# An arithmetic Bernstein-Kushnirenko inequality

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Let $\mathcal{A} := \{a_0, \dots, a_N\} \subset Z^n$ be a finite set of integer vectors. We compute the succesive minima of the projective toric variety $X_{\mathcal{A}} \subset \mathbb{P}^N$. Based on the work of S. Zhang on the Bogomolov conjecture [Zh95], we obtain from this an estimation for the height of $X_{\mathcal{A}}$.

This estimate allows us to prove the following arithmetic analogue of the Bernstein-Kushnirenko theorem, which improves upon a previous result of V. Maillot [Ma00]:

Let $f_1, \dots, f_n \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be Laurent polynomials with integer coefficients. Let $Q_0 \subset \mathbb{R}^n$ be a rational polytope, and set $Q_i := \mathrm{NP}(f_i) \subset \mathbb{R}^n$ for the Newton polytope of $f_i$ for $i = 1, \dots, n$. Let $V(f_1, \dots, f_n)_0$ be the set of isolated zeros of $V(f_1, \dots, f_n) \subset (\mathbb{C}^*)^n$. Then

$$\widehat{h}_{Q_0}(V(f_1, \dots, f_n)_0) \ \le \ \sum_{i=1}^{n} \mathrm{MV}(Q_0, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n)\, h_1(f_i).$$

Here $\widehat{h}_{Q_0}$ denotes the height induced by the inclusion $(\mathbb{C}^*)^n \hookrightarrow \bigoplus_{a \in \mathcal{A}} \mathbb{C}^*$, where $\mathcal{A} := Q_0 \cap \mathbb{Z}^n$.

The height of a Laurent polynomial $f = \sum_{a \in \mathcal{A}} f_a\, x^a$ is defined as the log of its $\mathcal{L}^1$-norm, that is $h_1(f) := \log ||f||_1 = \log(\sum_{a \in \mathcal{A}} |f_a|)$.

This result shows that the bit length complexity of the solution set of a sparse polynomial system is controlled by the mixed volumes of the considered polytopes.

References

[Ma00] V. Maillot, *Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables.* Mém. Soc. Math. Fr. **80** (2000), vi+129 pp.

[Zh95] S. Zhang, *Positive line bundles on arithmetic varieties.* J. Amer. Math. Soc. **8** (1995), 187-221.