the other hand, since both functions blow up at the same time the comparison principle implies that this number cannot be zero. Therefore we conclude that \( u_0(t) \leq y_0(t) = w_0(T_h - t)^{-1/(q-2)} \), where \( w_0 \) is independent of \( h \). This end the proof of Theorem 1.4.
Proof of Lemma 4.1. Multiplying (1.6) by \((\omega_{j+1} - \omega_{j-1})/2\) and performing the sum, we get, for every \(k \geq 1\),

\[
0 = \sum_{j=1}^{k} \left\{ \frac{h^{-p}}{p} \omega_j + \omega_{j-1}^{p-1} - \frac{1}{p-2} \omega_j \right\} \frac{\omega_{j+1} - \omega_{j-1}}{2} \tag{4.18}
\]

where \(F(x) = \frac{1}{p}x^p - \frac{1}{2(p-2)}x^2\), see for instance [2].

On the other hand, we note that if \(w_0\) is large then \(w_1 < w_0\). We now suppose that there exists a node \(x_K\) such that \(w_{K-1} > w_K < w_{K+1}\). Then from the equation (1.6) at node \(x_K\) we obtain

\[
\omega_{K-1}^{p-1} - \frac{1}{p-2} \omega_K = h^{-p} D_p \omega_K > 0.
\]

Therefore \(w_K \leq (p - 2)^{-1/(p-2)}\) (which implies \(F(\omega_K) < 0\)). From the same equation it also follows that

\[
h^{-p}(\omega_K - \omega_{K-1})^{p-1} \leq \omega_{K-1}^{p-1} - \frac{1}{p-2} \omega_K \leq C,
\]

and consequently \(((w_K - w_{K-1})/h)^{p-1} \leq Ch\). Substituting this inequality into (4.18) we finally get \(F(\omega_K) = F(\omega_0) + O(h)\), which gives a contradiction since \(w_0\) is large.

There must exist a node \(x_{j_0}\) such that \(F(\omega_{j_0}) \leq 0\). If on the contrary,

\[
w_k > \left( \frac{p}{2(p-2)} \right)^{1/p-2} > \left( \frac{1}{(p-2)} \right)^{1/p-2}
\]

for all \(k \geq 0\), the functions \(V(t) = (T_h - t)^{-1/(p-2)} W_h\) and \(\tilde{V}(t) = (T_h - t)^{-1/(p-2)} (p - 2)^{-1/(p-2)}\) would be ordered having the same blow-up time, which is a contradiction. Since \(W_h\) is decreasing we have

\[
\omega_k \leq \omega_{j_0-1} - Ch(F(\omega_0)) \quad \forall k > j_0.
\]

This implies that for each \(h\) fixed, there must exist a first node \(x_J\) such that \(\omega_J < 0\). Considering (4.18) for this node we can conclude that there must exist a sequence \(h_n \to 0\) such that \(x_J \to \tilde{L}, w_J \to 0\) and \((\omega_J - \omega_{J-1})/h_n \to -\Lambda < 0\).
Therefore, we consider the auxiliary problem
\[
\begin{aligned}
&\begin{cases}
(z'p^{-2}z')' + z^{p-1} - \frac{1}{p-2}z = 0, \\
z(0) = \omega_0, \\
z(\tilde{L}) = 0. 
\end{cases} \\
&x \in [0, \tilde{L}],
\end{aligned}
\]  
(4.19)

It is known that this problem has a unique solution for certain \(\tilde{L}\). By classical theory we have the convergence of \(W_h\) to \(z\) in \((0, \tilde{L}]\). Observing that at the degeneration point \(x = 0\), we have \(\omega_0 = z(0)\), the convergence is in the whole interval.

On the other hand, multiplying by \(z'\) the equation in (4.19) we obtain
\[
\left(1 - \frac{1}{p}\right) |z'|^p + F(z) = \left(1 - \frac{1}{p}\right) \Lambda^p.
\]
Taking the limit in (4.18) with \(k = J\), we get \(F(\omega_0) = (1 - 1/p)\Lambda^p\), therefore \(z'(0) = 0\). 

Now, we turn our attention to the blow-up time.

**Proof of Theorem 1.5.** Let \(U\) be a blowing-up solution. From (4.15) there exists a constant \(C\) independent of \(h\) such that
\[
T_h - t \leq Cu_0^{2-q}.
\]
On the other hand, as \(u\) blows up at time \(T\) we can choose \(t_0\) such that
\[
T - t_0 \leq \varepsilon \quad \text{and} \quad u(0, t_0) > C_1 \varepsilon^{-1/(q-2)}.
\]
If \(h\) is small enough, by the convergence result we have \(u_0(t_0) > C_1 \varepsilon^{-1/(q-2)}/2\) and hence, \(T_h - t_0 < C \varepsilon\).

Finally, \(|T_h - T| \leq |T_h - t_0| + |T - t_0| < C \varepsilon\). 

**5 Blow-up sets**

Now we turn our interest to the blow-up set of the numerical solution so we devote this section to the proof of Theorem 1.6, considering separately both cases, \(q > p\) and \(q = p\). For a fixed \(h\) we want to look at the set of nodes \(x_k\) such that \(u_k(t) \to +\infty\) as \(t \not\to T_h\). This task will be carried out by means
of the self-similar variables introduced in (1.7), which transforms our system into

\[
\begin{aligned}
y_0'(s) &= -2e^{-s^{q-2}h}y_0(s) - y_1(s) + y_0^{-1}(s) - \frac{1}{q-2}y_0(s), \\
y_k'(s) &= e^{-s^{q-2}h}D_p y_k(s) + y_k^{-1}(s) - \frac{1}{q-2}y_k(s), \quad k > 0, \\
y_N(s) &= 0.
\end{aligned}
\]

We start with the case \( q > p \), where reaction is stronger than diffusion. From (4.16) we have that

\[
\lim_{s \to \infty} y_0(s) = C_q = \left( q - 2 \right)^{-1/(q-2)}.
\]

Moreover, the only node that behaves like \( C_q(T_h - t)^{-1/(q-2)} \) is \( x = 0 \), see [22]. Using that \( u_1(t)(T_h - t)^{1/(q-2)} \to 0 \) in the expression verified by \( y_1(s) \) we get

\[
y_1' = e^{-s^{q-2}h}D_p y_1(s) + y_1^{-1}(s) - \frac{1}{q-2}y_1(s) \sim C e^{-s^{q-2}} - \frac{1}{q-2}y_1(s).
\]

By integration we get

\[
0 \leq y_1(s) \sim \begin{cases} 
C e^{-s^{q-2}}, & q < p + 1, \\
C s e^{-s^{q-2}}, & q = p + 1, \\
C e^{-s^{q-2}}, & q > p + 1,
\end{cases}
\]

where it is easily observed that in all cases \( y_1(s) \to 0 \). Translating this behaviour into the \( U \) variables,

\[
u_1(s) \sim \begin{cases} 
C (T_h - t)^{\frac{q-p}{q-2}}, & q < p + 1, \\
-C \ln(T_h - t), & q = p + 1, \\
C, & q > p + 1,
\end{cases}
\]

which shows that if \( q \leq p + 1 \) the node \( u_1(t) \) blows up with different rate than \( u_0 \), and for \( q > p + 1 \) it is even bounded.

We also obtain that \( y_k(s) \to 0 \), for all \( k \neq 0 \), using the same argument for the other nodes. Moreover, once we know the asymptotic behaviour of one node we can obtain the behaviour of the following repeating the previous procedure, which also give us the number of nodes that do blow up, the only integer \( K \) that verifies the following expression,

\[
\frac{\sum_{i=0}^{K+1} (p-1)^i}{\sum_{i=0}^{K} (p-1)^i} < q - 1 \leq \frac{\sum_{i=0}^{K} (p-1)^i}{\sum_{i=0}^{K-1} (p-1)^i}.
\]
that is
\[ K = \left\lceil \frac{\ln((q-2)/(q-p))}{\ln(p-1)} \right\rceil. \] (5.20)

Now we consider the case \( q = p \) and prove the convergence of the function \( Y_h(s) \) to a stationary profile. To do that, we consider the following Lyapunov functional
\[
\Phi(Y(s)) = -\frac{1}{p} \langle D_p Y(s), Y(s) \rangle - \frac{1}{p} \langle Y(s)^p - 1, Y(s) \rangle + \frac{1}{2(p-2)} \langle Y(s), Y(s) \rangle,
\]
which is decreasing along the orbits. A classical argument gives us the convergence result, see [14]. Moreover, if we begin with a nonnegative and decreasing initial data \( Y(0) \), then for a fixed \( s \) the vector \( Y(s) \) is nonnegative and decreasing. Hence we have to look at a nonnegative and nonincreasing stationary solution.

This solution must be positive for all \( 0 < k < N \). If not, i.e., if \( \omega_j = 0 \) for some \( 0 < j < N \), we would have that \( \omega_k = 0 \) for all \( k \geq j \), since \( W_h \) is nonnegative and nonincreasing, which also implies that \( \omega_k = 0 \) for all \( k \). But this is no possible since \( \omega_0 > 0 \).

Therefore, in the original variable
\[ u_k \sim (T_h - t)^{-1/(p-2)} \omega_k \rightarrow \infty, \quad 0 < k < N. \]

We have obtained in this way global blow-up for every \( L > L_0 \). This gives us the expected blow-up set if \( L_0 < L \leq L_1 \). In order to recover the expected regional blow-up set for \( L > L_1 \) we prove that for every node \( x_k \) that lies in the interval \([L_1, L]\), we have that \( \omega_k \rightarrow 0 \) as \( h \rightarrow 0 \). This is a consequence of the convergence of the discrete profiles to the unique continuous one satisfying (4.17), \( z(x) \geq 0, \forall x \in [0, L] \) and \( z(L) = 0 \). This convergence can be proven in a similar way than in the previous section (see also [13]) and allows us to recover in this sense the regional blow-up. This finishes the proof of Theorem 1.6.

6 Numerical experiments

In this section we illustrate our previous results with some numerical experiments which show some of the properties observed for the numerical solutions. In all cases we take the initial data \( \varphi(x) = L^2 - x^2 \). To perform
the integration in time we use an adaptative ODE solver for stiff problems (Matlab ODE23s).

First we deal with the case \( q = p \). Figure 1 shows the evolution of the numerical solution corresponding to the values \( q = p = 2.3 \) and \( L = 2 \), whose blow-up time turns out to be \( T_h = 4.993189 \). In figure 2 we display \( \ln(u_i) \) versus \(-\ln(T_h - t)\) for \( i = 0, N - 1 \) in order to show that with our scheme every node blows up with the same rate.

![Figure 1. Evolution of the numerical solution \((q = p, L < L_1)\).](image1)

![Figure 2. Blow-up rates \((q = p)\).](image2)

However for larger values of \( L \) we expect regional blow-up. To see that, we perform the experiment with \( q = p = 2.3 \) and \( L = 15 \). In Figure 3 we compare the discrete rescaled solution in self-similar variables near \( T_h \) with the self-similar profile. From this picture one can appreciate that at the nodes that lay outside of the expected blow-up set for the continuous solution, the discrete rescaled solution is positive but small. And it is in this sense that we recover the regional blow-up for the numerical solution, whose evolution is depicted in Figure 4.
Finally, we consider the case $q = 3$, $p = 2.3$. We find that the corresponding blow-up time is $T_h = 0.00441$ and $K = 1$. We show the evolution of the numerical approximation in Figure 5. In order to obtain the numerical blow-up rates and the numerical blow-up set. In Figure 6 we display again $\ln(u_i)$ versus $-\ln(T_h - t)$ for $i = 0, 1, 2$. It can be observed that the first curve (corresponding to $\max u_k = u_0$) has slope 1, the second (for $u_1$) has slope 0.3 and the last one (for $u_2$) has zero slope. This means in particular that the third node ($u_2$) does not blow up and first two nodes blow up with different rates. These behaviours correspond to the expected results given in the previous sections.
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References


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