SEMILINEAR BIHARMONIC PROBLEMS WITH A SINGULAR TERM.

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Abstract. The aim of this work is to study the optimal exponent \( p \) to have solvability of problem

\[
\begin{aligned}
\Delta^2 u &= \lambda \frac{u}{|x|^4} + u^p + cf \quad \text{in } \Omega, \\
u > 0 &\quad \text{in } \Omega, \\
u = -\Delta u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( p > 1, \lambda > 0, c > 0, \) and \( \Omega \subset \mathbb{R}^N, N > 4, \) is a smooth and bounded domain such that \( 0 \in \Omega. \)

We consider \( f \geq 0 \) under some hypothesis that we will precise later.

1. Introduction

The main concern of this work is to determine a critical threshold exponent \( p \) to have solvability of the following problem,

\[
\begin{aligned}
\Delta^2 u &= \lambda \frac{u}{|x|^4} + u^p + cf \quad \text{in } \Omega, \\
u > 0 &\quad \text{in } \Omega, \\
u = -\Delta u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( p > 1, \lambda > 0, c > 0, \) and \( \Omega \subset \mathbb{R}^N, N > 4, \) is a smooth and bounded domain such that \( 0 \in \Omega. \)

There exists a large literature dealing with the case \( \lambda = 0. \) The differential equation \( \Delta^2 u = f \) is called the Kirchhoff-Love model for the vertical deflection of a thin elastic plate. For a more elaborate history of the biharmonic problem and the relation with elasticity from an engineering point of view one may consult a survey of Meleshko [19]. In particular, the Kirchhoff-Love model with Navier conditions, \( u = -\Delta u = 0 \) on \( \partial \Omega, \) corresponds to the hinged plate model and allows rewriting these fourth order problems as a second order system.

Among nonlinear problems for fourth order elliptic equations with \( \lambda = 0, \) there exist several results concerning to semilinear equations with power type nonlinear sources (see for example [17]). A crucial role is played by the critical (Sobolev) exponent, \( \frac{N+4}{N-4}, \) which appears whenever \( N > 4 \) (see [5]).

The case \( \lambda > 0 \) is quite different. The singular term \( \frac{u}{|x|^4} \) is related to the Hardy inequality:
Let $u \in C^2(\Omega)$ and $N > 4$, then it holds that

$$
\Lambda_N \int_{\Omega} \frac{u^2}{|x|^4} \, dx \leq \int_{\Omega} |\Delta u|^2 \, dx,
$$

where $\Lambda_N = \left(\frac{N^2(N-4)^2}{16}\right)$ is optimal (see Theorem 2.2 below).

First of all, it is not difficult to show that any positive supersolution of (1) is unbounded near the origin and then additional hypotheses on $p$ are needed to ensure existence of solutions. We will say that problem (1) blows-up completely if the solutions to the truncated problems (with the weight $\frac{\lambda}{|x|^4 + \frac{n}{4}}$ instead of the Hardy type term $\frac{\lambda}{|x|^4}$) tend to infinity for every $x \in \Omega$ as $n \to \infty$.

The main objective of this work is to explain the influence of the Hardy type term on the existence or nonexistence of solutions and to determine the threshold exponent $p_+ (\lambda)$ to have a complete blow-up phenomenon if $p \geq p_+ (\lambda)$.

The corresponding elliptic semilinear case with the Laplacian operator was studied in [8, 14], where the authors show the existence of a critical exponent $p^* (\lambda) > 1$ such that the problem has no local distributional solution if $p \geq p^* (\lambda)$. Furthermore, they prove the existence of solutions with $p < p^* (\lambda)$ under some suitable hypothesis on the datum.

In fourth order problems, Navier boundary conditions play an important role to prove existence results. The problem can be rewritten as a second order system with Dirichlet boundary conditions. By classical elliptic theory, we easily prove a Maximum Principle. As a consequence, we deduce a Comparison Principle that allows us to prove the existence of solutions for $p < p_+ (\lambda)$ as a limit of approximated problems.

The paper is organized as follows.

In Section 2 we briefly describe the natural functional framework for our problem and the embeddings we will use throughout the paper.

Section 3 is devoted to some definitions and preliminary results. First, we describe the radial solutions to the homogeneous problem that allow us to know the singularity of our supersolutions near the origin and prove nonexistence results with local arguments. The notion of solution we are going to consider in the nonexistence results is local in nature, we just ask the regularity needed to give distributional sense to the equation. In this section, we prove a Moreau type decomposition and a Picone inequality that will be used in the nonexistence proofs and which are interesting themselves. However, to prove existence of solutions to (1) with $L^1$ data, we will consider the solution obtained as limit of approximations. For this, we use a comparison result that immediately follows from Navier boundary conditions.

In Section 4 we prove the existence of a threshold exponent for existence of solutions, namely, when we consider an exponent over the critical, there do not exist positive solutions even in the sense of distributions. Furthermore, in Section 5, we prove that a complete blow-up phenomenon occurs over the critical exponent.

Section 6 deals with the complementary interval of powers. In this range it is shown that, under some suitable hypotheses on $f$, problem (1) has a positive solution.
2. Functional Framework

We briefly describe the natural framework to treat the solutions to the problem in consideration. Let \( \Omega \subset \mathbb{R}^N \) denote a bounded and smooth domain. We define the Sobolev space
\[
W^{k,p}(\Omega) = \{ u \in L^p(\Omega), \ D^\alpha u \in L^p(\Omega) \text{ for all } 1 \leq |\alpha| \leq k \},
\]
edowed with the norm
\[
\|u\|_{W^{k,p}(\Omega)} = \left( \int_\Omega |u|^p dx + \int_\Omega \sum_{1 \leq |\alpha| \leq k} |D^\alpha u|^p dx \right)^{\frac{1}{p}},
\]
which is equivalent to (2). Using interpolation theory one can get rid of the intermediate derivatives and find that
\[
\|u\|_{W^{k,p}(\Omega)} = \|u\|_p + \sum_{1 \leq |\alpha| \leq k} \|D^\alpha u\|_p,
\]
where \( \| \cdot \|_p \) denotes the usual norm in \( L^p(\Omega) \) for \( 1 \leq p < \infty \).

Taking the closure of \( C_0^k(\Omega) \) in \( W^{k,p}(\Omega) \) gives rise to the following Sobolev space \( W_0^{k,p}(\Omega) \), with the norm
\[
\|u\|_{W_0^{k,p}(\Omega)} = \left( \int_\Omega \sum_{1 \leq |\alpha| \leq k} |D^\alpha u|^p dx \right)^{\frac{1}{p}},
\]
equivalent to (2).

In fact, using interpolation theory one can get rid of the intermediate derivatives and find that
\[
\|u\|_{W_0^{k,p}(\Omega)} = \left( \int_\Omega \sum_{1 \leq |\alpha| \leq k} |D^\alpha u|^p dx \right)^{\frac{1}{p}},
\]
defines a norm which is equivalent to (2), and similarly
\[
\|u\|_{W_0^{k,p}(\Omega)} = \left( \int_\Omega |D^k u|^p dx \right)^{\frac{1}{p}},
\]
see for instance [2].

The following embeddings will be useful in the forthcoming arguments.

**Theorem 2.1** (Rellich-Kondrachov's Theorem). Let \( k \in \mathbb{N}^+ \) and \( 1 \leq p < +\infty \). Suppose that \( \Omega \subset \mathbb{R}^N \) is a Lipschitz domain. Then,
\[
W^{k+j,p}(\Omega) \hookrightarrow W^{j,p}(\Omega) \text{ and } W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \text{ for all } 1 \leq q \leq \frac{Np}{N-kp} = p^*,
\]
with the convention that \( \frac{Np}{N-kp} = +\infty \) if \( kp \geq N \). Moreover, this embedding is compact if \( q < \frac{Np}{N-kp} \).

The statement of the previous theorem also holds if we replace \( W^{m,p}(\Omega) \) with its proper subspace \( W_0^{m,p}(\Omega) \).

In particular, the natural space for our problem is
\[
V = \{ u \in W^{2,2}(\Omega), \text{ such that } u = 0, -\Delta u = 0 \text{ on } \partial \Omega \} \subset W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).
\]
Let us briefly discuss that for certain domains, $\mathcal{H} := W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ is a Hilbert space (so it is $V$ as a closed subset of $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$), endowed with the following scalar product

(4) $\langle u, v \rangle_{\mathcal{H}} = \int_{\Omega} \Delta u \Delta v \, dx,$

which induces the norm

(5) $\|u\|_{\mathcal{H}} = \|\Delta u\|_{L^2(\Omega)},$

equivalent to (3) with $k = p = 2$. Note that in (5) we ignore not only the intermediate derivatives, but also some of the highest order derivatives in a larger space than $W^{2,2}_0(\Omega)$.

Trivially, for any $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ it holds that

$$|D^2 u| = \sum_{i,j=1}^N \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \geq \frac{1}{N} \left( \sum_{i=1}^N \left( \frac{\partial^2 u}{\partial x_i^2} \right) \right)^2 \geq \frac{1}{N} (\Delta u)^2.$$  

To show the reverse inequality we refer to Theorem 2.2 in [3], where it is shown that under certain conditions on the domain, in particular for smooth domains, there exists a positive constant, independent of $u$, such that

$$\|u\|_{W^{2,2}(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)}.$$

From Rellich-Kondrachov’s Theorem it follows that,

$$W^{2,2}(\Omega) \hookrightarrow W^{1,r}(\Omega) \text{ with } r \leq \frac{2N}{N-2} = 2^*$$

(6) $$W^{2,2}(\Omega) \hookrightarrow L^q(\Omega) \text{ with } q \leq \frac{2N}{N-4} = 2^*,$$

where the embeddings are compact with strict inequalities on $r$ and $q$. There exists an extensive literature about this field. For more details and properties of higher order spaces and some applications we refer for example to the books [14, 17] and references therein. See also the classic books [2, 18]. We denote by $W^{k,p}(\Omega)$ as the set of functions belonging to $W^{k,p}(\Omega')$, for any $\Omega' \subset \subset \Omega$.

Furthermore, we need to understand deeply the behavior of the linear operator:

$$\Delta^2(\cdot) - \frac{\lambda}{|x|^4}(\cdot) : W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \rightarrow W^{-2,2}_0(\Omega)$$

It is well known that this operator is coercive if $0 \leq \lambda < \Lambda_N$, noting by $\Lambda_N = \left( \frac{N^2(N-4)^2}{16} \right)$. This coercivity is a direct consequence of the the optimality of the constant $\Lambda_N$ in the Rellich inequality, stated in the next theorem. The proof of this result is due to Rellich in [25], but see also [13] for alternative proofs. For convenience of the reader, we include here an elementary proof of this inequality (based in a proof of María J. Esteban for the Laplacian operator, see also [20]).

**Theorem 2.2.** For any $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ and $N > 4$, it holds that

(7) $$\Lambda_N \int_{\Omega} \frac{u^2}{|x|^4} \, dx \leq \int_{\Omega} |\Delta u|^2 \, dx,$$

where $\Lambda_N = \left( \frac{N^2(N-4)^2}{16} \right)$ is optimal.
Proof. We consider the following identities that follow from integrating by parts

\[ 0 \leqslant \left\| \Delta u \frac{x}{|x|} + \lambda u \frac{x}{|x|^3} \right\|_{L^2(\Omega)}^2 = \int_\Omega (\Delta u)^2 \, dx + \lambda^2 \int_\Omega \frac{u^2}{|x|^2} \, dx + 2\lambda \int_\Omega \frac{u\Delta u}{|x|^2} \, dx \]

(8)

\[ = \int_\Omega (\Delta u)^2 \, dx + \lambda^2 \int_\Omega \frac{u^2}{|x|^2} \, dx - 2\lambda \int_\Omega (\nabla u, \nabla \left( \frac{u}{|x|^2} \right)) \, dx = \int_\Omega (\Delta u)^2 \, dx \]

\[ + \lambda^2 \int_\Omega \frac{u^2}{|x|^4} \, dx - 2\lambda \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx + 4\lambda \int_\Omega \frac{\langle \nabla u, x \rangle u}{|x|^4} \, dx. \]

Noticing that

\[ \sum_{i=1}^N \frac{\partial u}{\partial x_i} x_i = \sum_{i=1}^N \frac{\partial (ux_i)}{\partial x_i} - Nu, \]

we can write

\[ \int_\Omega \frac{\langle \nabla u, x \rangle u}{|x|^4} \, dx = -\int_\Omega (\nabla \left( \frac{u}{|x|^2} \right), x) u \, dx - N \int_\Omega \frac{u^2}{|x|^2} \, dx \]

\[ = -\int_\Omega (\nabla u, x) u \, dx - (N - 4) \int_\Omega \frac{u^2}{|x|^4} \, dx. \]

Hence,

\[ \int_\Omega \frac{\langle \nabla u, x \rangle u}{|x|^4} \, dx = -\frac{(N - 4)}{2} \int_\Omega \frac{u^2}{|x|^4} \, dx. \]

On the other hand, thanks to Caffarelli-Kohn-Nirenberg inequalities (see [10]) with \( p = 2, \gamma = 1 \),

\[ \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx \geqslant \left( \frac{N - 4}{2} \right)^2 \int_\Omega \frac{u^2}{|x|^4} \, dx. \]

Substituting these last two expressions in (8) we obtain

\[ 0 \leqslant \int_\Omega (\Delta u)^2 \, dx + \left( \lambda^2 + 2\lambda \frac{N(N - 4)}{4} \right) \int_\Omega \frac{u^2}{|x|^4} \, dx. \]

The polynomial \( p(\lambda) = -\lambda^2 - 2\lambda \frac{N(N - 4)}{4} \) reaches its maximum at \( \lambda = \frac{N(N - 4)}{4} \) and (7) follows. \( \square \)

This constant \( \Lambda_N \) is optimal but not achieved. For literature on this topic see for instance [13, 16] and the references therein.

3. SOME DEFINITIONS AND PRELIMINARY RESULTS

For \( 0 \leqslant \lambda < \Lambda_N \), let us consider the homogenous equation,

\[ \Delta^2 u - \lambda \frac{u}{|x|^4} = 0 \text{ in } \mathbb{R}^N. \]

(9)
We denote by

\[ \alpha_{\pm}(\lambda) = \frac{N-4}{2} \pm \frac{1}{2} \sqrt{(N^2 - 4N + 8) - 4\sqrt{N^2 - 4N + 8} + 4(N - 2)^2 + \lambda} \]

\[ \beta_{\pm}(\lambda) = \frac{N-4}{2} \pm \frac{1}{2} \sqrt{(N^2 - 4N + 8) + 4\sqrt{N^2 - 4N + 8} + \lambda} \]

the four real roots which give the radial solutions \(|x|^{-\alpha_{\pm}}\), \(|x|^{-\beta_{\pm}}\) to equation (9). It follows that \(\beta_{-}(\lambda) \leq \alpha_{-}(\lambda) \leq \alpha_{+}(\lambda) \leq \beta_{+}(\lambda)\).

For \(0 \leq \lambda < \Lambda_N\), then \(\alpha_{-} \in [0, \frac{N-4}{2}]\) and \(\alpha_{+} \in (\frac{N-4}{2}, N-4]\) and if \(\lambda = \Lambda_N = \frac{N^2(N-4)^2}{16}\), then \(\alpha_{-} = \alpha_{+} = \frac{N-4}{2}\). Moreover, notice that if \(0 \leq \lambda < \Lambda_N\), then \(\beta_{-} \leq -2\), and \(\beta_{+} \geq (N-2)\).

We also remark that \(|x|^{-\beta_{-}}\) is the unique bounded solution and \(|x|^{-\alpha_{-}}\) is the unique singular solution that belongs to the Sobolev Space \(W^{2,2}(\Omega)\) for \(\lambda < \Lambda_N\).

We give the notion of solution we are going to consider in the nonexistence results.

**Definition 3.1.** We say that \(u \in L^1_{\text{loc}}(\Omega)\) is a supersolution (subsolution) to

\[ \begin{cases} \Delta^2 u = g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \]

in the sense of distributions if \(g \in L^1_{\text{loc}}(\Omega)\) and for all positive \(\phi \in C^\infty_0(\Omega)\), we have that

\[ \int_{\Omega} \Delta u \Delta \phi \, dx \geq (\leq) \int_{\Omega} g(u) \phi \, dx. \]

If \(u\) is a supersolution and subsolution in the sense of distributions, then we say that \(u\) is a distributional solution to (1).

In particular, in problem (1), we consider \(g(u) = (\lambda \frac{u^{p+1}}{u^r} + u^p + cf) \in L^1_{\text{loc}}(\Omega)\).

In the previous Definition 3.1 we just ask the regularity needed to give distributional sense to the equation. As we will prove below, Remark 3.11, if \(u\) is a supersolution to (1) in the sense of distributions, then \(g\) must satisfy a regularity condition. We will use this general framework to prove nonexistence of positive solutions \(u\) satisfying \(-\Delta u \geq 0\) in \(\Omega\).

We note here that from Navier boundary conditions, we can straightforward deduce a Strong Maximum Principle and as a consequence, a comparison result.

**Lemma 3.2. Strong Maximum Principle** Let us consider \(u\) to be a nontrivial supersolution to

\[ \begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = -\Delta u = 0 & \text{on } \partial\Omega. \end{cases} \]
Then \(-\Delta u > 0\) and \(u > 0\) in \(\Omega\).

**Proof.** Considering the change of variables \(-\Delta u = v\), if \(u\) is a supersolution to (11), then \(v\) is a supersolution to

\[
\begin{aligned}
-\Delta v &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Applying the known Strong Maximum Principle to the Laplacian operator, it immediately follows that \(v > 0\) in \(\Omega\) and then \(u > 0\) in \(\Omega\). \(\square\)

**Lemma 3.3. Comparison Principle** Let \(u\) and \(v\) satisfy the following

\[
\begin{aligned}
\Delta^2 u &\geq \Delta^2 v \text{ in } \Omega, \\
u &\geq v \text{ on } \partial \Omega, \\
-\Delta u &\geq -\Delta v \text{ on } \partial \Omega.
\end{aligned}
\]

Then, \(-\Delta v \leq -\Delta u\) and \(v \leq u\) in \(\Omega\).

**Proof.** It is sufficient to apply to \(w = u - v\), a supersolution to (11), the previous Strong Maximum Principle.

Notice that from the associated system (12), we can also easily obtain a Weak Harnack’s type inequality.

**Lemma 3.4. Weak Harnack inequality** Let \(u\) be a positive distributional supersolution to (11), then for any \(B_R(x_0) \subset \subset \Omega\), there exists a positive constant \(C = C(\theta, \rho, q, R)\), \(0 < q < \frac{N}{N - 2}\), \(0 < \theta < \rho < 1\), such that

\[
\|u\|_{L^q(B_{\rho R}(x_0))} \leq C \inf_{B_{\theta R}(x_0)} u.
\]

Related to comparison of solutions we have the following decomposition due to Moreau, see [21].

For details of the proof considering more general operators, we refer for instance to [15, 17].

However, for completeness, we perform the proof in the case of the bilaplacian operator in the Hilbert Space \(H = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\), with the scalar product defined in (4), see [14, 17].

**Proposition 3.5.** Let \(u \in H\). Then, there exist unique \(u_1, u_2 \in H\) such that \(u = u_1 + u_2\), satisfying \(u_1 \geq 0\), and \(u_2 \leq 0\) in \(\Omega\), and \(\int_\Omega \Delta u_1 \Delta u_2 dx = 0\).

**Proof.** The arguments of this proof are based on the fact that \(W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\) is a Hilbert Space endowed with the following scalar product

\[
(u, v)_H = \int_\Omega \Delta u \Delta v dx.
\]

Then, if we consider the cone of the a.e positive functions defined in \(\Omega\),

\[
C = \{ v \in H, \text{ such that } v(x) \geq 0 \text{ a.e in } \Omega \}
\]

the corresponding dual cone with respect to the scalar product above is defined as

\[
C^* = \{ w \in H, \text{ such that } \int_\Omega \Delta u \Delta w dx \leq 0, \text{ for every } u \in C \}.
\]

Let us take as \(u_1\) the orthogonal projection of \(u \in H\) on \(C\), namely, let \(u_1\) be such that

\[
\int_\Omega |\Delta(u - u_1)|^2 dx = \min_{v \in C} \int_\Omega |\Delta(u - v)|^2 dx.
\]
Letting \( u_2 = u - u_1 \), for all \( t \geq 0 \) and \( \nu \in \mathcal{C} \), it holds that
\[
\int_{\Omega} |\nabla(u - u_1)|^2 dx \leq \int_{\Omega} |\nabla(u - (u_1 - t\nu))|^2 dx = \int_{\Omega} |\nabla(u - u_1)|^2 dx - 2t \int_{\Omega} \nabla(u - u_1) \nabla \nu dx + t^2 \int_{\Omega} |\nabla \nu|^2 dx.
\]

Therefore,
\[
(13) \quad 2t \int_{B_R(0)} \nabla(u - u_1) \nabla \nu dx \leq t^2 \int_{B_R(0)} |\nabla \nu|^2 dx.
\]

Choosing \( t > 0 \), simplifying and making then \( t \to 0 \), we obtain that
\[
\int_{\Omega} \nabla u_2 \nabla \nu dx \leq 0,
\]
for any \( \nu \in \mathcal{C} \), hence \( u_2 \in \mathcal{C}^* \).

In particular, we can put \( \nu = u_1 \), thus
\[
\int_{\Omega} \nabla u_1 \nabla u_2 dx \leq 0.
\]

Arguing analogously, for some \( t \in [-1, 0) \) in (13), and letting \( t \) tend to 0, we get the reverse inequality, and hence
\[
\int_{\Omega} \nabla u_1 \nabla u_2 dx = 0.
\]

Let us show next the uniqueness. Assume that \( u = u_1 + u_2 = v_1 + v_2 \) with \( u_1, v_1 \in \mathcal{C} \) and \( u_2, v_2 \in \mathcal{C}^* \). Then
\[
0 = \langle u_1 - v_1 + u_2 - v_2, u_1 - v_1 + u_2 - v_2 \rangle_H = \|u_1 - v_1\|_H^2 + \|u_2 - v_2\|_H^2 + 2(u_1 - v_1, u_2 - v_2)_H
= \|u_1 - v_1\|_H^2 + \|u_2 - v_2\|_H^2 - 2(u_1, v_2)_H - 2(v_1, u_2)_H
\geq \|u_1 - v_1\|_H^2 + \|u_2 - v_2\|_H^2.
\]
This implies that \( u_1 = v_1 \) and \( u_2 = v_2 \) as desired.

To conclude the proof we show that every function \( w \in \mathcal{C}^* \) is non positive and, in particular, \( u_2 \leq 0 \). For every arbitrary nonnegative \( h \in \mathcal{C}_0^0(\Omega) \), consider the solution to the following problem
\[
\begin{align*}
\nabla^2 v &= h & \text{in } \Omega, \\
v &= 0, \quad -\Delta v &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

By the Maximum Principle \( v \in \mathcal{C} \). But then,
\[
0 \geq \int_{\Omega} \Delta v \Delta w dx = \int_{\Omega} hw dx,
\]
for every nonnegative function \( h \in \mathcal{C}_0^0(\Omega) \). By density, we conclude that \( w(x) \leq 0 \) a. e. \( x \in \Omega \) as we wanted to prove. \( \square \)

To show existence of solutions to (1) with \( L^1 \) data, we will consider the solution obtained as limit of approximations (see [6, 7, 11, 23] for the Laplacian operator case).
We denote
\[ T_k(s) = \begin{cases} s, & |s| \leq k \\ k \text{sign}(s), & |s| > k, \end{cases} \]
the usual truncation operator. Consider \( f \in L^1(\Omega), f \geq 0 \), and let \( \{f_n\} \) be a sequence of functions such that, \( f_n(x) \leq f(x), x \in \Omega \), and \( f_n \to f \) in \( L^1(\Omega) \).

Let \( u_n \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \) be the weak solution to problem,
\[
\begin{cases}
\Delta^2 u_n = f_n & \text{in } \Omega, \\
u_n > 0 & \text{in } \Omega, \\
u_n = \Delta u_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(14)

Then \( u_n \) verifies
\[
\int_{\Omega} (-\Delta u_n)(-\Delta \phi) \, dx = \int_{\Omega} f_n \phi \, dx, \quad \forall \phi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).
\]

Noting by \(-\Delta u_n = v_n\) such that \(-\Delta v_n = f_n\), we obtain that \( \{v_n\} \) is bounded in \( W_0^{1,q}(\Omega) \), for any \( 1 \leq q < \frac{N}{N-1} \). Furthermore, for all \( k > 0 \), \( \{T_k(v_n)\} \) is bounded in \( W_0^{1,2}(\Omega) \) (see [6]).

Consequently, up a subsequence, there exists \( v \in W_0^{1,q}(\Omega) \), for all \( 1 \leq q < \frac{N}{N-1} \), such that \( v_n \to v \), weakly in \( W_0^{1,q}(\Omega), \forall q \in [1, \frac{N}{N-1}) \).

With these previous properties one can prove that \( \nabla v_n \to \nabla v \) a.e. in \( \Omega \). Therefore by Fatou’s and Vitali’s Theorems it follows that \( v_n \to v \) strongly in \( W_0^{1,q}(\Omega), \forall q \in [1, \frac{N}{N-1}) \) and \( T_k(v_n) \to T_k(v) \) strongly in \( W_0^{1,2}(\Omega) \), for all \( k > 0 \). Hence \( v \) is a positive solution of the problem \(-\Delta v = f\).

Taking into account that \(-\Delta u_n = v_n\) and applying Embedding Theorem 2.1, we conclude that there exists \( u \in W^{2,q}(\Omega) \), for all \( 1 \leq q < \frac{N}{N-1} \), with \( v = -\Delta u \), such that
\[
u_n \to u \quad \text{strongly in } W^{2,q}(\Omega), \quad \forall q \in [1, \frac{N}{N-1}).
\]

By the Strong Maximum Principle for the truncated problems (14), we find that for \( f \geq 0 \), then \( u_n > 0 \) and \( u > 0 \). This shows that \( u \) is a positive solution of the problem
\[
\begin{cases}
\Delta^2 u = f & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = -\Delta u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The positive supersolution obtained above allows us to construct a positive solution to (1). This is the core of the following lemma.

**Lemma 3.6.** Let \( \bar{u} \in L_{\text{loc}}^1(\bar{\Omega}) \) be a positive distributional supersolution to (1) with \( \lambda \leq \Lambda_N \), \( f \in L^1(\Omega), f \geq 0 \), and \( \bar{\Omega} \supset \Omega \), such that \(-\Delta \bar{u} \geq 0\) in the distributional sense. Then, there exists a positive minimal solution \( v \in L^1(\Omega) \cap L^p(\Omega) \) to problem (1) obtained as limit of approximations.

**Proof.** Let \( \bar{u} \) be a positive supersolution to (1) with \( \lambda \leq \Lambda_N \). For \( f_n = T_n(f) \), we construct recursively a sequence
\[
\{v_n\} \in L^1(\Omega) \cap L^p(\Omega),
\]
starting with
\[
\begin{cases}
\Delta^2 v_1 = f_1 & \text{in } \Omega, \\
v_1 > 0 & \text{in } \Omega, \\
v_1 = -\Delta v_1 = 0 & \text{on } \partial \Omega.
\end{cases}
\]
By the comparison principle in Lemma 3.3, it follows that \( v_1 \leq \bar{u} \) in \( \Omega \). By iteration, we define for \( n > 1 \),
\[
\begin{cases}
\Delta^2 v_n = \lambda \frac{v_{n-1}}{|x|^4} + v_{n-1} + f_{n-1} & \text{in } \Omega, \\
v_n > 0 & \text{in } \Omega, \\
v_n = -\Delta v_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(15)
As above, it follows that \( 0 \leq v_1 \leq \ldots \leq v_{n-1} \leq v_n \leq \bar{u} \) in \( \Omega \), so we obtain the pointwise limit,
\[
v(x) = \lim_{n \to \infty} v_n(x),
\]
which verifies that \( 0 \leq v \leq \bar{u} \) and
\[
\begin{cases}
\Delta^2 v = \lambda \frac{v}{|x|^4} + v^p + f & \text{in } \Omega, \\
v > 0 & \text{in } \Omega, \\
v = -\Delta v = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Moreover, \( v \) has the regularity of a solution obtained as limit of approximations to (1) in \( \Omega \).

We state some inequalities we will use along the paper. We start formulating an extension of a well-known Picone identity, that in the case of regular functions and the Laplacian operator was obtained by Picone in [24] (see also [4] and [1] for an extension to positive Radon measures and the \( p \)-Laplacian with \( p > 1 \)).

Lemma 3.7. Let \( u, v \in C^2(\Omega) \) satisfy \( v > 0 \) and \( -\Delta v \geq 0 \) in \( \Omega \). The following functionals,
\[
L(u, v) = \left( \Delta u - \frac{u}{v} \Delta v \right)^2,
\]
\[
R(u, v) = |\Delta u|^2 - \Delta v \Delta \left( \frac{u^2}{v} \right),
\]
verify that \( R(u, v) \geq L(u, v) \geq 0 \) in \( \Omega \).

Proof. Notice that,
\[
R(u, v) = |\Delta u|^2 - \Delta v \Delta \left( \frac{u^2}{v} \right)
= |\Delta u|^2 - 2 \frac{u}{v} \Delta u \Delta v + \frac{u^2}{v^2} |\Delta v|^2 - 2 \Delta v \nabla \left( \frac{u}{v} \right) \nabla u + 2 \frac{u}{v} \Delta v \nabla \left( \frac{u}{v} \right) \nabla v
= \left( \Delta u - \frac{u}{v} \Delta v \right)^2 - 2 \frac{\Delta v}{v} \left( \nabla u - \frac{u}{v} \nabla v \right)^2 \geq L(u, v) \geq 0.
\]

\( \Box \)

Lemma 3.8. Let \( v \in W^{2,2}(\Omega) \) be such that \( v \geq \delta > 0 \) in \( \Omega \). Then for all \( u \in C^\infty_0(\Omega) \),
\[
\int_\Omega |\Delta u|^2 \, dx \geq \int_\Omega \frac{\Delta^2 v}{v} u^2 \, dx.
\]
Proof. Since \( v \in W^{2,2}(\Omega) \) and \( v \geq \delta > 0 \) in \( \Omega \), there exists a family of regular functions \( v_n \) such that
\[
v_n \to v \text{ in } W^{2,2}(\Omega), v_n \in C^2(\Omega), v_n \to v, \text{ a.e., and } v_n \geq \frac{\delta}{2} \text{ in } \Omega.
\]
Furthermore, \( \Delta^2 \) is a continuous operator from \( W^{2,2}(\Omega) \) to \( W^{-2,2}(\Omega) \), see [17]. Hence \( \Delta^2 v_n \to \Delta^2 v \) in \( W^{-2,2}(\Omega) \). By the previous lemma applied to \( v_n \), we deduce
\[
|\Delta u|^2 \geq \Delta v_n \Delta \left( \frac{u^2}{v_n} \right).
\]
Thanks to (16), integrating by parts,
\[
\int_{\Omega} \frac{\Delta^2 v_n}{v_n} u^2 \, dx = \int_{\Omega} \Delta v_n \Delta \left( \frac{u^2}{v_n} \right) \, dx \leq \int_{\Omega} |\Delta u|^2 \, dx.
\]
Using the hypothesis on \( v_n \) and Lebesgue dominated convergence theorem, we obtain
\[
\int_{\Omega} \frac{\Delta^2 v}{v} u^2 \, dx \leq \int_{\Omega} |\Delta u|^2 \, dx, \quad u \in \mathcal{C}_0^\infty(\Omega),
\]
and the result follows.

\( \square \)

**Theorem 3.9.** Picone’s inequality. Consider \( u, v \in W^{2,2}(\Omega) \cap W^{4,2}_0(\Omega) \) such that \( \Delta^2 v = \mu \), where \( \mu \) is a positive bounded Radon measure, \( v = \Delta v = 0 \) on \( \partial \Omega \) and \( v \geq 0 \). Then,
\[
\int_{\Omega} |\Delta u|^2 \, dx \geq \int_{\Omega} \frac{\Delta^2 v}{v} u^2 \, dx.
\]

**Proof.** The Strong Maximum principle yields that \( v > 0 \) in \( \Omega \). Set \( \hat{v}_m(x) = v(x) + \frac{1}{m}, m \in \mathbb{N} \). Note that \( \Delta^2 \hat{v}_m = \Delta^2 v \) and \( \hat{v}_m \to v \) in \( W^{2,2}(\Omega) \) and almost everywhere. The general result follows from the previous lemma and a density argument. Namely, let \( u_n \in \mathcal{C}_0^\infty(\Omega) \) be such that \( u_n \to u \) in \( W^{2,2}(\Omega) \cap W^{4,2}_0(\Omega) \). By Lemma 3.8, it holds that
\[
\int_{\Omega} \frac{\Delta^2 v}{\hat{v}_m} u_n^2 \, dx = \int_{\Omega} \frac{\Delta^2 \hat{v}_m}{\hat{v}_m} u_n^2 \, dx \leq \int_{\Omega} |\Delta v_n|^2 \, dx.
\]
Taking limits \( m \to \infty, n \to \infty \) and applying Fatou’s Lemma we obtain the desired result. \( \square \)

In the following result we find some estimates on the supersolutions to our problem (1). They will be used in the next section to show nonexistence of solutions when \( p \) is above the threshold \( p^+(\lambda) \).

**Lemma 3.10.** Assume that \( u \) is a nonnegative function defined in \( \Omega \) such that \( u \not\equiv 0, u \in L_{loc}^4(\Omega) \) and \( \frac{u}{|x|^2} \in L_{loc}^4(\Omega) \). If \( u \) satisfies \(-\Delta u \geq 0, \Delta^2 u - \frac{\lambda u}{|x|^4} \geq 0 \) in the sense of distributions, then there exists a small ball \( B_\eta(0) \subset \Omega \) such that
\[
uu u \geq \hat{c}|x|^{-\alpha_-} \quad \text{and} \quad -\Delta u \geq c^* |x|^{-(\alpha_- + 2)} \quad \text{in } B_\eta(0),
\]
where \( \hat{c} = \hat{c}(\eta) \), \( c^* = c^*(\eta) \) and \( \alpha_- \) is defined in (10).

**Proof.** Since \( u \) is a nonnegative function satisfying \(-\Delta u \geq 0 \) in the sense of distributions and \( \lambda > 0 \), using the Strong Maximum Principle 3.2, it is not difficult to obtain that \( u \geq \delta > 0 \) and \(-\Delta u \geq \nu > 0 \) in a small ball \( B_\eta(\Omega) \).
Fixed $\eta > 0$, in order to get the above estimate, let $v \in W^{2,2}(B_\eta(0))$ be the unique solution to:

$$
\begin{cases}
\Delta^2 v - \lambda \frac{v}{|x|^4} = 0 & \text{in } B_\eta(0), \\
v > 0 & \text{in } B_\eta(0), \\
v = \delta, -\Delta v = \nu & \text{on } \partial B_\eta(0).
\end{cases}
$$

By a direct computation we obtain that $v(r) = C_1 r^{4} - \beta + C_2 r^{-\alpha}$ with $C_1$ and $C_2$ satisfying

$$
\begin{cases}
C_1 \eta^{-\beta} + C_2 \eta^{-\alpha} = \delta, \\
C_1 \beta - (N-2-\beta) \eta^{-\beta-2} + C_2 \alpha - (N-2-\alpha) \eta^{-\alpha-2} = \nu.
\end{cases}
$$

Since $u$ is a positive supersolution to problem (17) and using a Comparison Principle, we conclude that $u \geq v$ and $-\Delta u \geq -\Delta v$ in $B_\eta(0)$, so $u \geq C|x|^{-\alpha}$ in $B_\eta(0)$, and $-\Delta u \geq c^*|x|^{-\alpha-2}$ in $B_\eta(0)$.

We search a solution $u = A|x|^{-\alpha}$ of the associated radial elliptic problem in $B_R(0)$,

$$
\Delta^2 u - \lambda \frac{u}{|x|^4} = u^p \text{ in } B_R(0).
$$

Hence by a direct computation, we obtain that

$$
|x|^{-\alpha-4}(\alpha^4 - 2\alpha^3(N-4) + (N^2 - 10N + 20)\alpha^2 + 2(N-2)(N-4)\alpha - \lambda) = A^{p-1}|x|^{-p\alpha} \text{ in } B_R(0).
$$

Therefore,

$$
\alpha = \frac{4}{p-1} \text{ and } A^{p-1} = \alpha^4 - 2\alpha^3(N-4) + (N^2 - 10N + 20)\alpha^2 + 2(N-2)(N-4)\alpha - \lambda.
$$

It is clear that $\alpha^4 - 2\alpha^3(N-4) + (N^2 - 10N + 20)\alpha^2 + 2(N-2)(N-4)\alpha - \lambda > 0$ if and only if:

- $\alpha_- < \alpha < \alpha_+$, which means $p_- (\lambda) < p < p_+ (\lambda)$ with $p_+ (\lambda) = 1 + \frac{4}{\alpha_-}$ and $p_- (\lambda) = 1 + \frac{4}{\alpha_+}$.
- $\alpha > \beta_+$ which implies $p < p_- (\lambda)$.

We will see that if we perturb the bilaplacian operator with the Hardy potential, the critical power is $p_+ (\lambda)$.

Some properties of $p_- (\lambda)$ and $p_+ (\lambda)$ are,

$$
p_+ (\lambda) \to \frac{N+4}{N-4} \text{ as } \lambda \to \Lambda_N, \quad p_+ (\lambda) \to \infty \text{ as } \lambda \to 0,
$$

$$
p_- (\lambda) \to \frac{N+4}{N-4} \text{ as } \lambda \to \Lambda_N, \quad p_- (\lambda) \to \frac{N}{N-4} \text{ as } \lambda \to 0.
$$
It is clear that $p_+(\lambda)$ and $p_-(\lambda)$ are respectively decreasing and increasing functions on $\lambda$ and then,

$$1 < p_-(\lambda) \leq 2^* - 1 \leq p_+(\lambda).$$

Recall that $W^{2,2} (\Omega) \hookrightarrow L^q (\Omega)$ with $q \leq \frac{2N}{N-4} = 2^*.$

**Remark 3.11.** Notice that if $w$ satisfies

$$\Delta^2 w - \lambda \frac{w}{|x|^4} \geq g, \quad -\Delta w \geq 0,$$

in the sense of distributions in $\tilde{\Omega},$ with $g \in L^1_{loc} (\tilde{\Omega}),$ $g(x) \geq 0,$ $\tilde{\Omega} \supset \supset \Omega,$ and $\lambda \leq \Lambda_N,$ then $g$ must satisfy $g|x|^{-\alpha-} \in L^1_{loc} (\Omega).$

Indeed, we can construct a sequence $\{\varphi_n\}$ as follows. Take $\varphi_1$ such that $\Delta^2 \varphi_1 = 1$ in $\Omega,$ $\varphi_1 > 0$ in $\Omega$ and $\varphi_1 = \Delta \varphi_1 = 0$ on $\partial \Omega.$ The rest of the sequence is determined by

$$\begin{cases}
\Delta^2 \varphi_{n+1} = \lambda \frac{\varphi_n}{|x|^4 + \frac{4}{n}} + 1 & \text{in } \Omega, \\
\varphi_{n+1} > 0 & \text{in } \Omega, \\
\varphi_{n+1} = -\Delta \varphi_{n+1} = 0 & \text{on } \partial \Omega,
\end{cases}$$

such that $\varphi_n \leq \varphi_{n+1} \leq \varphi,$ where $\varphi$ is the positive solution to $\Delta^2 \varphi - \lambda \frac{\varphi}{|x|^4} = 1$ in $\Omega,$ with $\varphi = \Delta \varphi = 0$ on $\partial \Omega.$ Thanks to Lemma 3.10 there exists a positive constant $c$ and a small ball $B_R(0) \subset \Omega$ such that $\varphi(x) \geq c|x|^{-\alpha-}$ in $B_R(0),$ where $\alpha-$ is defined in (10).

As in Lemma 3.6, we can construct a minimal solution to problem

$$\begin{cases}
\Delta^2 w - \lambda \frac{w}{|x|^4} = g & \text{in } \Omega, \\
w > 0 & \text{in } \Omega, \\
w = -\Delta w = 0 & \text{on } \partial \Omega,
\end{cases}$$

obtained as limit of approximations. Then we observe that

$$\infty \geq \int_{\Omega} w dx = \int_{\Omega} w \left( \Delta^2 \varphi_{n+1} - \lambda \frac{\varphi_n}{|x|^4 + \frac{4}{n}} \right) dx$$

$$\geq \int_{\Omega} \varphi_n \left( \Delta^2 w - \lambda \frac{w}{|x|^4} \right) dx = \int_{\Omega} g \varphi_n dx \geq \int_{B_R(0)} g \varphi_n dx.$$
Therefore, \( \{g_{\varphi_n}\} \) is an increasing sequence uniformly bounded in \( L_{\text{loc}}^1(\Omega) \). The result follows by the Monotone Convergence Theorem and
\[
c \int_{B_R(0)} |x|^{-\alpha} - g \, dx \leq \int_{B_R(0)} g_{\varphi} \, dx < \infty.
\]

Since we are considering positive solutions to problem (1), then by setting \( g = u^p + cf \), we obtain that
\[
\int_{B_R(0)} (u^p + cf)|x|^{-\alpha -} \, dx < \infty \text{ for all } B_R(0) \subset\subset \Omega.
\]
This necessary condition will be useful in the forthcoming arguments.

4. Nonexistence results

We devote this section to show that \( p_+ (\lambda) \) is the threshold exponent for existence of solutions, namely, when we consider an exponent over the critical, there do not exist positive solutions even in the sense of distributions. To this end we argue \textit{ad contrarium}, we assume that there exists a positive supersolution to (1) and this will yield a contradiction with Hardy inequality.

**Theorem 4.1.** If \( \lambda > \Lambda_N \) and \( p > 1 \), or \( 0 < \lambda \leq \Lambda_N \) and \( p \geq p_+ (\lambda) \), then the problem (1) has no positive distributional supersolution \( u \) satisfying \(-\Delta u \geq 0 \) in the sense of distributions. In case that \( f \equiv 0 \), the unique nonnegative distributional supersolution is \( u \equiv 0 \).

**Proof.** Without loss of generality, we can assume \( f \in L^\infty(\Omega) \). Arguing by contradiction, we suppose that \( \tilde{u} \) is a positive supersolution satisfying \(-\Delta \tilde{u} \geq 0 \) in the sense of distributions. By Lemma 3.6, the problem (1) has a minimal solution \( u \) obtained as a limit of solutions \( u_n \) of truncated problems (15). We consider some different cases.

**Case** \( \lambda > \Lambda_N \).

It suffices to consider \( \tilde{u} \) as a positive distributional supersolution to the problem
\[
\begin{aligned}
\Delta^2 v - \Lambda_N \frac{v}{|x|^2} &= (\lambda - \Lambda_N) \frac{v}{|x|^4} + g & \text{in } \Omega, \\
v > 0 & \text{in } \Omega, \\
v = -\Delta v = 0 & \text{on } \partial \Omega,
\end{aligned}
\]
where \( g(x) = u^p + cf \). By Remark 3.11, necessarily
\[
\left( \lambda - \Lambda_{N,s} \right) \frac{\tilde{u}}{|x|^4} \left| x \right|^{\frac{N-4}{2}} \in L^1 (B_r(0))
\]
for all \( B_r(0) \subset\subset \Omega \). In particular, this implies
\[
\left( \lambda - \Lambda_{N,s} \right) \frac{\tilde{u}}{|x|^4} \left| x \right|^{\frac{N-4}{2}} \in L^1 (B_r(0)),
\]
and hence, applying Lemma 3.10,
\[
\left( \lambda - \Lambda_{N,s} \right) \left| x \right|^{-N} \in L^1 (B_r(0)),
\]
which is a contradiction. Therefore, there does not exist a positive supersolution if \( \lambda > \Lambda_{N,s} \).

**Case** \( 0 < \lambda \leq \Lambda_N \) and \( p > p_+ (\lambda) \).
Notice that the minimal positive solution \( u \) satisfies \(-\Delta u \geq 0\) and \( \Delta^2 u - \lambda \frac{u}{|x|^4} \geq 0\) in \( \mathcal{D}'(\Omega) \). Therefore, by Lemma 3.10, there exists a positive constant \( C \) and a small ball \( B_R(0) \subset \Omega \) such that \( u(x) \geq C|x|^{-\alpha_-} \) in \( B_R(0) \), where \( \alpha_- \) is defined in (10).

Let us consider the positive solutions \( u_n \) to the truncated problems (15). Since \( u_n > 0 \) in \( \Omega \), \( -\Delta u_n \geq 0 \) in \( \Omega \), using \( \frac{|\phi|^2}{u_n} \) with \( \phi \in C_0^\infty(B_R(0)) \) as a test function in the approximated problems (15), and applying Picone’s inequality,

\[
\int_{B_R(0)} \frac{u_n^{p-1}}{u_n} \phi^2 \, dx \leq \int_{B_R(0)} \Delta^2 u_n \frac{\phi^2}{u_n} \, dx \leq \int_{B_R(0)} |\Delta \phi|^2 \, dx.
\]

Passing to the limit as \( n \to \infty \), by Fatou’s Lemma and thanks to Lemma 3.10, we deduce that \( C \int_{B_R(0)} \frac{\phi^2}{|x|^{(p-1)\alpha_-}} \, dx \leq \int_{B_R(0)} |\Delta \phi|^2 \, dx \).

Since \( p > p_+(\lambda) \), then \( (p-1)\alpha_- > 4 \) and we obtain a contradiction with the Rellich inequality.

**Case** \( \lambda < \Lambda_N \) and \( p = p_+(\lambda) \). This case is more delicate because we consider the threshold exponent. In the sequel, the constant \( C \) may vary from line to line.

Since the minimal positive solution \( u \) satisfies \(-\Delta u \geq 0\) and \( \Delta^2 u - \lambda \frac{u}{|x|^4} \geq 0\) in \( \mathcal{D}'(\Omega) \), then thanks to Lemma 3.10, there exists \( \eta \) small enough and \( \hat{c} = \hat{c}(\eta) \) such that \( u(x) \geq \hat{c}|x|^{-\alpha_-} \) in \( B_\eta(0) \), where \( \alpha_- \) is defined in (10). Without loss of generality, from now on, we can fix \( \eta < e^{-1} \).

Next we define

\[
w = A|x|^{-a} \left( \log \left( \frac{1}{|x|} \right) \right)^b \quad \text{in} \quad B_\eta(0), \quad \text{with} \quad A = \left( \log \left( \frac{1}{\eta} \right) \right)^{b-\alpha_-}
\]

where \( a = \alpha_- \), and \( b \in (0, 1) \) will be chosen below conveniently small. Since \( \lambda < \Lambda_N \), then \( w \in W^{2,2}(B_\eta(0)) \). By a direct but tedious computation, we get

\[
\Delta^2 w - \lambda \frac{w}{|x|^4} = Ab \left( \log \left( \frac{1}{|x|} \right) \right)^{b-4} |x|^{-a-4} C_1(a, b, N) + Ab \left( \log \left( \frac{1}{|x|} \right) \right)^{b-3} |x|^{-a-4} C_2(a, b, N)
\]

\[
+ Ab \left( \log \left( \frac{1}{|x|} \right) \right)^{b-2} |x|^{-a-4} C_3(a, b, N) + Ab \left( \log \left( \frac{1}{|x|} \right) \right)^{b-1} |x|^{-a-4} C_4(a, b, N)
\]

\[
+ A \left( \log \left( \frac{1}{|x|} \right) \right)^{b} |x|^{-a-4} \left\{ a^4 - 2a^3(N - 4) + a^2(N^2 - 10N + 20) + 2a(N - 2)(N - 4) - \lambda \right\}.
\]

Notice that \( a^4 - 2a^3(N - 4) + a^2(N^2 - 10N + 20) + 2a(N - 2)(N - 4) - \lambda = 0 \), thus the term of order \((\log(1/|x|))^{b} |x|^{-a-4}\) cancels. Then, since we are taking \( \eta < e^{-1} \), the leading term has order \((\log(1/|x|))^{b-1} |x|^{-a-4}\), so one can conclude that

\[
\Delta^2 w - \lambda \frac{w}{|x|^4} \leq AbC \left( \log \left( \frac{1}{|x|} \right) \right)^{b-1} |x|^{-a-4} \quad \text{in} \quad B_\eta(0),
\]

for some \( C = C(a, b, N) \) small if \( b \ll 1 \). On the other hand, we observe that

\[
w^{p_+(\lambda)} = A^{p_+(\lambda)}|x|^{-p_+(\lambda)a} \left( \log \left( \frac{1}{|x|} \right) \right)^{bp_+(\lambda)} \quad \text{in} \quad B_\eta(0).
\]
Consider the function
\[ h(x) = A^{-(p_+)(\lambda) - 1} \left( \log \left( \frac{1}{|x|} \right) \right)^{-p_+(\lambda)b}. \]
Since $|x| < e^{-1}$ in $B_\eta(0)$, we have that $h > 0$ and $h \in L^\infty(B_\eta(0))$ being
\[ \|h\|_\infty = \eta^{\alpha - (p_+)(\alpha) - 1} \log \left( \frac{1}{\eta} \right)^{-b}. \]
Furthermore, $h$ satisfies
\[ hw^{p_+}(\lambda) = A|x|^{-\alpha - 4}, \]
and then thanks to (18),
\[ (19) \quad \Delta^2 w - \frac{w}{|x|^2} \leq Cbh w^{p_+}(\lambda) \text{ in } B_\eta(0). \]

Let us construct one supersolution of (19). Define $u_1 = c_1 u$.

Observe that $w = \eta^{-\alpha} - \partial B_\eta(0)$. It is easy to check that there exists $C' > 0$ such that $-\Delta w \leq C' \eta^{-\alpha - 2}$ on $\partial B_\eta(0)$. It immediately follows that $u_1 \geq c_1 \hat{w}$ on $\partial B_\eta(0)$ and $-\Delta u_1 \geq \frac{\eta}{\eta^2} (-\Delta w)$ on $\partial B_\eta(0)$. Noticing $C_2 = \min \{ \hat{c}, \frac{1}{\eta^2} \}$, it is sufficient to take $c_1 C_2 > 1$ ensuring that $u_1 \geq w$ on $\partial B_\eta(0)$ and $-\Delta u_1 \geq -\Delta w$ on $\partial B_\eta(0)$.

We claim that $u_1 \geq w$ in $B_\eta(0)$. Indeed, we can choose $b$ sufficiently small such that $c_1^{-p_+}(\lambda) \geq b\|h\|_L^\infty(B_\eta(0))$, so that
\[ \Delta^2 u_1 - \lambda \frac{u_1}{|x|^2} \geq c_1^{-p_+}(\lambda) u_1^{p_+}(\lambda) \geq bhu_1^{p_+}(\lambda) \text{ in } B_\eta(0). \]
Define by $\varpi = w - u_1$. This function verifies
\[ \Delta^2 \varpi - \lambda \frac{\varpi}{|x|^2} \leq Cbh \left( w^{p_+}(\lambda) - u_1^{p_+}(\lambda) \right). \]

Now we need to use Proposition 3.5, a type Moreau decomposition. To this end we define $\theta = \varpi + \varphi$, with $\varphi$ satisfying
\[ \begin{cases} 
\Delta^2 \varphi = 0 & \text{in } B_\eta(0), \\
\varphi \geq 0 & \text{in } B_\eta(0), \\
-\Delta \varphi = \Delta \varpi & \text{on } \partial B_\eta(0), \\
\varphi = -\varpi & \text{on } \partial B_\eta(0).
\end{cases} \]
Since $\varpi \leq 0$ on $B_\eta(0)$ and $-\Delta \varpi \leq 0$ on $B_\eta(0)$, it follows that $\theta \geq \varpi$ and $\theta \in W^{2,2}(B_\eta(0)) \cap W_0^{1,2}(B_\eta(0))$ satisfies
\[ (20) \quad \begin{cases} 
\Delta^2 \theta - \lambda \frac{\theta}{|x|^2} \leq Cbh \left( w^{p_+}(\lambda) - u_1^{p_+}(\lambda) \right), \\
\theta = -\Delta \theta = 0 & \text{on } \partial B_\eta(0).
\end{cases} \]

It suffices to show that $\theta \leq 0$ in $B_\eta(0)$.

Thanks to Proposition 3.5, we can decompose it as $\theta = \varphi + \varphi$ being $v_1, v_2 \in W^{2,2}(B_\eta(0)) \cap W_0^{1,2}(B_\eta(0))$ such that $v_1 \geq 0$ and $v_2 \leq 0$. Taking $v_1$ as a test function in (20) yields
\[ \int_{B_\eta(0)} \Delta \theta \Delta v_1 dx = \lambda \int_{B_\eta(0)} \frac{\theta v_1}{|x|^2} dx \leq \int_{B_\eta(0)} bh(w^{p_+}(\lambda) - u_1^{p_+}(\lambda)) v_1 dx. \]
Taking into account that \( \int_{B_{r}(0)} \eta \Delta v_{1} \Delta v_{2} \, dx = 0 \), we deduce that

\[
\int_{B_{r}(0)} |\Delta v_{1}|^{2} \, dx - \lambda \int_{B_{r}(0)} \frac{\theta v_{1}}{|x|^{4}} \, dx \leq \int_{B_{r}(0)} bh(w^{p_{+}(\lambda)} - u_{1}^{p_{+}(\lambda)})v_{1} \, dx.
\]

Using now that \( v_{2} \leq 0 \) and \( \theta \leq v_{1} \), together with Rellich inequality in the left hand side, it follows that

\[
(21) \quad \int_{B_{r}(0)} |\Delta v_{1}|^{2} \, dx \leq c_{\lambda} \int_{B_{r}(0)} p_{+}(\lambda)bh w^{p_{+}(\lambda)-1}v_{1}^{2} \, dx.
\]

Notice that taking \( b \) enough small, there exists \( 0 < \epsilon < 1 \) such that

\[
hw^{p_{+}(\lambda)-1} \leq \frac{c\Lambda_{N}}{c_{\lambda}p_{+}(\lambda)b} |x|^{-4} \text{ in } B_{r}(0),
\]

so applying again Rellich inequality, we get that

\[
\int_{B_{r}(0)} |\Delta v_{1}|^{2} \, dx \leq c_{\lambda} \int_{B_{r}(0)} p_{+}(\lambda)bh w^{p_{+}(\lambda)-1}v_{1}^{2} \, dx \leq \epsilon \int_{B_{r}(0)} |\Delta v_{1}|^{2} \, dx.
\]

It implies that \( \|v_{1}\|_{W^{2,2}(B_{r}(0))} \cap W^{1,2}_{0}(B_{r}(0)) = 0 \), hence \( v_{1} = 0 \), and then \( \theta = v_{2} \leq 0 \) as we wanted to show.

Once we have that \( u_{1} \geq w \) in \( B_{r}(0) \) we reach the desired contradiction just observing that, as before,

\[
\int_{B_{r}(0)} u_{1}^{p_{+}(\lambda)-1} \phi^{2} \, dx \leq \int_{B_{r}(0)} |\Delta \phi|^{2} \, dx.
\]

But then

\[
\int_{B_{r}(0)} \left( \log \left( \frac{1}{|x|} \right) \right)^{(p_{+}(\lambda)-1)b} \phi^{2} \frac{|x|^{4}}{|x|^{4}} \, dx \leq \int_{B_{r}(0)} |\Delta \phi|^{2} \, dx,
\]

which contradicts the optimality of the constant in Rellich inequality.

**Case** \( p = p_{+}(\lambda) \) and \( \lambda = \Lambda_{N} \). In this case we have that \( \alpha_{-} = \frac{N-4}{2} \) and \( p_{+}(\Lambda_{N}) = \frac{N+4}{N-4} \). If \( \tilde{u} \) is a supersolution to (1), then thanks to Lemma 3.10 and Remark 3.11, we have that

\[
C^{p} \int_{B_{r}(0)} |x|^{-\alpha_{-}(p_{+}(\Lambda_{N})+1)} \, dx \leq \int_{B_{r}(0)} |x|^{-\alpha_{-} - \tilde{p}_{+}(\Lambda_{N})} \, dx < \infty.
\]

Since \( \alpha_{-}(p_{+}(\Lambda_{N})+1) = N \), we reach the desired contradiction.
5. Complete blow up results

The nonexistence result obtained above for \( p \geq p_+ (\lambda) \) is very strong in the sense that a complete blow-up phenomenon occurs in two different senses.

a) If \( u_n \) is the solution to the approximated problem with \( p \geq p_+ (\lambda) \), where the Hardy potential is substituted by the bounded weight \( (|x|^4 + \frac{1}{n})^{-1} \), then \( u_n (x) \to \infty \) as \( n \to \infty \).

b) If \( u_n \) is the solution to the approximated problem with \( p = p_n < p_+ (\lambda) \) and \( p_n \to p_+ (\lambda) \) as \( n \to \infty \), then \( u_n (x) \to \infty \) as \( n \to \infty \).

5.1. Blow up for the approximated problems when \( \lambda > \Lambda_N \) and \( p > 1 \), or \( 0 < \lambda \leq \Lambda_N \) and \( p \geq p_+ (\lambda) \). We get a blow-up behavior for the following approximated problems.

**Theorem 5.1.** Let \( u_n \in L^1 (\Omega) \cap L^p (\Omega) \) be a positive solution to the problem

\[
\begin{cases}
\Delta^2 u_n = \frac{u_n^p}{1 + \frac{p}{n} u_n^p} + \lambda a_n (x) u_n + cf & \text{in } \Omega, \\
u_n = -\Delta u_n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with \( f \neq 0 \), and \( a_n (x) = \frac{1}{|x|^4 + \frac{1}{n}} \). We consider \( \lambda > \Lambda_N \) and \( p > 1 \), or \( 0 < \lambda \leq \Lambda_N \) and \( p \geq p_+ (\lambda) \). Then \( u_n (x_0) \to \infty \), \( \forall x_0 \in \Omega \).

**Proof.** Without loss of generality, we can assume that \( f \in L^\infty (\Omega) \). The existence of a positive solution to problem (22) follows using classical sub-supersolution arguments. Thanks to the monotonicity of the nonlinear term and the coefficient \( a_n \) we can assume the existence of minimal solution \( u_n \) such that \( u_n \leq u_{n+1} \) for all \( n \geq 1 \). Therefore to get the blow-up result we have just to show the complete blow-up for the family of minimal solutions denoted by \( u_n \).

Applying weak Harnack inequality in Lemma 3.4, we conclude that for any \( B_R (x_0) \subset \subset \Omega \), there exists a positive constant \( C = C (\theta, \rho, R) \), which may vary from line to line, with \( 0 < \theta < \rho < 1 \), such that

\[
\| u_n \|_{L^1 (B_{\rho R} (x_0))} \leq C \operatorname{ess} \inf_{B_{\rho R} (x_0)} u_n \leq C u_n (x_0) \leq C.
\]

We claim that, in particular, there exists \( r > 0 \) and a positive constant \( C = C (r) \), such that

\[
\int_{B_r (0)} u_n \, dx \leq C, \quad \text{uniformly in } n \in \mathbb{N}.
\]

If \( 0 \in B_{\rho R} (x_0) \), the claim follows directly by (23) for any \( B_r (0) \subset \subset B_{\rho R} (x_0) \). If \( 0 \not\in B_{\rho R} (x_0) \), let us consider a smooth curve \( \Gamma \in \Omega \) joining \( x_0 \) with the origin. We define

\[
r = \frac{1}{2} \min_{x \in \Gamma} \operatorname{dist} (x, \partial \Omega),
\]

to ensure that \( B_r (x) \subset \subset \Omega \), for every \( x \in \Gamma \). Applying Weak Harnack inequality, we get

\[
\int_{B_r (x_0)} u_n \, dx \leq C u_n (x_0) \leq C.
\]

Therefore, thanks to the uniform integral estimate (24), we can conclude that \( \forall D \subset B_r (x_0) \),

\[
\inf_D u_n \leq \frac{1}{|D|} \int_D u_n (x) \, dx \leq \frac{1}{|D|} \int_{B_r (x_0)} u_n (x) \, dx \leq \frac{C}{|D|}.
\]
Now, we take $x_1 \in B_r(x_0) \cap \Gamma$ and $D_1 = B_r(x_0) \cap B_{\theta r}(x_1)$, with $0 < \theta < 1$. Once more, Weak Harnack inequality yields,

$$\int_{B_r(x_1)} u_n \, dx \leq C \inf_{B_{\theta r}(x_1)} u_n \leq \inf_{D_1} \inf u_n \leq \frac{C}{|D_1|}.$$ 

Recursively, in a finite number of steps, we conclude the claim. Furthermore, by the monotone convergence theorem, there exists $u \geq 0$ such that $u_n \uparrow u$ strongly in $L^1(B_r(0))$.

Let us take $\varphi$, the solution to the problem

$$\begin{cases}
\Delta^2 \varphi = \chi_{B_r(0)} & \text{in } \Omega, \\
\varphi = -\Delta \varphi = 0 & \text{on } \partial\Omega,
\end{cases}$$

as a test function in (22). Then we have

$$C' \geq \int_{B_r(0)} u_n(x) \, dx = \int_\Omega \frac{u_n^p}{1 + \frac{1}{n} u_n^p} \varphi \, dx + \lambda \int_\Omega a_n(x) u_n \varphi \, dx + c \int_\Omega f \varphi \, dx.$$ 

By the monotone convergence theorem and Fatou’s Lemma,

$$\frac{u_n^p}{1 + \frac{1}{n} u_n^p} \to u^p \text{ in } L^1_{\text{loc}}(B_r(0)), \\
a_n(x) u_n \to \frac{u}{|x|^4} \text{ in } L^1_{\text{loc}}(B_r(0)).$$

Thus it follows that $u$ is a very weak supersolution to (1) in $B_{r_1}(0) \subset \subset B_r(0)$, a contradiction with Theorem 4.1. \hfill \Box

5.2. Blow up when $p_n \to p_+(\lambda)$. We prove now another strong blow-up result when the power $p_n \uparrow p_+(\lambda)$.

Theorem 5.2. Assume that $p_n$ satisfies $p_n < p_+(\lambda)$ and $p_n \to p_+(\lambda)$ as $n \to \infty$ and $f \geq 0$. Let $u_n \in L^1_{\text{loc}}(\Omega)$ be a very weak supersolution to the problem

$$\begin{cases}
\Delta^2 u_n \geq \lambda \frac{u_n}{|x|^4} + u_n^p + f & \text{in } \Omega, \\
u_n = -\Delta u_n = 0 & \text{on } \partial\Omega.
\end{cases}$$

Then $u_n(x_0) \to \infty, \forall x_0 \in \Omega$.

Proof. Without loss of generality we can assume that $f \in L^\infty(\Omega)$. Suppose by contradiction that there exists a subsequence denoted by $p_n$ and a supersolution $u_n$ such that for some point $x_0 \in \Omega$ we have $u_n(x_0) \to C_0 < \infty, \forall n \in \mathbb{N}$. It is not restrictive taking $p_n(\lambda) = 1 + \frac{4}{\alpha - 1 - n}$. As before, thanks to the weak Harnack inequality, we can deduce the existence of $r > 0$ and a positive constant $C = C(r)$ such that

$$\int_{B_r(0)} u_n(x) \, dx \leq C, \text{ uniformly in } n \in \mathbb{N}.$$ 

If $u_n \in L^1_{\text{loc}}(\Omega)$ is a supersolution to problem (25), then it can be shown as in Lemma 3.6 that there exists a minimal solution to (25) in $\Omega_1 \subset \subset \Omega$ with $0 \in \Omega_1 \subset \subset \Omega$ obtained by approximation.
Let us denote by \( v_n \leq u_n \) this minimal solution. Then \( v_n \) solves
\[
\begin{cases}
\Delta^2 v_n = \lambda \frac{v_n}{|x|^4} + v_n^p + f \text{ in } \Omega_1, \\
v_n = -\Delta v_n = 0 \text{ on } \partial \Omega_1.
\end{cases}
\]

Take \( \phi \), the solution to the problem
\[
\begin{cases}
\Delta^2 \phi = 1 \text{ in } \Omega_1, \\
\phi = -\Delta \phi = 0 \text{ on } \partial \Omega_1,
\end{cases}
\]
as a test function in (26). Since \( v_n \leq u_n \), there results that
\[
C \geq \int_{\Omega_1} v_n(x) \, dx = \int_{\Omega_1} g_n(x) \phi \, dx,
\]
where \( g_n(x) = \lambda \frac{v_n}{|x|^4} + v_n^p + f \). Thus,
\[
\int_{\Omega_1} g_n \phi \, dx \leq C \text{ for all } n.
\]
This implies that \( g_n \) is uniformly bounded in \( L^1_{\text{loc}}(\Omega_1) \) and then \( g_n \rightharpoonup \mu \) in the sense of measures.

Using the usual change of variables \( \Delta^2 v_n = (-\Delta)(-\Delta v_n) = -\Delta w_n \) in (26) we get,
\[
\begin{cases}
(-\Delta) w_n = \lambda \frac{v_n}{|x|^4} + v_n^p + f \text{ in } \Omega_1, \\
v_n = u_n = 0 \text{ on } \partial \Omega_1.
\end{cases}
\]

Now take \( T_k(w_n) \cdot \varphi \) with \( \varphi \in C_0^\infty(\Omega_1) \) as a test function in (27). Invoking the previous boundedness, we get
\[
\int_{\Omega_1} |\nabla T_k(w_n)|^2 \varphi \, dx + \int_{\Omega_1} \Theta_k(w_n)(-\Delta \varphi) \, dx = \int_{\Omega_1} g_n(x) \varphi T_k(w_n) \, dx \leq k \int_{\Omega_1} g_n(x) \varphi \, dx \leq C,
\]
where \( \Theta_k(s) = \int_s^\infty T_k(\sigma) \, d\sigma \).

Hence, \( T_k(w_n) \rightharpoonup T_k(w) \) weakly in \( W^{1,2}_{\text{loc}}(B_r(0)) \). Applying the renormalized theory, the Embedding Theorem 2.1, and the fact that \(-\Delta v_n = w_n\), we conclude that there exists a non-negative function \( v \) such that \( v_n \rightharpoonup v \) strongly in \( L^q_{\text{loc}}(B_r(0)), q < \frac{N}{N-3} \).

Let \( \psi \in C_0^\infty(B_r(0)) \) be a nonnegative function. By Fatou Lemma it follows that
\[
\lim_{n \to \infty} \int_{B_r(0)} g_n(x) \psi \, dx \geq \int_{B_r(0)} v^{p_s(\lambda)} \psi \, dx + \lambda \int_{B_r(0)} \frac{v \psi}{|x|} \, dx + \int_{B_r(0)} f \psi \, dx.
\]

Therefore, using \( \psi \) as a test function in (26) and passing to the limit as \( n \to \infty \), we conclude that
\[
\int_{B_r(0)} v \Delta^2 \psi \, dx \geq \int_{B_r(0)} v^{p_s(\lambda)} \psi \, dx + \lambda \int_{B_r(0)} \frac{v \psi}{|x|} \, dx + \int_{B_r(0)} f \psi \, dx.
\]
Hence \( v \) is a positive distributional supersolution to (1) and then we reach a contradiction.

The goal of this section is to prove that, under some suitable hypotheses on $f$, problem (1) has a positive solution, if we consider the complementary interval of powers, namely, $1 < p < p_+(\lambda)$. For the existence result, we first analyze the case $f \equiv 0$.

We distinguish some cases:

(A) $0 < \lambda < \Lambda_N$.

(I) If $1 < p < \frac{N+4}{N-4}$ and $\Omega$ is a bounded domain, there exists a positive solution to problem (1) with $f \equiv 0$ using the classical Mountain Pass Theorem in the Sobolev space $W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$ (see for example [5, 17, 22]).

(II) If $\frac{N+4}{N-4} < p < p_+(\lambda)$, there exists a positive solution in $\mathbb{R}^N$. By setting $w(x) = A|x|^{-\alpha}$ with $\alpha = \frac{4}{p-4}$ and $A^{p-1} = \alpha^4 - 2\alpha^2(N-4) + (N^2 - 10N + 20)\alpha^2 + 2(N-2)(N-4)\alpha - \lambda$, then $w$ is a supersolution to (1) in any domain $\Omega$. It is clear that if $p_-(\lambda) < p < p_+(\lambda)$, then $A^{p-1} > 0$. Notice that $w \in L^p_{\text{loc}}(\mathbb{R}^N)$, $\frac{w}{|x|^4} \in L^1_{\text{loc}}(\mathbb{R}^N)$ and the solution could be verified in distributional sense. This is the way in which the critical powers $p_-(\lambda)$ and $p_+(\lambda)$ appear.

(B) If $\lambda = \Lambda_N$, then $p_+(\Lambda_N) = \frac{N+4}{N-4} = p_-(\Lambda_N)$. Define the Hilbert space $H(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||\phi||^2 = \int_\Omega (|\Delta \phi|^2 - \Lambda_N \frac{\phi^2}{|x|^4})dx.$$ 

Invoking the improved Hardy-Sobolev inequality (see for example [9, 16]), then classical variational methods in the space $H(\Omega)$ allow us to prove the existence of a positive solution $w$ to problem (1) with $f \equiv 0$.

Remark 6.1. In the presence of a source term $f \geq 0$, if $f(x) \leq \frac{c_0}{|x|^{2\gamma}}$ with $c_0 > 0$ and sufficiently small, by similar arguments as above we can show the existence of a supersolution. Then, the existence of a minimal solution to problem (1) follows for all $p < p_+(\lambda)$.

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