SOME PROBLEMS ON SKEW BRACES AND THE YANG–BAXTER EQUATION

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Abstract. This is an extended abstract of the talk given in the Oberwolfach mini-workshop "Algebraic Tools for Solving the Yang–Baxter Equation" in November of 2019.

The first problem I would like to mention comes from the theory of Hopf–Galois extensions and it was first formulated by Byott [4].

Problem 1 (Byott)**.** Let A be a finite skew left brace with solvable additive group. Is the multiplicative group of A solvable?

To state the second problem I will first recall the following question: Which finite solvable groups are IYB groups? Recall that IYB-groups are those group that are multiplicative groups of skew left braces of abelian type. This problem was solved by Bachiller in [1], following the ideas of Rump [12]. However, I would like to understand better the ideas behind this proof.

Problem 2. Which is the minimal size of a finite solvable group that is a non-IYB-group?

The following two similar problems appear in [5]:

Problem 3 (Cedó–Jespers–Okniński)**.** Is every finite nilpotent group of nilpotecy class two the multiplicative group of a (two-sided) skew brace of abelian type?

Problem 4 (Cedó–Jespers–Okniński)**.** Which finite nilpotent groups are multiplicative groups of (two-sided) skew braces of abelian type?

Problem 4 is interesting even in the particular case of nilpotency class ≤ 3.

Problem 5 (Rump)**.** Is there an example of a non-IYB finite group where all the Sylow subgroups are IYB groups?

Now I mention some problems related to involutive multipermutation solutions. For a finite non-degenerate involutive solution (X, r), the following properties are equivalent:

(1) (X, r) is a multipermutation solution.
(2) The structure group $G(X, r)$ is left orderable
(3) The structure group $G(X, r)$ is diffuse.

The implication 1) $\implies$ 2) was proved by Jespers and Okniński [9] and independently by Chouraqui [6]. The implications 2) $\implies$ 1) and 3) $\implies$ 1) were proved in [2] and in [10], respectively. All these results answer a question of Gateva–Ivanova [8]. The followin problem appears in [10].

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Problem 6. Let (X, r) be a finite involutive non-degenete solution to the YBE. When $G(X, r)$ has the unique product property?

Benson proved in [3] that groups that contain a finite index subgroup isomorphic to $\mathbb{Z}^n$ have rational growth series. The structure group of a finite solutions has a finite index subgroup isomorphic to $\mathbb{Z}^n$, see [7, 11, 13].

Problem 7. Let (X, r) be a finite non-degenerate solution. Compute the growth series of the structure group $G(X, r)$.

Finally, let me discuss a problem related to multipermutation solutions. It seems that "almost all" non-degenerate involutive solutions to the Yang–Baxter equation of size ≤ 8 are multipermutation solutions:

$$\frac{\text{number of multipermutation solutions}}{\text{number of solutions}} = \frac{36115}{38698} > 0.93.$$

For these calculation one needs the list of small solutions computed of [7].

Problem 8. Is it true that

$$\frac{\text{number of multipermutation solutions of size } n}{\text{number of solutions of size } n} \underset{n\to\infty}{\longrightarrow} 1?$$

This question makes sense for non-involutive solutions as well.

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