Soliton decomposition of the Box-Ball System

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Resources

Simulation

https://mate.dm.uba.ar/~leorolla/simulations/bbs.html

(on a 1d torus wrapped around a 2d torus)

These slides

http://mate.dm.uba.ar/~leorolla/bbs-slides.pdf

Article

https://arxiv.org/abs/1806.02798

(and references therein)

Extended abstract

http://mate.dm.uba.ar/~leorolla/bbs-abstract.pdf

Ball-Box System (Takahashi-Satsuma 1990)

Ball configuration $\eta \in \{0,1\}^{\mathbb{Z}}$

$$\eta(x) = 0 \leftrightarrow \text{empty box}, \qquad \eta(x) = 1 \leftrightarrow \text{ball at } x$$

Carrier picks balls from occupied boxes

Carrier deposits one ball, if carried, at empty boxes

0	0	1	0	1	1	0	0	0	1	1	1	0	0	1	0	0	0	0	0	0	η
0	0	0	1	0	0	1	1	0	0	0	0	1	1	0	1	1	0	0	0	0	$T\eta$
0	0	0	0	1	0	0	0	1	1	0	0	0	0	1	0	0	1	1	1	0	$T^2\eta$

$T\eta$: configuration after the carrier visited all boxes.

Formal definition

We say that x is an *excursion point* if, for some $z \leqslant y$,

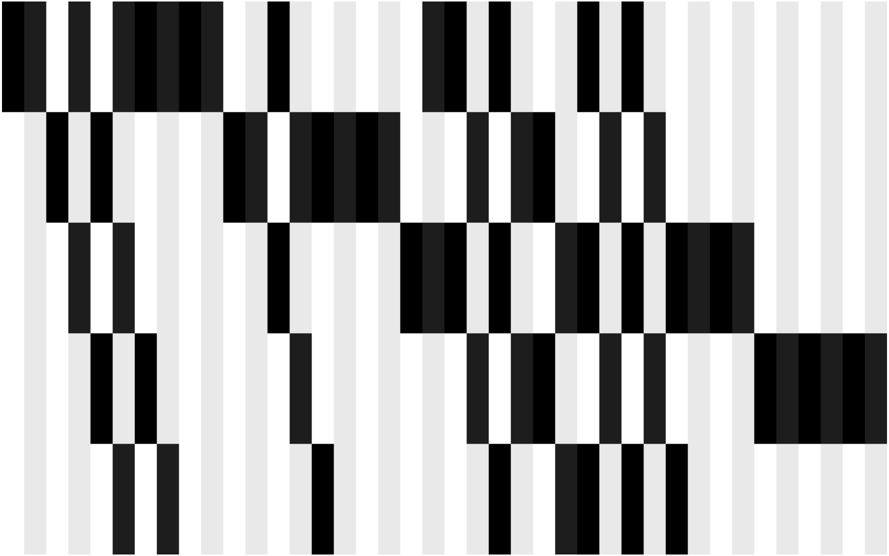
$$\sum_{y=z}^{x} \eta(y) \geqslant \sum_{y=z}^{x} [\, 1 - \eta(y)\,],$$

otherwise x is a *record*.

Now we define

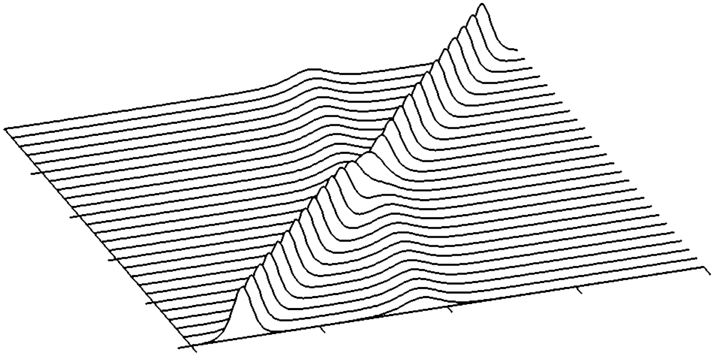
$$T\eta(x) = \begin{cases} 0, & x \text{ is a record,} \\ 1 - \eta(x), & \text{otherwise.} \end{cases}$$

Example



Motivation: Korteweg & de Vries equation

$$\dot{u} = u''' + u\,u'$$



Soliton: a solitary wave that propagates with little loss of energy and retains its shape and speed after colliding with another such wave

Take-home messages

Ergodic Theory $\leftrightarrow$ Integrable System $\leftrightarrow$ Algebraic Structures?

Identifying solitons and hierarchical structures

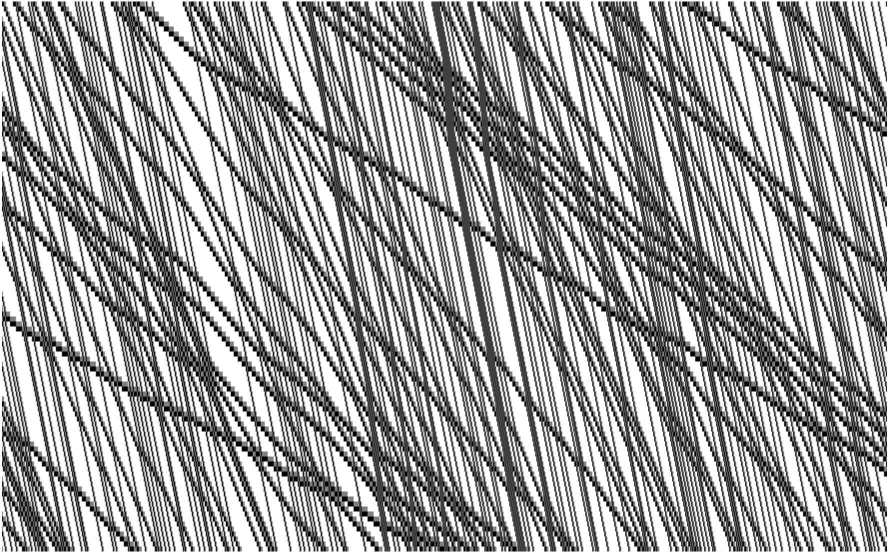
Interaction $\rightsquigarrow$ asymptotic speeds

Many conservations $\rightsquigarrow$ many invariant measures

Complete description of invariant measures still missing

Uniqueness of solutions to speed equations still missing

Solitons in the BBS



Outline of the talk

1) Conservation of k-**solitons** and how to identify them (T&S)

2) Asymptotic **speed** of k-solitons

3) k-**slots** and k-**components**

4) **Invariant measures** for T from independent k-components

5) Evolution of k-components is a **hierarchical translation**

6) **Reconstruction** from k-components

Solitons

Conservation of solitons

k-soliton: set of k successive ones followed by k zeros (for now)

Isolated k-solitons travel at speed k and conserve shape and distance:

```
000001110000000000000000000111000000000000
000000001110000000000000000000111000000000
000000000001110000000000000000000111000000
000000000000001110000000000000000000111000
000000000000000001110000000000000000000111
000000000000000000001110000000000000000000
000000000000000000000001110000000000000000
000000000000000000000000001110000000000000
000000000000000000000000000001110000000000
000000000000000000000000000000001110000000
000000000000000000000000000000000001110000
000000000000000000000000000000000000001110
```

Conservation of solitons

k-solitons and distances are conserved after interacting with m-solitons:

000001110000001000000000000111000000000000000000000000
000000001110000100000000000001110000000000000000000000
000000000001110010000000000000001110000000000000000000
000000000000001101100000000000000000111000000000000000
000000000000000010011100000000000000000011100000000000
000000000000000001000011100000000000000000001110000000
000000000000000000100000011100000000000000000000011100
000000000000000000010000000111000000000000000000000111
000000000000000000001000000000111000000000000000000000
000000000000000000000100000000000111000000000000000000
000000000000000000000010000000000000111000000000000000

Conservation of solitons

Isolated k-solitons travel at speed k and conserve the distances:

```
.....111000...............111000..........
........111000...............111000.......
...........111000...............111000....
..............111000...............111000.
```

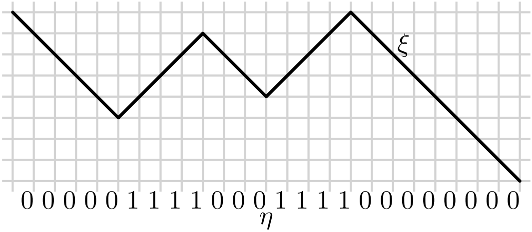
k-solitons and distances are conserved after interacting with m-solitons:

```
.....111000...10............111000...................
........111000.10..............111000................
...........11100100...............111000.............
..............11011000..............111000...........
................10.111000..............111000........
.................10...111000..............111000.....
..................10.....111000..............111000..
```

Conservation of solitons

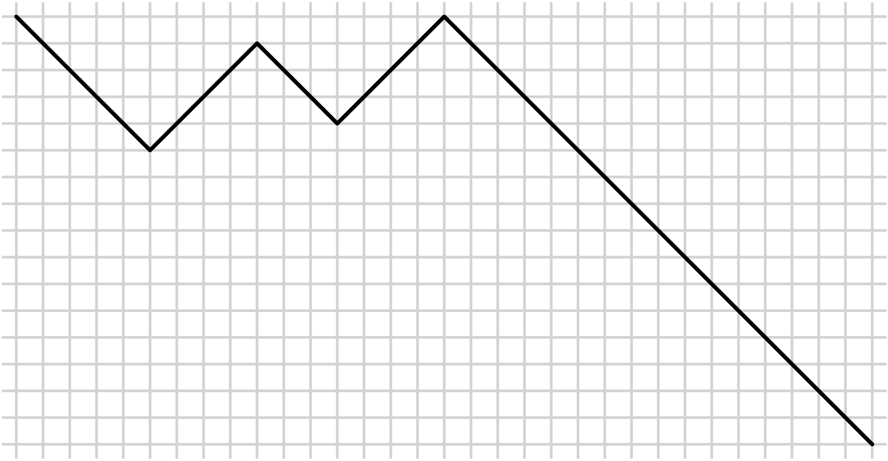
```
00...1100....1100..1100.11001100...1100...1010..10...10.10.....10....................................................................
1100...1100....1100..1100.11001100...1100..1010..10...10.10.....10...................................................................
..1100...1100....1100..1100.11001100...1100.1010..10...10.10.....10..................................................................
....1100...1100....1100..1100.11001100...11001010..10...10.10.....10.................................................................
......1100...1100....1100..1100.11001100...11010100.10...10.10.....10................................................................
........1100...1100....1100..1100.11001100...1010110010...10.10.....10...............................................................
..........1100...1100....1100..1100.11001100..1010.110100..10.10.....10..............................................................
............1100...1100....1100..1100.11001100.1010..101100.10.10.....10.............................................................
..............1100...1100....1100..1100.110011001010..10.110010.10.....10............................................................
................1100...1100....1100..1100.110011010100.10..11010010.....10...........................................................
..................1100...1100....1100..1100.11001010110010...10110100....10..........................................................
....................1100...1100....1100..1100.11010100110100..10.101100...10.........................................................
......................1100...1100....1100..1100.10101100101100.10.10.1100..10........................................................
........................1100...1100....1100..11001010.110100110010.10..1100.10.......................................................
..........................1100...1100....1100..11010100.10110011010010...110010......................................................
............................1100...1100....1100..1010110010.110010110100...110100....................................................
..............................1100...1100....1100.1010.110100.110100101100...101100..................................................
................................1100...1100....11001010..101100.101101001100..10.1100................................................
..................................1100...1100....11010100.10.110010.1011001100.10..1100..............................................
....................................1100...1100....1010110010..11010010.1100110010...1100............................................
......................................1100...1100...1010.110100..10110100.1100110100...1100..........................................
........................................1100...1100..1010..101100.10.101100.1100101100...1100........................................
..........................................1100...1100.1010..10.110010.10.1100.1101001100...1100......................................
............................................1100...11001010..10..11010010..1100.1011001100...1100....................................
..............................................1100...11010100.10...10110100..110010.11001100...1100..................................
................................................1100...1010110010...10.101100..110100.11001100...1100................................
..................................................1100..1010.110100..10.10.1100..101100.11001100...1100..............................
....................................................1100.1010..101100.10.10..1100.10.1100.11001100...1100............................
......................................................11001010..10.110010.10...110010..1100.11001100...1100..........................
........................................................11010100.10..11010010....110100..1100.11001100...1100........................
..........................................................1010110010...10110100....101100..1100.11001100...1100......................
...........................................................1010.110100..10.101100...10.1100..1100.11001100...1100....................
............................................................1010..101100.10.10.1100..10..1100..1100.11001100...1100..................
.............................................................1010..10.110010.10..1100.10...1100..1100.11001100...1100................
..............................................................1010..10..11010010...110010....1100..1100.11001100...1100..............
...............................................................1010..10...10110100...110100....1100..1100.11001100...1100............
................................................................1010..10...10.101100...101100....1100..1100.11001100...1100..........
.................................................................1010..10...10.10.1100..10.1100....1100..1100.11001100...1100........
..................................................................1010..10...10.10..1100.10..1100....1100..1100.11001100...1100......
...................................................................1010..10...10.10...110010...1100....1100..1100.11001100...1100....
....................................................................1010..10...10.10....110100...1100....1100..1100.11001100...1100..
.....................................................................1010..10...10.10.....101100...1100....1100..1100.11001100...1100
```

Walk representation

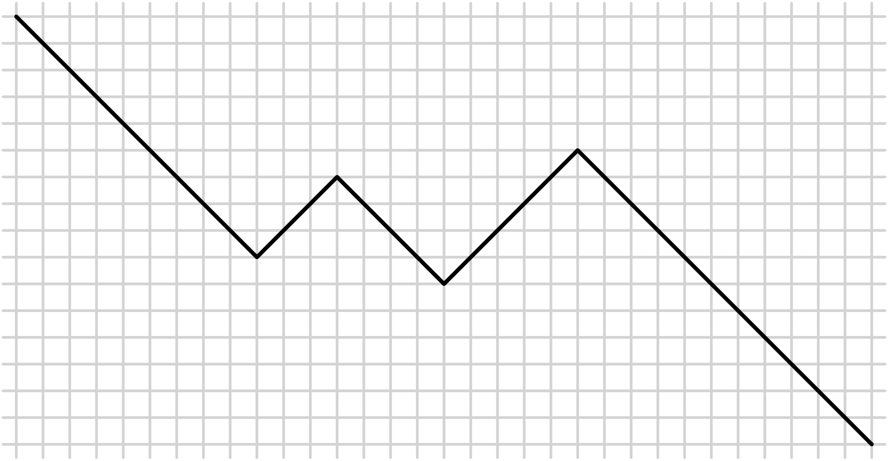


Excursion: configuration between two successive **records**

Pitman transformation



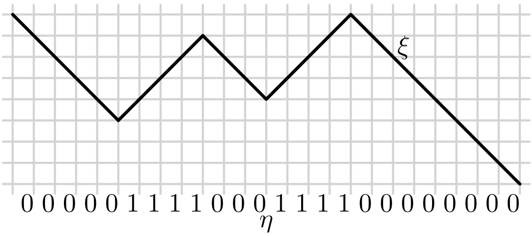
Pitman transformation



Pitman transformation

Identifying solitons (Takahashi-Satsuma)

Call *runs* the segments induced by broken lines in the walk representation



Explore runs from left to right. If a run has length $k \leqslant$ length of the next run, then its k boxes and the first k boxes of the next run form a k-soliton. Remove these sites and start again exploring from the left.

Identifying solitons (Takahashi-Satsuma)

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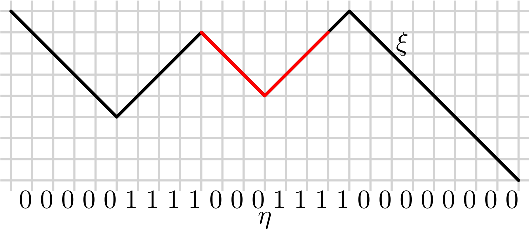

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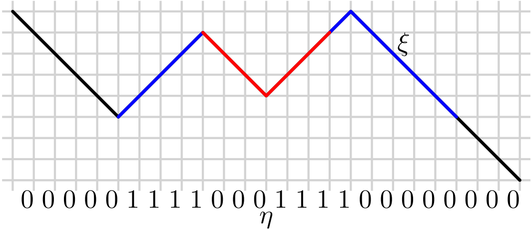

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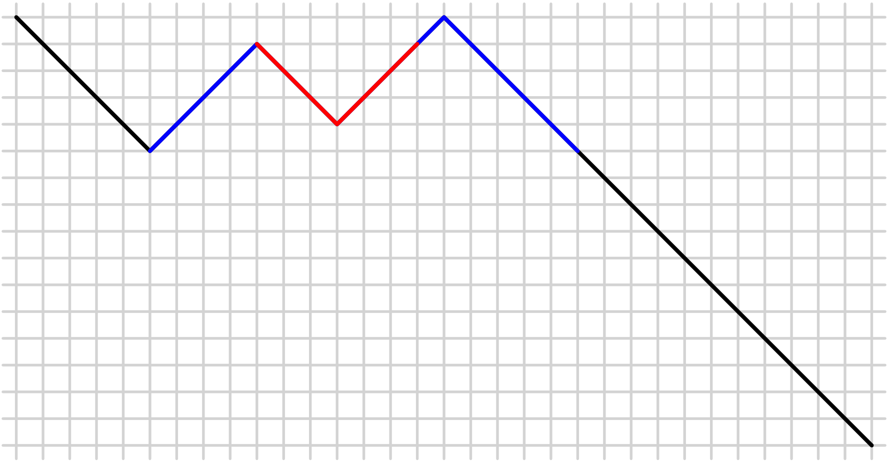
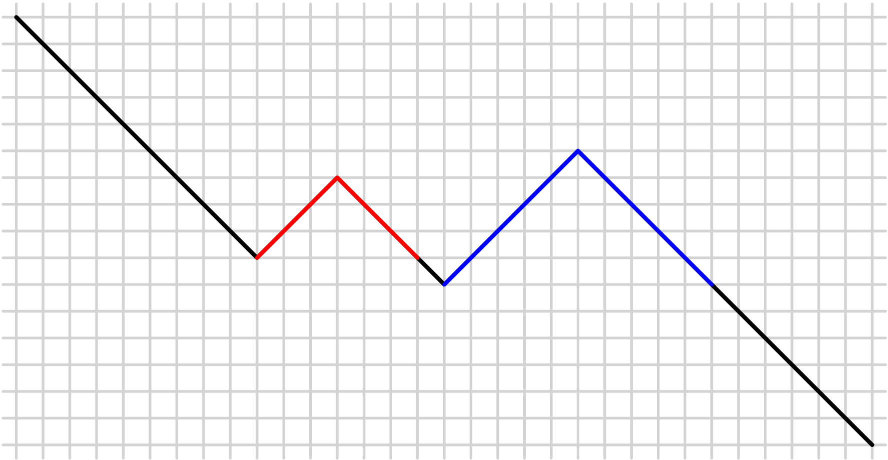
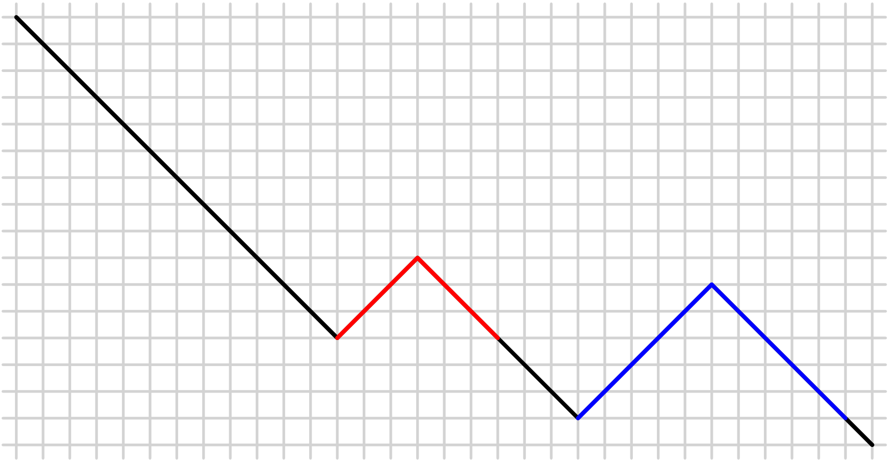
Following solitons



Following solitons



Following solitons



Speed

Asymptotic speed of solitons

η shift-ergodic and T-invariant, $\rho_k :=$ number of k-solitons per excursion

Theorem. *There exists deterministic $v = (v_k)_{k\geqslant 1}$ such that, a.s.,*

$$\lim_{t\to\infty} \frac{\textit{Position of } \gamma^t}{t} = v_k$$

for every k-soliton γ of η. The sequence v solves

$$v_k = k + \sum_{m<k} 2m\rho_m(v_k - v_m) - \sum_{m>k} 2k\rho_m(v_m - v_k).$$

and can be computed explicitly from $(\rho_k)_{k\geqslant 1}$.

Examples

```
10111101000010...10111101000010...10111101000010...
010...10111101000010...10111101000010...10111101000
101000010...10111101000010...10111101000010...10111
.10111101000010...10111101000010...10111101000010..
001010...101111010000...10111101000010...1011110100
110101000010...101111000010...10111101000010...1011
..10111101000010...10111101000010...10111101000010.
0001010...101111010000...10111101000010...101111010
1110101000010...101111000010...10111101000010...101
...10111101000010...10111101000010...10111101000010
```

$$v_1 = \frac{1}{3},\ v_4 = 6$$

Examples

```
101111010000010...1011110100010...1011110100010...
010...1011110100010...1011110100010...1011110100
101000010...1011110100010...1011110100010...10111
.1011110100010...1011110100010...1011110100010..
001010...1011101000...1011110100010...1011110100
11010100010...101111000010...1011110100010...1011
..1011110100010...1011110100010...1011110100010.
0001010...1011101000...1011110100010...10111010
1110101000010...101111000010...1011110100010...101
...1011110100010...1011110100010...1011110100010
```

$$v_1 = \frac{1}{3},\ v_4 = 6$$

Examples

1011110110001000.1011110110001000.1011110110001000.
01000010.1110111001000010.1110111001000010.11101110
1011101000010 0.1101111010000100.1101111010000100.1
01000.1011110110001000.1011110110001000.10111101100
1011100100010.1110111001000010.1110111001000010.11
0100.1101111010000100.1101111010000100.110111101000
10110001000.1011110110001000.1011110110001000.10111
010.1110111001000010.1110111001000010.111011100100
01010000100.1101111010000100.1101111010000100.110111
.1011110110001000.1011110110001000.1011110110001000

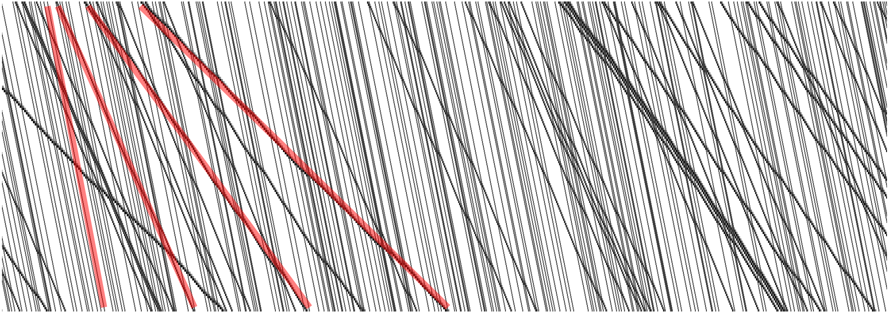
$$v_1 = \frac{1}{9},\ v_5 = \frac{23}{3}$$

Examples

1011110110001000.1011110110001000.1011110110001000.
01000010.1110111001000010.1110111001000010.11101110
1011110100000100.1101111010000100.1101111010000100.1
01000.1011110110001000.1011110110001000.10111101100
10111001000010.1110111001000010.1110111001000010.11
0100.1101111010000100.1101111010000100.110111101000
10110001000.1011110110001000.1011110110001000.10111
010.1110111001000010.1110111001000010.1110111001000
1010000100.1101111010000100.1101111010000100.110111
.1011110110001000.1011110110001000.1011110110001000

$$v_1 = \frac{1}{9},\ v_5 = \frac{23}{3}$$

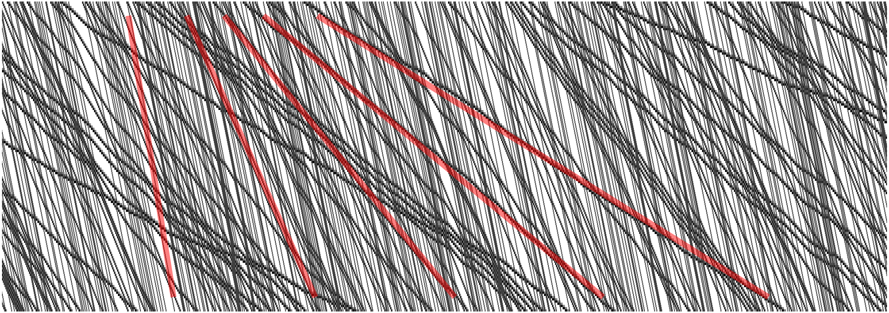
Simulations



2000×140, i.i.d. with density $\lambda = .15$

red straight lines are deterministic and computed by the theorem

Simulations



2000×140, i.i.d. with density $\lambda = .25$

red straight lines are deterministic and computed by the theorem

Equation for speeds

Why

$$v_k = k + \sum_{m<k} 2m\rho_m(v_k - v_m) - \sum_{m>k} 2k\rho_m(v_m - v_k) \ ?$$

Isolated k-solitons have speed k

When a k-soliton encounters an m-soliton:

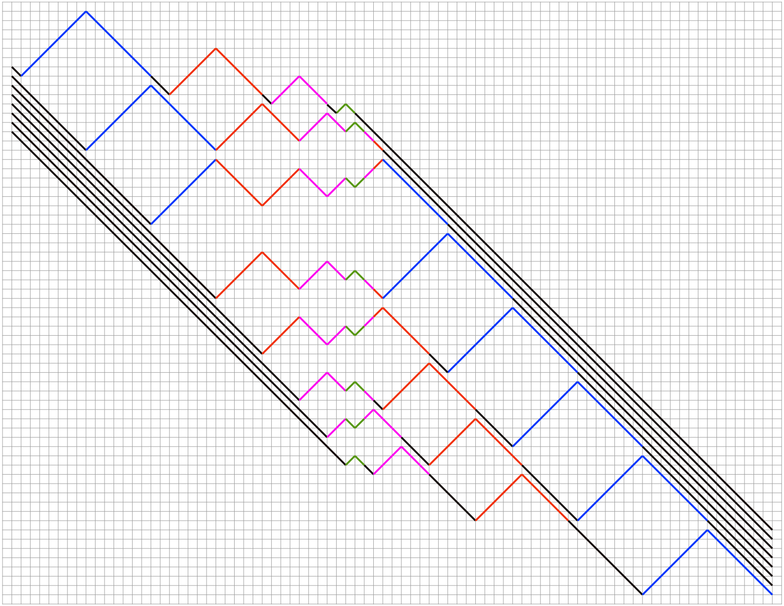
- it advances $2m$ extra units if $m < k$ or
- it is delayed by 2 time steps if $m > k$.

$\rho_m|v_k - v_m|$ is the frequency of such encounters as seen from a k-soliton.

Slots and components

Nesting Solitons

```
.11111110000000..1111100000.111000.10..................................................
........111111100000001111100000111001000..............................................
...............111111100000011110001101110000000.......................................
......................11111000011100100011111110000000.................................
...........................111100011011100000..11111110000000..........................
...............................11100100.1111100000....11111110000000...................
..................................11011000...1111100000......11111110000000............
....................................10.111000.....1111100000........11111110000000.....
```

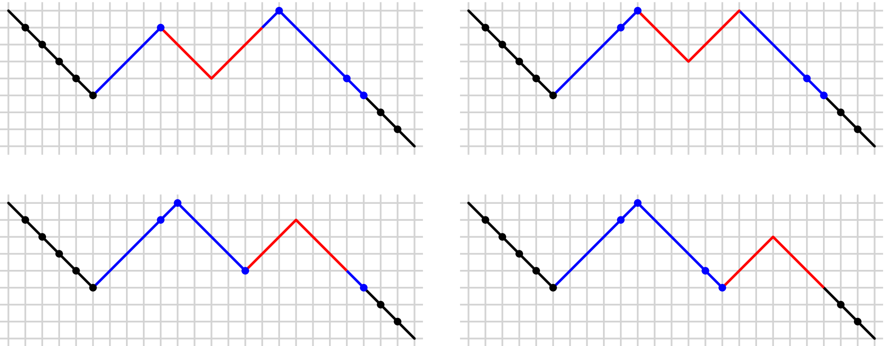


Nesting Solitons

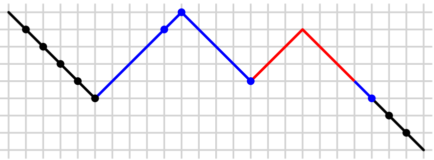
```
.7{}..5[].3().1..................................................
........7{5[3(1)]}...............................................
...............7{5[3(1)]}........................................
......................5[3(1)]7{}.................................
...........................5[3(1)]..7{}..........................
...............................3(1).5[]....7{}...................
..................................3(1)...5[]......7{}............
....................................1.3().....5[]........7{}.....
```

Slots

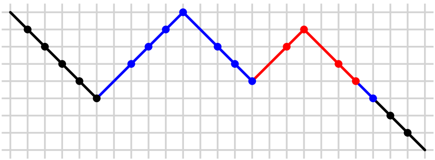
3-slots: sites where a 3-soliton can be appended



Enumerating the k-slots

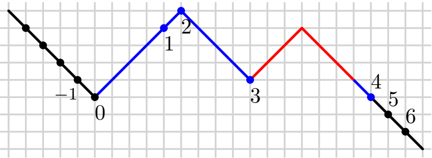


enumerating 3-slots

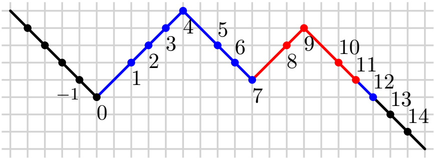


enumerating 1-slots

Enumerating the k-slots

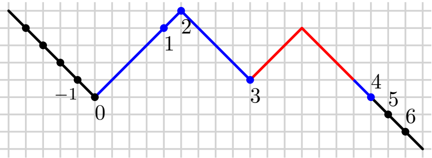


enumerating 3-slots

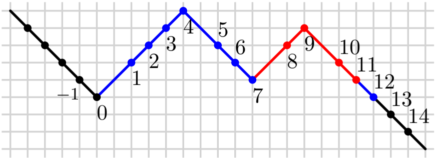


enumerating 1-slots

Enumerating the k-slots



enumerating 3-slots



enumerating 1-slots

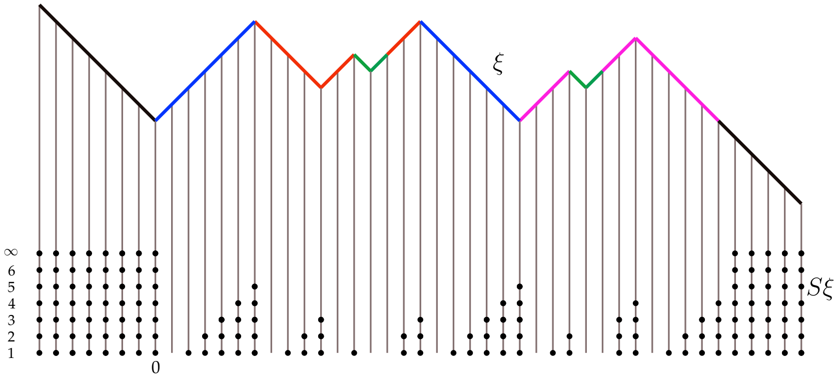
Requires an arbitrary:

labeling of records

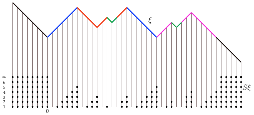
or choice of "Record 0"

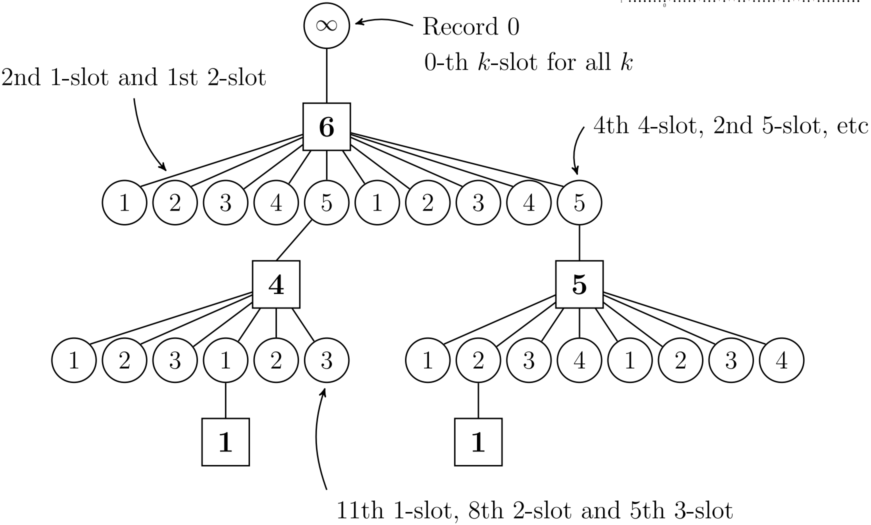
or walk representation

Slot configuration

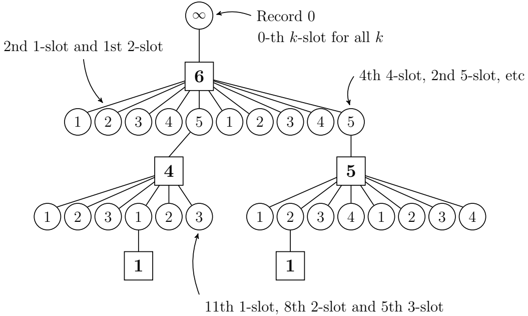


Nested solitons





Soliton components $\zeta = M\xi \in (\mathbb{N}^{\mathbb{Z}})^{\mathbb{N}_*}$




```
1.....................................................
..1...................................................
..1...................................................
......................................................
......................................................
.........1........1...................................
```

Invariant measures

Invariant measures with specified components

Theorem. *Let* $\zeta = (\zeta_k \in \mathbb{N}^{\mathbb{Z}})_{k \geqslant 1}$ *be independent fields with shift-invariant law for each* k*, and whose average density* α_k *satisfy*

$$\sum_k k\alpha_k < \infty.$$

Then there is a random η *whose law* μ *is shift-invariant and such that* $M\eta$ *is distributed as* ζ*. This law* μ *is unique, and it is* T*-invariant.*

If moreover ζ_k *is an i.i.d. field for each* k*, then* μ *is also shift-ergodic.*

Counter-examples

Invariant ergodic state whose components are not independent

10111101000010...**10111101000010**...10111101000010...
010...10111101000010...**10111101000010**...10111101000
101000010...10111101000010...**10111101000010**...10111
.10111101000010...10111101000010...**10111101000010**..

Invariant non-ergodic state with independent ergodic components

1011110110001000.1011110110001000.1011110110001000.
01000010.111011100100010.111011100100010.11101110
10111101000100.110111101000100.110111101000100.1
01000.**1011110110001000**.1011110110001000.10111101100

Evolution of components

Evolution of components

```
.11111110000000..1111100000.111000.10..................................................
........11111110000000111110000111001000...............................................
...............11111110000011110001101110000000........................................
......................11111000011100100011111110000000.................................
...........................111100011011100000..11111110000000..........................
...............................11100100.1111100000....11111110000000...................
..................................11011000...1111100000......11111110000000............
....................................10.111000.....1111100000........11111110000000.....
```

Evolution of components

```
.7{}..5[].3().1..................................................
........7{5[3(1)]}...............................................
...............7{5[3(1)]}........................................
......................5[3(1)]7{}.................................
...........................5[3(1)]..7{}..........................
...............................3(1).5[]....7{}...................
..................................3(1)...5[]......7{}............
....................................1.3().....5[]........7{}.....
```

Evolution of components

Ball configuration

```
.11111110000000..1111100000.111000.10..................................................
```

Fuzzy representation

```
.7{}..5[].3().1..................................................
```

Components

```
1.....................................................
......................................................
......1...............................................
......................................................
...............1......................................
......................................................
............................1.........................
```

Evolution of components

Ball configuration

........11111110000000111110000111001000...

Fuzzy representation

.........7{5[3(1)]}...

Components

.......1..

....

...........1..

....

..................1...................................

....

.............................1........................

Evolution of components

Ball configuration

```
...............11111110000011110001101110000000........................................
```

Fuzzy representation

```
...............7{5[3(1)]}............................................
```

Components

```
..............1.......................................
......................................................
................1.....................................
......................................................
.....................1................................
......................................................
..............................1.......................
```

Evolution of components

Ball configuration

....................11111000011100100011111110000000................................

Fuzzy representation

.....................5[3(1)]7{}................................

Components

....................1................................

....

....................1................................

....

.......................1.............................

....

..............................1......................

Evolution of components

Ball configuration

```
...........................111100011011100000..1111111000000..........................
```

Fuzzy representation

```
...........................5[3(1)]..7{}..........................
```

Components

```
............................1.........................
......................................................
..........................1...........................
......................................................
...........................1..........................
......................................................
................................1.....................
```

Evolution of components

Ball configuration

```
...............................11100100.1111100000....11111110000000...................
```

Fuzzy representation

```
...............................3(1).5[]....7{}...................
```

Components

```
...................................1..................
......................................................
...............................1......................
..............................1.......................
......................................................
.................................1....................
......................................................
```

Evolution of components

Ball configuration

```
..................................11011000...1111100000......11111110000000............
```

Fuzzy representation

```
..................................3(1)...5[]......7{}.............
```

Components

```
...........................................1...........
.......................................................
.....................................1.................
.......................................................
..................................1....................
.......................................................
...................................1...................
```

Evolution of components

Ball configuration

....................................10.111000....1111100000.......1111111000000.....

Fuzzy representation

....................................1.3().....5[].........7{}.....

Components

....1....

....

....1............

....

....................................1.................

....

...................................1..................

Evolution of components

Ball configuration

....................................10.111000....1111100000.......1111110000000.....

Fuzzy representation

....................................1.3().....5[]........7{}.....

Components

1......1......1......1......1......1......1......1....
....
......1....1....1....1....1....1....1....1............
....
...............1..1..1..1..1..1..1..1.................
....
............................11111111..................

Dynamics of components is a hierarchical translation

Theorem. *Components of $\hat{T}^t\xi$ are shifts components of ξ:*

$$M_k\hat{T}^t\xi = \theta^{o_k^t(\xi)+kt}M_k\xi$$

More concisely:

$$\hat{T} = M^{-1}\,\Theta\,M$$

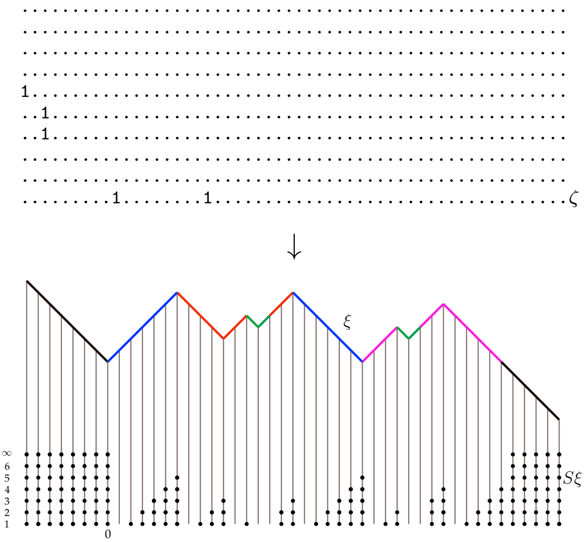
$J_m^t(\xi) :=$ Flow of m-solitons through Record 0

$o_k^t(\xi) := \sum_{m>k} 2(m-k)J_m^t(\xi)$

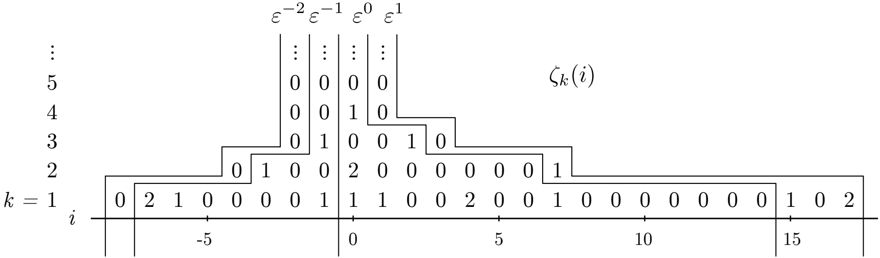
So o_k^t is determined by $(M_m\xi)_{m>k}$

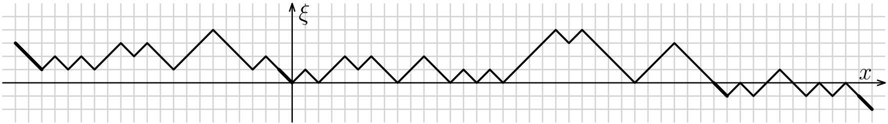
Reconstruction

Reconstruction algorithm $\xi = M^{-1}\zeta$

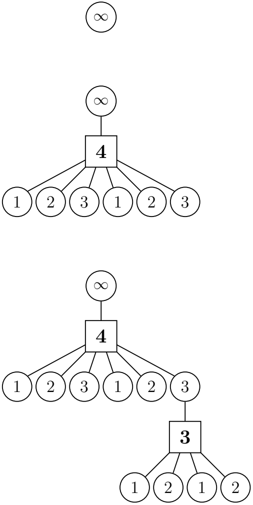


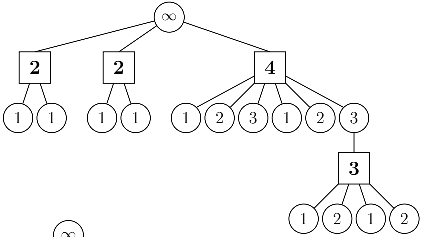
Reconstruction algorithm $\xi = M^{-1}\zeta$



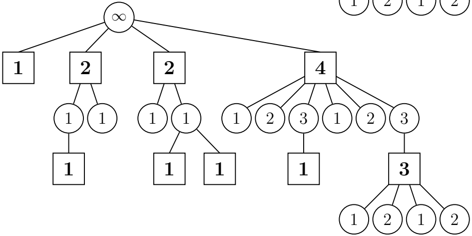


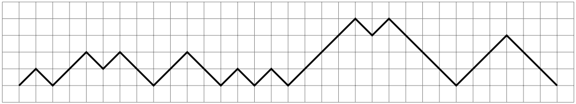
Reconstruction of ε^0





ε^0





Reconstruction of the other excursions: