

SYMMETRIC INTERPOLATION, EXCHANGE LEMMA AND SYLVESTER SUMS

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ABSTRACT. The theory of symmetric multivariate Lagrange interpolation is a beautiful but rather unknown tool that has many applications. Here we derive from it an Exchange Lemma that allows to explain in a simple and natural way the full description of the double sum expressions introduced by Sylvester in 1853 in terms of subresultants and their Bézout coefficients.

Keywords: Sylvester double sums. Subresultants. Symmetric Lagrange interpolation.

1. INTRODUCTION

W. Chen and J. Louck proved in [Chen & Louck 1996, Th.2.1] a beautiful interpolation result which describes the Lagrange interpolation basis for all multivariate symmetric polynomials in $m - d$ variables of multidegree bounded by (d, \dots, d) , for $0 \leq d < m$, see Section 2 below for a precise statement. They use their result to recover identities involving symmetric functions, generalizing for instance the following polynomial identity for a finite set $A = \{\alpha_1, \dots, \alpha_m\}$ contained in a field, and a finite set of variables $X = \{x_1 \dots, x_{m-d}\}$:

$$x_1 \cdots x_{m-d} = \sum_{A' \subset A, |A'|=d} \left(\prod_{\alpha_j \notin A'} \alpha_j \right) \frac{\prod_{x_j \in X, \alpha_i \in A'} (x_j - \alpha_i)}{\prod_{\alpha_j \notin A', \alpha_i \in A'} (\alpha_j - \alpha_i)}.$$

Here we take another direction and derive from this symmetric interpolation the following Exchange Lemma (see Corollary 3.2 below): given $0 \leq d \leq m$ and $0 \leq r \leq m - d$, for any finite sets A and B contained in a field, satisfying $|A| = m$ and $|B| \geq d$, and any set of variables X with $|X| = r$, one has the following polynomial identity

$$\sum_{\substack{A' \subset A \\ |A'|=d}} \mathcal{R}(A \setminus A', B) \frac{\mathcal{R}(X, A')}{\mathcal{R}(A', A \setminus A')} = (-1)^{d(m-d)} \sum_{\substack{B' \subset B \\ |B'|=d}} \mathcal{R}(A, B \setminus B') \frac{\mathcal{R}(X, B')}{\mathcal{R}(B', B \setminus B')},$$

where $\mathcal{R}(Y, Z) := \prod_{y \in Y, z \in Z} (y - z)$ (and $\mathcal{R}(Y, Z) := 1$ if Y or Z is empty).

In the particular case that $|B| = d$, our Exchange Lemma reads

$$\sum_{A' \subset A, |A'|=d} \mathcal{R}(A \setminus A', B) \frac{\mathcal{R}(X, A')}{\mathcal{R}(A \setminus A', A')} = \mathcal{R}(X, B),$$

which is exactly the statement of [Lascoux, Lem.Rt1], proved there using Schur functions. However, it does not seem possible to directly recover our Exchange Lemma from Lascoux result.

In these pages, we use symmetric interpolation and the Exchange Lemma to show in a very natural way the different and somehow puzzling relationships between Sylvester single and double sums, subresultants and their Bézout coefficients.

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Double sums were introduced in [Sylvester 1853]: for $A = \{\alpha_1, \dots, \alpha_m\}$ and $B = \{\beta_1, \dots, \beta_n\}$ contained in a field, he defined for $0 \leq p \leq m$, $0 \leq q \leq n$

$$\text{Syl}_{p,q}(A, B)(x) := \sum_{\substack{A' \subset A, B' \subset B \\ |A'|=p, |B'|=q}} \mathcal{R}(A', B') \mathcal{R}(A \setminus A', B \setminus B') \frac{\mathcal{R}(x, A') \mathcal{R}(x, B')}{\mathcal{R}(A', A \setminus A') \mathcal{R}(B', B \setminus B')},$$

which is a polynomial in x of degree bounded by $p + q =: d$. When $p = 0$ or $q = 0$, the resulting expression is called a *single sum*.

Note that the previous Exchange Lemma applied to $X = \{x\}$ reads

$$(1) \quad \text{Syl}_{d,0}(A, B)(x) = (-1)^{d(m-d)} \text{Syl}_{0,d}(A, B)(x),$$

which is one of the relationships between single sums that we can derive from Sylvester's statements in his original work.

Subresultants were also introduced by Sylvester in the same article. For $f(x) = \sum_{i=0}^m f_i x^i$ with $f_m \neq 0$, $g(x) = \sum_{i=0}^n g_i x^i$ with $g_n \neq 0$, and $d \leq \min\{m, n\}$ when $m \neq n$ or $d < m = n$,

$$\text{Sres}_d(f, g)(x) := \det \begin{array}{c} \begin{array}{cccc} & & & m+n-2d \\ f_m & \cdots & \cdots & f_{d+1-(n-d-1)} & x^{n-d-1} f(x) \\ & \ddots & & \vdots & \vdots \\ & & f_m & \cdots & f_{d+1} & f(x) \\ \hline g_n & \cdots & \cdots & g_{d+1-(m-d-1)} & x^{m-d-1} g(x) \\ & \ddots & & \vdots & \vdots \\ & & g_n & \cdots & g_{d+1} & g(x) \end{array} \\ \begin{array}{c} n-d \\ \cdot \\ m-d \end{array} \end{array}.$$

This turns out to be a polynomial in x of degree bounded by d .

Associating to A and B the monic polynomials $f(x) := (x - \alpha_1) \cdots (x - \alpha_m)$ and $g(x) := (x - \beta_1) \cdots (x - \beta_n)$, one has for instance

$$\begin{aligned} \text{Syl}_{0,0}(A, B) &= \mathcal{R}(A, B) = \text{Res}(f, g), \\ \text{Syl}_{m,0}(A, B) &= \mathcal{R}(x, A) = f \quad (= \text{Sres}_m(f, g) \text{ if } m < n), \\ \text{Syl}_{0,n}(A, B) &= \mathcal{R}(x, B) = g \quad (= \text{Sres}_n(f, g) \text{ if } n < m), \\ \text{Syl}_{m,n}(A, B) &= \mathcal{R}(A, B) \mathcal{R}(x, A) \mathcal{R}(x, B) = \text{Res}(f, g) f g, \end{aligned}$$

where $\text{Res}(f, g) := \text{Sres}_0(f, g)$ is the resultant of f and g , which is well-known to satisfy the Poisson formula $\text{Res}(f, g) = \prod_{\alpha \in A} g(\alpha) = \mathcal{R}(A, B)$.

Sylvester also mentions in his article [Sylvester 1853] the link between double sums and subresultants, and many other expressions for $\text{Syl}_{p,q}(A, B)$ depending on the values of p and q , see also [Lascoux and Pragasz 2003, D'Andrea et al. 2007, Roy & Szpirglas 2011]. The full description of $\text{Syl}_{p,q}(A, B)$ for all possible cases of $0 \leq p \leq m$, $0 \leq q \leq n$ is given as follows:

Theorem 1.1. (See [D'Andrea et al. 2009, Main Th. 1] and also [Krick & Szanto 2012, Th. 1].)

Let $1 \leq m \leq n$, $0 \leq p \leq m$, $0 \leq q \leq n$, and set $d := p + q$ and $k := m + n - d - 1$.

Lagrange interpolation formulas, as it is for the case $p = 0$, $q = n - 1$ explained below. This is in fact the way we recovered Chen and Louck's symmetric interpolation result [Chen & Louck 1996], as we were unaware of its existence. We also note that Lascoux in [Lascoux 2003, Section 3.5] mentions the possibility of using Lagrange interpolation techniques to prove identities for Sylvester single sums, without developing it. On the other hand, the Exchange Lemma shows that in fact a natural relationship exists between single and double sums (Propositions 3.3 and 3.8 below), therefore yielding all remaining expressions in Theorem 1.1.

Section 2 below treats the particular case of the Sylvester single sum $\text{Syl}_{0,d}$, which not only motivates the use of the interpolation technique we are referring to, but ends up being a key case for the general case treated in Section 3. We emphasize in Section 2 the very simple symmetric Lagrange interpolation Proposition 2.3, see [Chen & Louck 1996, Th.2.1], which is the basis for all our development, and Proposition 2.9, which allows to make the link between the single sum and the subresultant. In Section 3 we stress again the crucial Exchange Lemma 3.1, also obtained as a consequence of the symmetric Lagrange interpolation Proposition 2.3, that seems to be novel and allows to express all cases of Sylvester's double sums $\text{Syl}_{p,q}(A, B)$ in terms of the particular cases $\text{Syl}_{0,d}(A, B)$ and $\text{Syl}_{m,d-m}(A, B)$, where $d := p + q$. Theorem 1.1 is then obtained as a consequence of Corollaries 3.4 and 3.8. In addition, in Corollary 3.10 we obtain expressions in roots for the polynomials $F_k(f, g)$ and $G_k(f, g)$ in Bezout identity (2) below. Finally, we show in Corollary 3.13 that $F_k(f, g)$ and $G_k(f, g)$ are symmetric polynomials in $A \cup \{x\}$ and $B \cup \{x\}$, respectively.

2. SYLVESTER'S SINGLE SUMS AND SYMMETRIC LAGRANGE INTERPOLATION

We keep the following notation for the whole paper:

$$A = (\alpha_1, \dots, \alpha_m), \quad f = (x - \alpha_1) \cdots (x - \alpha_m) = \sum_{i=0}^m f_i x^i, \quad \text{where } f_m = 1,$$

$$B = (\beta_1, \dots, \beta_n), \quad g = (x - \beta_1) \cdots (x - \beta_n) = \sum_{i=0}^n g_i x^i, \quad \text{where } g_n = 1.$$

The double sum expression specializes when $p = 0$ and $q = d \leq n$ to the following *single sum* expression:

$$\begin{aligned} \text{Syl}_{0,d}(A, B)(x) &= \sum_{B' \subset B, |B'|=d} \mathcal{R}(A, B \setminus B') \frac{\mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')} \\ &= (-1)^{(m-d)(n-d)} \sum_{B' \subset B, |B'|=d} \mathcal{R}(B \setminus B', A) \frac{\mathcal{R}(x, B')}{\mathcal{R}(B \setminus B', B')} \\ &= (-1)^{(m-d)(n-d)} \sum_{B' \subset B, |B'|=d} \left(\prod_{\beta \notin B'} f(\beta) \right) \frac{\prod_{\beta \in B'} (x - \beta)}{\mathcal{R}(B \setminus B', B')}. \end{aligned}$$

In this section we investigate the relationship between this single sum expression and a specific multivariate symmetric Lagrange interpolation instance. As we will see in next section, the symmetric interpolation tools that we describe here for the single sum expressions will be crucial to tackle the claims about Sylvester's double sums.

As a motivation for what follows, we note that when $d = n - 1$ we get

$$\text{Syl}_{0,n-1}(A, B)(x) = (-1)^{m+n-1} \sum_{1 \leq i \leq n} f(\beta_i) \frac{\prod_{j \neq i} (x - \beta_j)}{\prod_{j \neq i} (\beta_i - \beta_j)},$$

where it is well-known that the set

$$\left\{ \frac{\prod_{j \neq i} (x - \beta_j)}{\prod_{j \neq i} (\beta_i - \beta_j)}; 1 \leq i \leq n \right\}$$

forms the so-called Lagrange basis, used to define the solution of the Lagrange interpolation problem, that is the unique polynomial of degree $\leq n - 1$ that takes given values at the n nodes β_1, \dots, β_n . Therefore, $\text{Syl}_{0, n-1}(A, B)(x)$ equals the unique polynomial h_{n-1} of degree $\leq n - 1$ which satisfies the n conditions $h_{n-1}(\beta_1) = (-1)^{m+n-1} f(\beta_1), \dots, h_{n-1}(\beta_n) = (-1)^{m+n-1} f(\beta_n)$. In particular, when $m < n$, $\text{Syl}_{0, n-1}(A, B)(x) = (-1)^{m+n-1} f(x)$.

When trying to generalize this to the case when $d < n - 1$, the difficulty is that the polynomials

$$\left\{ \mathcal{R}(x, B') = \prod_{\beta \in B'} (x - \beta); B' \subset B, |B'| = d \right\}$$

are linearly dependent in the vector space $K_d[x]$ of polynomials of degree bounded by d , since there are $\binom{n}{d} > d + 1$ of them (here $K = \mathbb{Q}(A, B)$, a field containing A and B).

This can be fixed for $0 \leq d \leq n - 1$ by considering a symmetric multivariate interpolation problem.

2.1. Symmetric Lagrange interpolation.

Notation 2.1. We denote by $S_{(\ell, d)}$ the K -vector space of all symmetric polynomials h in ℓ variables x_1, \dots, x_ℓ of multidegree bounded by (d, \dots, d) , i.e. such that $\deg_{x_i}(h) \leq d$ for $1 \leq i \leq \ell$ (with no specified bound for the total degree of h).

Lemma 2.2. $\dim_K(S_{(\ell, d)}) = \binom{\ell+d}{d}$.

Proof. It is well-known by the fundamental theorem of elementary symmetric polynomials that the symmetric polynomials in ℓ -variables are generated as an algebra by the elementary symmetric polynomials

$$e_1(x_1, \dots, x_\ell) = x_1 + \dots + x_\ell, \dots, e_\ell(x_1, \dots, x_\ell) = x_1 \cdots x_\ell,$$

all homogeneous of degree 1 in each variable x_i . Therefore, each symmetric polynomial of multidegree bounded by (d, \dots, d) can be uniquely expressed as $h = \sum_{\mathbf{a}} c_{\mathbf{a}} e_1^{a_1} \cdots e_\ell^{a_\ell}$ satisfying $|\mathbf{a}| := a_1 + \dots + a_\ell \leq d$. Thus it corresponds to a polynomial in e_1, \dots, e_ℓ of total-degree bounded by d : the space of such polynomials has dimension $\binom{\ell+d}{d}$. \square

We note that when $d < n$ and $\ell := n - d$, then $\dim_K(S_{(n-d, d)}) = \binom{n}{d}$.

Next proposition was proved in [Chen & Louck 1996, Th.2.1], but we include its proof here for sake of completeness. It shows that the set $\left\{ \frac{\mathcal{R}(X, B')}{\mathcal{R}(B \setminus B', B')}, B' \subset B, |B'| = d \right\}$ is the Lagrange interpolation basis for all symmetric polynomials in $n - d$ variables $X = \{x_1, \dots, x_{n-d}\}$ of multidegree bounded by (d, \dots, d) .

Proposition 2.3. Set $0 \leq d \leq n - 1$ and $X := (x_1, \dots, x_{n-d})$. Given $B = \{\beta_1, \dots, \beta_n\}$, the set

$$\mathcal{B} := \left\{ \mathcal{R}(X, B'); B' \subset B, |B'| = d \right\} \subset S_{(n-d, d)}$$

is a basis of $S_{(n-d, d)}$.

Moreover, any polynomial $h(X) \in S_{(n-d, d)}$ satisfies

$$h(X) = \sum_{B' \subset B, |B'|=d} h(B \setminus B') \frac{\mathcal{R}(X, B')}{\mathcal{R}(B \setminus B', B')}$$

where $h(B \setminus B') := h(\beta_{i_1}, \dots, \beta_{i_{n-d}})$ for $B \setminus B' = \{\beta_{i_1}, \dots, \beta_{i_{n-d}}\}$.

Proof. Since there are exactly $\binom{n}{d} = \dim_K(S_{(n-d,d)})$ elements in \mathcal{B} , it is enough to prove that all the elements are linearly independent. Suppose

$$\sum_{B' \subset B, |B'|=d} c_{B'} \mathcal{R}(X, B') = 0.$$

For each subset $\{\beta_{i_1}, \dots, \beta_{i_{n-d}}\} \subset B$, if we evaluate the left hand side at $x_1 = \beta_{i_1}, \dots, x_{n-d} = \beta_{i_{n-d}}$, every term in the sum vanishes except for $B' := B \setminus \{\beta_{i_1}, \dots, \beta_{i_{n-d}}\}$, where it gives $c_{B'} \mathcal{R}(B \setminus B', B')$. Since $\mathcal{R}(B \setminus B', B') \neq 0$, we get that $c_{B'} = 0$, proving linear independence. The second claim follows from the fact that $h(X) \in S_{(n-d,d)}$ is uniquely expressed in the basis \mathcal{B} , and its coordinates are uniquely defined by all evaluations at $\{\beta_{i_1}, \dots, \beta_{i_{n-d}}\} \subset B$. \square

2.2. Sylvester single sum.

Notation 2.4. Set $0 \leq d \leq n-1$ and $X := \{x_1, \dots, x_{n-d}\}$. We define

$$\text{MSyl}_{0,d}(A, B)(X) := (-1)^{(m-d)(n-d)} \sum_{B' \subset B, |B'|=d} \left(\prod_{\beta \notin B'} f(\beta) \right) \frac{\mathcal{R}(X, B')}{\mathcal{R}(B \setminus B', B')}.$$

Observation 2.5. By Proposition 2.3, $\text{MSyl}_{0,d}(A, B)(X) \in S_{(n-d,d)}$ is the unique polynomial in $S_{(n-d,d)}$ satisfying the $\binom{n}{d}$ conditions

$$\text{MSyl}_{0,d}(A, B)(B \setminus B') = (-1)^{(m-d)(n-d)} \prod_{\beta \in B \setminus B'} f(\beta) \text{ for all } B' \subset B, |B'| = d.$$

In particular,

$$(4) \quad \text{MSyl}_{0,d}(A, B)(X) = (-1)^{(m-d)(n-d)} f(x_1) \cdots f(x_{n-d}) \text{ when } \deg(f) \leq d.$$

The choice of the notation $\text{MSyl}_{0,d}(A, B)(X)$ for this polynomial stands for *multivariate Sylvester's sum*: the polynomial coincides with $\text{Syl}_{0,d}(A, B)(x)$ when $X = \{x\}$, i.e. $d = n-1$, or the latter is a coefficient of the former when there are two variables or more, i.e. $d < n-1$:

Remark 2.6. Set $0 \leq d \leq n-1$ and $X := (x_1, \dots, x_{n-d})$. Then

$$\text{Syl}_{0,d}(A, B)(x_{n-d}) = \begin{cases} \text{coeff}_{x_1^d \cdots x_{n-d-1}^d}(\text{MSyl}_{0,d}(A, B)(X)) & \text{for } 0 \leq d < n-1 \\ \text{MSyl}_{0,d}(A, B)(x_{n-d}) & \text{for } d = n-1. \end{cases}$$

Here $\text{coeff}_{x_1^d \cdots x_{n-d-1}^d}$ denotes the coefficient in $K[x_{n-d}]$ of the monomial $x_1^d \cdots x_{n-d-1}^d$ of $\text{MSyl}_{0,d}(A, B)(X)$.

Together with (4), this immediately implies:

Corollary 2.7. Set $0 \leq d \leq n-1$. If $m \leq d$ then

$$\text{Syl}_{0,d}(A, B) = \begin{cases} 0 & \text{for } m < d < n-1 \\ (-1)^{(m-d)(n-d)} f & \text{for } m < d = n-1 \text{ or } m = d \leq n-1. \end{cases}$$

Next we show a matrix formulation for the polynomial $\text{MSyl}_{0,d}(A, B)$ that will allow to recover the value of $\text{Syl}_{0,d}(A, B)(x)$ for the remaining case $d \leq m$. We need to introduce the following notations for the Vandermonde matrix.

Notation 2.8. Let $X = (x_1, \dots, x_k)$ be a k -tuple of (distinct) indeterminates or elements. We denote the (shifted) Vandermonde matrix of size $\ell \times k$ corresponding

We then evaluate the right-hand side expression into a $(n-d)$ -tuple $(\beta_{i_1}, \dots, \beta_{i_{n-d}})$ for fixed $1 \leq i_1 < \dots < i_{n-d} \leq n$. It is clear that the top determinant equals

$$(-1)^{(m-d)(n-d)} \det(V(\beta_{i_1}, \dots, \beta_{i_{n-d}})) f(\beta_{i_1}) \cdots f(\beta_{i_{n-d}}),$$

while the bottom determinant equals $\det(V(\beta_{i_1}, \dots, \beta_{i_{n-d}}))$. This concludes the proof. \square

Note that Proposition 2.9 is very similar in spirit to the matrix definition of the subresultant: actually they coincide when $d = n - 1$. This inspires the following result.

Proposition 2.10. *Set $0 \leq d \leq n - 1$. For $d \leq m$, one has*

$$\text{Syl}_{0,d}(A, B) = \text{Sres}_d(f, g).$$

Proof. We denote by $S(X)$ the polynomial in the numerator of the expression for $\text{MSyl}_{0,d}(A, B)(X)$ in the right-hand side of Lemma 2.9, and by $c_d(x_{n-d}) \in K[x_{n-d}]$ the coefficient of $x_1^d \cdots x_{n-d-1}^d$ in $\text{MSyl}_{0,d}(A, B)(X)$, that we want to show equals $\text{Sres}_d(f, g)(x_{n-d})$ according to Remark 2.6.

Since $\text{MSyl}_{0,d}(A, B)(X) = S(X) / \det(V(X))$, we get

$$\begin{aligned} S(X) &= \text{MSyl}_{0,d}(A, B)(X) \det(V(X)) \\ &= (c_d(x_{n-d}) x_1^d \cdots x_{n-d-1}^d + \cdots) (x_1^{n-d-1} x_2^{n-d-2} \cdots x_{n-d-1} + \cdots) \\ &= c_d(x_{n-d}) x_1^{n-1} x_2^{n-2} \cdots x_{n-d-1}^{d+1} + \cdots \end{aligned}$$

Therefore,

$$c_d(x_{n-d}) = \text{coeff}_{x_1^{n-1} x_2^{n-2} \cdots x_{n-d-1}^{d+1}}(S(X)).$$

It is clear that the coefficient of $x_1^{n-1} \cdots x_{n-d-1}^{d+1}$ in the determinant is obtained, by multilinearity, from the coefficients when the column with x_1 has all its exponents equal to $n - 1$, the column with x_2 has all its exponents equal to $n - 2$ up to the column with x_{n-d-1} has all its exponents equal to $d + 1$, that is $\text{coeff}_{x_1^{n-1} \cdots x_{n-d-1}^{d+1}}(S(X))$ equals

	$m-d$				$n-d$				
	f_m	\cdots	\cdots	f_{d+1}	f_d	\cdots	$f_{d+1-(n-d-1)}$	$x_{n-d}^{n-d-1} f(x_{n-d})$	
				\vdots	\vdots	\vdots	\vdots	\vdots	
			f_m	\cdots	f_n	\cdots	f_{d+1}	$f(x_{n-d})$	$n-d$
det	g_n	\cdots	\cdots	$g_{n-(m-d-1)}$	$g_{n-(m-d)}$	\cdots	$g_{d+1-(m-d-1)}$	$x_{n-d}^{m-d-1} g(x_{n-d})$	
				\vdots	\vdots	\vdots	\vdots	\vdots	
				g_n	g_{n-1}	\cdots	g_{d+1}	$g(x_{n-d})$	$m-d$

Thus

$$\text{coeff}_{x_1^{n-1} x_2^{n-2} \cdots x_{n-d-1}^{d+1}}(S(X)) = \text{Sres}_d(f, g)(x_{n-d}),$$

which implies $c_d(x_{n-d}) = \text{Sres}_d(f, g)(x_{n-d})$ as desired. \square

Putting together the information of Corollary 2.7, Proposition 2.10 and the value of $\text{Syl}_{0,d}$ for $d = n$, this interpolation technique therefore allowed us to recover very naturally the full description of Sylvester's single sums $\text{Syl}_{0,d}$ for any $0 \leq d \leq n$ and any m :

$$\text{Syl}_{0,d}(A, B) = \begin{cases} \text{Sres}_d(f, g) & \text{for } 0 \leq d \leq \{m-1, n\} \text{ or } d = m < n \\ 0 & \text{for } m < d < n-1 \\ (-1)^{m+n-1} f & \text{for } m < d = n-1 \\ g & \text{for } m \leq d = n. \end{cases}$$

3. SYLVESTER'S DOUBLE SUMS

We treat now the case of the general double sum expression, defined for $0 \leq p \leq m$, $0 \leq q \leq n$. Below we show how all cases reduce to the specific instances of $\text{Syl}_{0,d}$ and $\text{Syl}_{m,d-m}$.

The whole section flows from the following Exchange Lemma and its Corollary, which we could prove thanks to the interpolation Proposition 2.3 on symmetric polynomials.

Lemma 3.1. *Let A be any set with $|A| = m$, and set $0 \leq p \leq m$ and $X = \{x_1, \dots, x_{m-p}\}$. Let B be any set with $|B| \geq p$. Then*

$$\sum_{A' \subset A, |A'|=p} \mathcal{R}(A \setminus A', B) \frac{\mathcal{R}(X, A')}{\mathcal{R}(A \setminus A', A')} = \sum_{B' \subset B, |B'|=p} \mathcal{R}(A, B \setminus B') \frac{\mathcal{R}(X, B')}{\mathcal{R}(B', B \setminus B')}.$$

Proof. We observe that Proposition 2.3 implies that the polynomial $h(X) \in s_\lambda(m-p, p)$ on the left-hand side is the only symmetric polynomial satisfying the $\binom{m}{p}$ conditions $h(A \setminus A') = \mathcal{R}(A \setminus A', B)$. Since the polynomial on the right-hand side also belongs to $S_{(m-p, p)}$, it suffices to show that it satisfies the same specialization properties, i.e. that

$$\sum_{B' \subset B, |B'|=p} \mathcal{R}(A, B \setminus B') \frac{\mathcal{R}(A \setminus A', B')}{\mathcal{R}(B', B \setminus B')} = \mathcal{R}(A \setminus A', B), \quad \forall A' \subset A, |A'| = p.$$

But

$$\begin{aligned} \sum_{B' \subset B, |B'|=p} \mathcal{R}(A, B \setminus B') \frac{\mathcal{R}(A \setminus A', B')}{\mathcal{R}(B', B \setminus B')} &= \sum_{B' \subset B, |B'|=p} \mathcal{R}(A', B \setminus B') \mathcal{R}(A \setminus A', B \setminus B') \frac{\mathcal{R}(A \setminus A', B')}{\mathcal{R}(B', B \setminus B')} \\ &= \mathcal{R}(A \setminus A', B) \sum_{B' \subset B, |B'|=p} \frac{\mathcal{R}(A', B \setminus B')}{\mathcal{R}(B', B \setminus B')}. \end{aligned}$$

Consider for $Y = \{y_1, \dots, y_p\}$ the polynomial

$$\Psi(Y) = \sum_{B' \subset B, |B'|=p} \frac{\mathcal{R}(Y, B \setminus B')}{\mathcal{R}(B', B \setminus B')} \in S_{(p, |B|-p)}.$$

It is, again by Proposition 2.3, the only polynomial in $S_{(p, |B|-p)}$ satisfying the $\binom{|B|}{p}$ conditions $\Psi(B') = 1$, $\forall B' \subset B$, $|B'| = p$, and therefore $\Psi = 1$. In particular $\Psi(A') = 1$, which implies the statement. \square

Corollary 3.2. *Let A be any set with $|A| = m$, and set $0 \leq p \leq m$ and $X = (x_1, \dots, x_r)$, with $r \leq m-p$. Let B be any set with $|B| \geq p$. Then*

$$\sum_{A' \subset A, |A'|=p} \mathcal{R}(A \setminus A', B) \frac{\mathcal{R}(X, A')}{\mathcal{R}(A \setminus A', A')} = \sum_{B' \subset B, |B'|=p} \mathcal{R}(A, B \setminus B') \frac{\mathcal{R}(X, B')}{\mathcal{R}(B', B \setminus B')}.$$

Proof. The expressions above is simply the coefficient of $x_{r+1}^p \cdots x_{m-p}^p$ of the expressions in Lemma 3.1. \square

 3.1. The case $0 \leq d \leq n-1$.

In this section we set $d := p+q$, where $0 \leq p \leq n$ and $0 \leq q \leq n$, and we assume that it satisfies $d \leq n-1$.

We first deal with the case $d \leq \min\{m-1, n-1\}$. The Corollary 3.2 of the Exchange Lemma allows to relate $\text{Syl}_{p,q}(A, B)$ to $\text{Syl}_{0,p+q}(A, B)$, simply by a careful manipulation of terms.

Proposition 3.3. *Let $0 \leq p \leq m$, $0 \leq q \leq n$ and set $d := p + q$. Assume $d \leq \min\{m - 1, n - 1\}$. Then*

$$\text{Syl}_{p,q}(A, B) = (-1)^{p(m-d)} \binom{d}{p} \text{Syl}_{0,d}(A, B).$$

Proof. First note that

$$\text{Syl}_{p,q}(A, B) = (-1)^{p(m-d)} \sum_{\substack{B' \subset B \\ |B'|=q}} \left(\sum_{\substack{A' \subset A \\ |A'|=p}} \mathcal{R}(A \setminus A', B \setminus B') \frac{\mathcal{R}(x \cup B', A')}{\mathcal{R}(A \setminus A', A')} \right) \frac{\mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')}.$$

Applying the Exchange Corollary 3.2 to the coefficients inside the parenthesis of this expression for $X = x \cup B'$ of size $q + 1 \leq m - p$ and $B \setminus B'$ of size $n - q \geq p$ instead of B , we get

$$\sum_{\substack{A' \subset A \\ |A'|=p}} \mathcal{R}(A \setminus A', B \setminus B') \frac{\mathcal{R}(x \cup B', A')}{\mathcal{R}(A \setminus A', A')} = \sum_{\substack{C' \subset B \setminus B' \\ |C'|=p}} \mathcal{R}(A, B \setminus (B' \cup C')) \frac{\mathcal{R}(x \cup B', C')}{\mathcal{R}(C', B \setminus (B' \cup C'))}.$$

Therefore,

$$\begin{aligned} \text{Syl}_{p,q}(A, B) &= (-1)^{p(m-d)} \sum_{\substack{B' \subset B \\ |B'|=q}} \left(\sum_{\substack{C' \subset B \setminus B' \\ |C'|=p}} \mathcal{R}(A, B \setminus (B' \cup C')) \frac{\mathcal{R}(x \cup B', C')}{\mathcal{R}(C', B \setminus (B' \cup C'))} \right) \frac{\mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')} \\ &= (-1)^{p(m-d)} \sum_{\substack{B' \subset B, |B'|=q \\ C' \subset B \setminus B', |C'|=p}} \mathcal{R}(A, B \setminus (B' \cup C')) \frac{\mathcal{R}(B', C')}{\mathcal{R}(C', B \setminus (B' \cup C'))} \frac{\mathcal{R}(x, B' \cup C')}{\mathcal{R}(B', B \setminus B')} \\ &= (-1)^{p(m-d)} \sum_{\substack{B' \subset B, |B'|=q \\ C' \subset B \setminus B', |C'|=p}} \mathcal{R}(A, B \setminus (B' \cup C')) \frac{\mathcal{R}(x, B' \cup C')}{\mathcal{R}(B' \cup C', B \setminus (B' \cup C'))}. \end{aligned}$$

Finally, rewriting the sum over $B' \subset B, |B'| = q, C' \subset B \setminus B', |C'| = p$ as a sum over $D = B' \cup C' \subset B, |D| = d; C' \subset D, |C'| = p$, we get

$$\begin{aligned} \text{Syl}_{p,q}(A, B) &= (-1)^{p(m-d)} \sum_{\substack{D \subset B, |D|=d \\ C' \subset D, |C'|=p}} \mathcal{R}(A, B \setminus D) \frac{\mathcal{R}(x, D)}{\mathcal{R}(D, B \setminus D)} \\ &= (-1)^{p(m-d)} \binom{d}{p} \sum_{D \subset B, |D|=d} \mathcal{R}(A, B \setminus D) \frac{\mathcal{R}(x, D)}{\mathcal{R}(D, B \setminus D)} \\ &= (-1)^{p(m-d)} \binom{d}{p} \text{Syl}_{0,d}(A, B), \end{aligned}$$

by the definition of $\text{Syl}_{0,d}(A, B)$. □

Proposition 2.10 then immediately implies

Corollary 3.4. *Let $0 \leq p \leq m$, $0 \leq q \leq n$ and set $d := p + q$. Assume $d \leq \min\{m - 1, n - 1\}$. Then*

$$\text{Syl}_{p,q}(A, B) = (-1)^{p(m-d)} \binom{d}{p} \text{Sres}_d(f, g).$$

We now deal with the remaining case of this section, $m \leq d \leq n - 1$, where $d := p + q$ with $0 \leq p \leq m$, $0 \leq q \leq n$. We express the double sum $\text{Syl}_{p,q}(A, B)$ by

means of an interpolation problem as in Section 2. We rewrite $\text{Syl}_{p,q}(A, B)$ as

$$\begin{aligned} \text{Syl}_{p,q}(A, B)(x) &= (-1)^{(m-d)(n-q)} \sum_{\substack{A' \subset A \\ |A'|=p}} \left(\sum_{\substack{B' \subset B \\ |B'|=q}} \mathcal{R}(B \setminus B', A \setminus A') \frac{\mathcal{R}(x \cup A', B')}{\mathcal{R}(B \setminus B', B')} \right) \frac{\mathcal{R}(x, A')}{\mathcal{R}(A', A \setminus A')} \\ &= (-1)^{(m-d)(n-q)} \sum_{A' \subset A, |A'|=p} h_{A'}(x \cup A') \frac{\mathcal{R}(x, A')}{\mathcal{R}(A', A \setminus A')}, \end{aligned}$$

where

$$h_{A'}(x_1, x_2, \dots, x_{p+1}) := \sum_{B' \subset B, |B'|=q} \mathcal{R}(B \setminus B', A \setminus A') \frac{\mathcal{R}(\{x_1, \dots, x_{p+1}\}, B')}{\mathcal{R}(B \setminus B', B')},$$

is a symmetric polynomial in (x_1, \dots, x_{p+1}) of multidegree bounded by (q, \dots, q) . Since by hypothesis $p + q \leq n - 1$, we can complete the set (x_1, \dots, x_{p+1}) to the set $X = (x_1, \dots, x_{n-q})$. We have the following Lemma:

Lemma 3.5. *Let $0 \leq p \leq m$, $0 \leq q \leq n$, and set $d := p + q$. Assume $m \leq d \leq n - 1$ and define for $X = (x_1, \dots, x_{n-q})$ the polynomial*

$$H_{A'}(X) := \sum_{B' \subset B, |B'|=q} \mathcal{R}(B \setminus B', A \setminus A') \frac{\mathcal{R}(X, B')}{\mathcal{R}(B \setminus B', B')}.$$

Then

$$H_{A'}(X) = f_{A \setminus A'}(x_1) \cdots f_{A \setminus A'}(x_{n-q})$$

where $f_{A \setminus A'}(x) := \prod_{\alpha \in A \setminus A'} (x - \alpha)$.

Proof. Clearly $H_{A'}(X)$ has multidegree (q, \dots, q) , thus by Proposition 2.3, $H_{A'}$ is the only polynomial in $S_{(n-q, q)}$ satisfying the $\binom{n}{q}$ conditions

$$H_{A'}(B \setminus B') = \prod_{\beta \in B \setminus B'} f_{A \setminus A'}(\beta) \text{ for all } B' \subset B, |B'| = q,$$

which proves the claim since $\deg(f_{A \setminus A'}) = m - p \leq q$ by the hypothesis $m \leq d$. \square

Note the similarity of $H_{A'}$ with $h_{A'}$: they are the same when $d = n - 1$, or the latter is a coefficient of the former when there are two variables or more, i.e. $d < n - 1$:

Remark 3.6. Let $0 \leq p \leq m$, $0 \leq q \leq n$ and set $d := p + q$. Assume $m \leq d \leq n - 1$ and set $X = (x_1, \dots, x_{n-q})$. Then

$$h_{A'}(x_1, x_2, \dots, x_{p+1}) = \begin{cases} \text{coeff}_{x_{p+2}^q \cdots x_{n-q}^q} (H_{A'}(X)) & \text{for } 0 \leq d < n - 1 \\ H_{A'}(X) & \text{for } d = n - 1. \end{cases}$$

Here $\text{coeff}_{x_{p+2}^q \cdots x_{n-q}^q}$ denotes the coefficient in $K[x_1, \dots, x_{p+1}]$ of the monomial $x_{p+2}^q \cdots x_{n-q}^q$ of $H_{A'}$.

Together with Corollary 3.4, Lemma 3.5 and Remark 3.6 immediately imply the following full description of the case $d \leq n - 1$:

Corollary 3.7. *Let $0 \leq p \leq m$, $0 \leq q \leq n$ and set $d := p + q$. Assume $0 \leq d \leq n - 1$. Then,*

$$\text{Syl}_{p,q}(A, B) = \begin{cases} (-1)^{p(m-d)} \binom{d}{p} \text{Sres}_d(f, g) & \text{for } 0 \leq d \leq \min\{m, n - 1\} \\ 0 & \text{for } m < d < n - 1 \\ (-1)^{(p+1)(m+n-1)} \binom{m}{p} f & \text{for } m < d = n - 1. \end{cases}$$

Proof. The first case is direct from Corollary 3.4 for $d \leq m - 1$ while for $d = m$, $\text{Syl}_{m,0}(A, B) = f = \text{Sres}_m(f, g)$ when $m < n$.

The two following cases are a consequence of the definition of $\text{Syl}_{p,q}(A, B)$ in terms of $h_{A'}(x \cup A')$, Identity (3.5) and Remark 3.6. For the second case we observe that $h_{A'} = 0$ and for the third one, that the polynomial $f_{A \setminus A'}(x_1) \cdots f_{A \setminus A'}(x_{p+1})$ specialized into $x \cup A'$ equals $f_{A \setminus A'}(x) \mathcal{R}(A', A \setminus A')$. So, for $d = n - 1$,

$$\begin{aligned} \text{Syl}_{p,q}(A, B) &= (-1)^{(m-d)(n-q)} \sum_{A' \subset A, |A'|=p} h_{A'}(x \cup A') \frac{\mathcal{R}(x, A')}{\mathcal{R}(A', A \setminus A')} \\ &= (-1)^{(m-d)(n-q)} \sum_{A' \subset A, |A'|=p} \mathcal{R}(x, A \setminus A') \mathcal{R}(A', A \setminus A') \frac{\mathcal{R}(x, A')}{\mathcal{R}(A', A \setminus A')} \\ &= (-1)^{(m+n-1)(p+1)} \sum_{A' \subset A, |A'|=p} f(x) = (-1)^{(p+1)(m+n-1)} \binom{m}{p} f(x). \end{aligned}$$

□

3.2. The case $\max\{m, n\} \leq d \leq m + n$.

In the previous section we concluded the case $0 \leq d \leq n - 1$ and m arbitrary. By the symmetry mentioned in Identity (3), this also concludes the case $0 \leq d \leq m - 1$ and n arbitrary. Thus it only remains to consider the case $d \geq m$ and $d \geq n$, i.e. $\max\{m, n\} \leq d \leq m + n$.

First we observe that the case $d = m + n$, i.e. $p = m$ and $n = q$ is already solved, since $\text{Syl}_{m,n}(A, B) = \text{Res}(f, g) f g$, as was mentioned in the introduction.

In the rest of this section we restrict to the case $\max\{m, n\} \leq d \leq m + n - 1$. We show that we can express $\text{Syl}_{p,q}(A, B)$ in these cases in terms of $\text{Syl}_{0,k}(A, B)$ and $\text{Syl}_{m,d-m}(A, B)$, where $k := m + n - d - 1$.

Proposition 3.8. *Let $0 \leq p \leq m$, $0 \leq q \leq n$ and set $d := p + q$. Assume $\max\{m, n\} \leq d \leq m + n - 1$. Then*

$$\text{Syl}_{p,q}(A, B) = (-1)^e \binom{k}{n-q} \text{Syl}_{0,k}(A, B) + (-1)^e \binom{k+1}{m-p} \text{Syl}_{m,d-m}(A, B),$$

where $k := m + n - d - 1$, $c := (d - m)(n - q) + d - n$ and $e := (d - m)(q + 1)$.

Proof. We rewrite, for a fixed $B' \subset B$, $|B'| = q$,

$$\begin{aligned} \sum_{\substack{A' \subset A \\ |A'|=p}} \mathcal{R}(B', A') \mathcal{R}(B \setminus B', A \setminus A') \frac{\mathcal{R}(x, A')}{\mathcal{R}(A', A \setminus A')} &= \sum_{\substack{A' \subset A \\ |A'|=p}} \mathcal{R}(B' \cup x, A') \frac{\mathcal{R}(B \setminus B', A \setminus A')}{\mathcal{R}(A', A \setminus A')} \\ &= (-1)^{p(q+1)} \sum_{\substack{A'' \subset A \\ |A''|=m-p}} \mathcal{R}(A \setminus A'', B' \cup x) \frac{\mathcal{R}(B \setminus B', A'')}{\mathcal{R}(A \setminus A'', A'')} \\ &= (-1)^{p(q+1)} \sum_{\substack{C'' \subset B' \cup \{x\} \\ |C''|=m-p}} \mathcal{R}(A, (B' \cup x) \setminus C'') \frac{\mathcal{R}(B \setminus B', C'')}{\mathcal{R}(C'', (B' \cup x) \setminus C'')}, \end{aligned}$$

where for the last equality we used the Exchange Corollary 3.2 for $B' \cup \{x\}$ of size $q + 1 \geq m - p$ since $d \geq m - 1$ instead of B and $X = B \setminus B'$ of size $n - q \leq m - (m - p)$

since $d \geq n$. Therefore, setting $\zeta := (m-p)(n-q)$, we get

$$\begin{aligned} \text{Syl}_{p,q}(A, B)(x) &= (-1)^{\zeta+pq} \sum_{\substack{B' \subset B \\ |B'|=q}} \left(\sum_{\substack{A' \subset A \\ |A'|=p}} \mathcal{R}(B', A') \mathcal{R}(B \setminus B', A \setminus A') \frac{\mathcal{R}(x, A')}{\mathcal{R}(A', A \setminus A')} \right) \frac{\mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')} \\ &= (-1)^{\zeta+p} \sum_{\substack{B' \subset B \\ |B'|=q}} \left(\sum_{\substack{C'' \subset B' \cup \{x\} \\ |C''|=m-p}} \mathcal{R}(A, (B' \cup x) \setminus C'') \frac{\mathcal{R}(B \setminus B', C'')}{\mathcal{R}(C'', (B' \cup x) \setminus C'')} \right) \frac{\mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')}. \end{aligned}$$

Now we split this sum (without considering the sign $(-1)^{\zeta+p}$ for now) in two sums, according whether $x \in C''$ or $x \notin C''$. The first sum S_1 , when $x \in C''$, equals

$$\begin{aligned} S_1 &= \sum_{\substack{B' \subset B, |B'|=q \\ C' \subset B', |C'|=m-p-1}} \mathcal{R}(A, B' \setminus C') \frac{\mathcal{R}(B \setminus B', C' \cup x)}{\mathcal{R}(C' \cup x, B' \setminus C')} \frac{\mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')} \\ &= (-1)^{n-q+(n-q)q} \sum_{\substack{B' \subset B, |B'|=q \\ C' \subset B', |C'|=m-p-1}} \mathcal{R}(A, B' \setminus C') \frac{\mathcal{R}(x, (B \setminus B') \cup C')}{\mathcal{R}((B \setminus B') \cup C', B' \setminus C')} \\ &= (-1)^{n(q+1)} \sum_{\substack{B' \subset B, |B'|=q \\ C' \subset B', |C'|=m-p-1}} \mathcal{R}(A, B \setminus ((B \setminus B') \cup C')) \frac{\mathcal{R}(x, (B \setminus B') \cup C')}{\mathcal{R}((B \setminus B') \cup C', B \setminus ((B \setminus B') \cup C'))}. \end{aligned}$$

Finally, replacing the summation over B' by a sum over $D = (B \setminus B') \cup C' \subset B$, where $|D| = n - q + m - p - 1 = m + n - d - 1 = k$ (observe that $0 \leq k \leq \min\{m-1, n-1\}$ since $\max\{m, n\} \leq d \leq m + n - 1$), we get

$$\begin{aligned} S_1 &= (-1)^{n(q+1)} \sum_{\substack{D \subset B, |D|=k \\ C' \subset D, |C'|=m-p-1}} \mathcal{R}(A, B \setminus D) \frac{\mathcal{R}(x, D)}{\mathcal{R}(D, B \setminus D)} \\ &= (-1)^{n(q+1)} \binom{k}{m-p-1} \sum_{D \subset B, |D|=k} \mathcal{R}(A, B \setminus D) \frac{\mathcal{R}(x, D)}{\mathcal{R}(D, B \setminus D)} \\ &= (-1)^{n(q+1)} \binom{k}{n-q} \text{Syl}_{0,k}(A, B)(x). \end{aligned}$$

Let S_2 denote the second sum (without the sign $(-1)^{\zeta+p}$ for now), when $x \notin C''$:

$$\begin{aligned}
S_2 &= \sum_{\substack{B' \subset B, |B'|=q \\ C'' \subset B', |C''|=m-p}} \mathcal{R}(A, (B' \cup x) \setminus C'') \frac{\mathcal{R}(B \setminus B', C'')}{\mathcal{R}(C'', (B' \cup x) \setminus C'')} \frac{\mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')} \\
&= (-1)^\varepsilon f(x) \sum_{\substack{B' \subset B, |B'|=q \\ C'' \subset B', |C''|=m-p}} \mathcal{R}(A, B' \setminus C'') \frac{\mathcal{R}(x, B' \setminus C'')}{\mathcal{R}(B' \setminus C'', (B \setminus B') \cup C'')} \\
&= (-1)^\varepsilon f(x) \sum_{\substack{B' \subset B, |B'|=q \\ C'' \subset B', |C''|=m-p}} \mathcal{R}(A, B \setminus ((B \setminus B') \cup C'')) \frac{\mathcal{R}(x, B \setminus ((B \setminus B') \cup C''))}{\mathcal{R}(B \setminus ((B \setminus B') \cup C''), (B \setminus B') \cup C'')} \\
&= (-1)^\varepsilon f(x) \sum_{\substack{D \subset B, |D|=k+1 \\ C'' \subset D, |C''|=m-p}} \mathcal{R}(A, B \setminus D) \frac{\mathcal{R}(x, B \setminus D)}{\mathcal{R}(B \setminus D, D)} \\
&= (-1)^\varepsilon \binom{k+1}{m-p} f(x) \sum_{D \subset B, |D|=k+1} \mathcal{R}(A, B \setminus D) \frac{\mathcal{R}(x, B \setminus D)}{\mathcal{R}(B \setminus D, D)} \\
&= (-1)^\varepsilon \binom{k+1}{m-p} f(x) \sum_{D' \subset B, |D'|=d-m} \mathcal{R}(A, D') \frac{\mathcal{R}(x, D')}{\mathcal{R}(D', B \setminus D')} \\
&= (-1)^\varepsilon \binom{k+1}{m-p} \text{Syl}_{m, d-m}(A, B),
\end{aligned}$$

where $\varepsilon := m + m - p + (m - p)(n - q) + (m - p)(q + p - m) \equiv m + n(m - p) \pmod{2}$, and $D = (B \setminus B') \cup C''$ with $|D| = m + n - d - 1 + 1 = k + 1$ and $D' = B \setminus D$ with $|D'| = n - (k + 1) = d - m$, by the definition of $\text{Syl}_{m, d-m}$.

Finally we add up the signs

$$c := \zeta + p + n(q + 1) \equiv (d - m)(n - q) + d - n \pmod{2}$$

$$e := \zeta + p + \varepsilon = (m - p)(n - q) + m + n(m - p) \equiv (d - m)(q + 1) \pmod{2}$$

to get the expression in the claim. \square

We end up considering the only remaining case, $\text{Syl}_{m, d-m}(A, B)$ for $\max\{m, n\} \leq d \leq m + n - 1$. Remember that

$$\begin{aligned}
\text{Syl}_{m, d-m}(A, B)(x) &= \sum_{B' \subset B, |B'|=d-m} \mathcal{R}(A, B') \frac{\mathcal{R}(x, A) \mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')} \\
&= f(x) \sum_{B' \subset B, |B'|=d-m} \mathcal{R}(A, B') \frac{\mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')},
\end{aligned}$$

which is a polynomial of degree bounded by $m + (d - m) = d$.

Since $\max\{m, n\} \leq d \leq m + n - 1$, $k := m + n - d - 1$ satisfies $k \leq \min\{m - 1, n - 1\}$.

Thus, for these values of k we have the Bezout Identity (2)

$$\text{Sres}_k(f, g) = F_k(f, g)f + G_k(f, g)g.$$

Here $F_k(f, g)$ is a polynomial of degree bounded by $n - k - 1 = d - m$.

Proposition 3.9. *Let $\max\{m, n\} \leq d \leq m + n - 1$ and set $k := m + n - d - 1$. Then*

$$\text{Syl}_{m, d-m}(A, B) = (-1)^{(d-m)n+m+n-1} F_k(f, g) f.$$

Proof. Since $\text{Syl}_{m, d-m}(A, B)$ and $F_k(f, g) f$ are both polynomials of degree bounded by $d \leq m + n - 1$, it suffices to show the equality by interpolating them on the

indeterminates of $A \cup B$. It is clear that they both vanish –thus coincide– on any $\alpha \in A$. So it remains to evaluate both polynomials on $\beta \in B$. We have

$$\begin{aligned} \text{Syl}_{m,d-m}(A,B)(\beta) &= f(\beta) \sum_{B' \subset B, |B'|=d-m, \beta \notin B'} \mathcal{R}(A, B') \frac{\mathcal{R}(\beta, B')}{\mathcal{R}(B', B \setminus B')} \\ &= f(\beta) \sum_{B' \subset B \setminus \{\beta\}, |B'|=d-m} \mathcal{R}(A, B') \frac{\mathcal{R}(\beta, B')}{\mathcal{R}(B', \beta) \mathcal{R}(B', B \setminus (\beta \cup B'))} \\ &= (-1)^{d-m} f(\beta) \sum_{B' \subset B \setminus \{\beta\}, |B'|=d-m} \frac{\mathcal{R}(A, B')}{\mathcal{R}(B', B \setminus (\beta \cup B'))}. \end{aligned}$$

On the other hand, since $g(\beta) = 0$,

$$(F_k(f, g) f)(\beta) = (F_k(f, g) f + G_k(f, g) g)(\beta) = \text{Sres}_k(f, g)(\beta) = \text{Syl}_{0,k}(A, B)(\beta)$$

by Proposition 2.10. Therefore we need to compute $\text{Syl}_{0,k}(A, B)(\beta)$. But

$$\begin{aligned} \text{Syl}_{0,k}(A, B)(\beta) &= \sum_{D \subset B, |D|=k} \mathcal{R}(A, B \setminus D) \frac{\mathcal{R}(\beta, D)}{\mathcal{R}(D, B \setminus D)} \\ &= \sum_{D \subset B \setminus \{\beta\}, |D|=k} \mathcal{R}(A, \beta) \mathcal{R}(A, B \setminus (\beta \cup D)) \frac{\mathcal{R}(\beta, D)}{\mathcal{R}(D, \beta) \mathcal{R}(D, B \setminus (\beta \cup D))} \\ &= (-1)^{m-k} f(\beta) \sum_{D \subset B \setminus \{\beta\}, |D|=k} \frac{\mathcal{R}(A, B \setminus (\beta \cup D))}{\mathcal{R}(D, B \setminus (\beta \cup D))} \\ &= (-1)^{m-k} f(\beta) \sum_{B' \subset B \setminus \{\beta\}, |B'|=d-m} \frac{\mathcal{R}(A, B')}{\mathcal{R}(B \setminus (\beta \cup B'), B')} \\ &= (-1)^{m-k+k(d-m)} f(\beta) \sum_{B' \subset B \setminus \{\beta\}, |B'|=d-m} \frac{\mathcal{R}(A, B')}{\mathcal{R}(B', B \setminus (\beta \cup B'))}, \end{aligned}$$

setting $B' := B \setminus (\beta \cup D)$.

Therefore $\text{Syl}_{m,m-d}(A, B)(\beta) = (-1)^{m-k+(k+1)(d-m)} \text{Syl}_{0,k}(A, B)(\beta)$. The statement follows from $m - k + (k + 1)(d - m) \equiv (d - m)n + m + n - 1 \pmod{2}$. \square

The previous proposition allows to recover expressions in roots for the polynomials $F_k(f, g)$ and $G_k(f, g)$ appearing in Bezout Identity (2). Note that these identities already appeared in [Sylvester 1853, Art. 29], and more recently in [Krick & Szanto 2012, D'Andrea et al. 2015].

Corollary 3.10. *Let $0 \leq k \leq \min\{m - 1, n - 1\}$. Then*

$$\begin{aligned} F_k(f, g) &= (-1)^{m-k} \sum_{B' \subset B, |B'|=k+1} \mathcal{R}(A, B \setminus B') \frac{\mathcal{R}(x, B \setminus B')}{\mathcal{R}(B', B \setminus B')} \\ G_k(f, g) &= (-1)^{m-k+1} \sum_{A' \subset A, |A'|=k+1} \mathcal{R}(A \setminus A', B) \frac{\mathcal{R}(x, A \setminus A')}{\mathcal{R}(A \setminus A', A')}. \end{aligned}$$

Proof. The identity for $F_k(f, g)$ is simply obtained by comparing $\text{Syl}_{m, d-m}(A, B)$ and $F_k(f, g) f$ and simplifying f in both sides:

$$\begin{aligned} F_k(f, g) &= (-1)^{(d-m)n+m+n-1} \sum_{B' \subset B, |B'|=d-m} \mathcal{R}(A, B') \frac{\mathcal{R}(x, B')}{\mathcal{R}(B', B \setminus B')} \\ &= (-1)^{d-n+1} \sum_{B' \subset B, |B'|=d-m} \mathcal{R}(A, B') \frac{\mathcal{R}(x, B')}{\mathcal{R}(B \setminus B', B')} \\ &= (-1)^{m-k} \sum_{D \subset B, |D|=k+1} \mathcal{R}(A, B \setminus D) \frac{\mathcal{R}(x, B \setminus D)}{\mathcal{R}(D, B \setminus D)}. \end{aligned}$$

On the other hand, we have that

$$G_k(f, g) = (-1)^{(m-k)(n-k)} F_k(g, f)$$

which implies

$$\begin{aligned} G_k(f, g) &= (-1)^{(m-k+1)(n-k)} \sum_{A' \subset A, |A'|=k+1} \mathcal{R}(B, A \setminus A') \frac{\mathcal{R}(x, A \setminus A')}{\mathcal{R}(A', A \setminus A')} \\ &= (-1)^{m-k+1} \sum_{A' \subset A, |A'|=k+1} \mathcal{R}(A \setminus A', B) \frac{\mathcal{R}(x, A \setminus A')}{\mathcal{R}(A \setminus A', A')}. \end{aligned}$$

□

Propositions 3.8, 2.10 and 3.9 then imply

Corollary 3.11. *Let $0 \leq p \leq m$, $0 \leq q \leq n$ and set $d := p + q$. Assume $\max\{m, n\} \leq d \leq m + n - 1$. Then*

$$\text{Syl}_{p,q}(A, B) = (-1)^{(d-m)(n-q)+d-n} \left(\binom{k}{n-q} \text{Sres}_k(f, g) - \binom{k+1}{m-p} F_k f \right),$$

where $k := m + n - d - 1$.

Proof. We have

$$\begin{aligned} \text{Syl}_{p,q}(A, B) &= (-1)^c \binom{k}{n-q} \text{Syl}_{0,k}(A, B) + (-1)^e \binom{k+1}{m-p} \text{Syl}_{m, d-m}(A, B) \\ &= (-1)^c \binom{k}{n-q} \text{Sres}_k(f, g) + (-1)^e (-1)^{(d-m)n+m+n-1} \binom{k+1}{m-p} F_k(f, g) f, \end{aligned}$$

where $c := (d-m)(n-q) + d - n$ and $e := (d-m)(q+1)$. The statement follows from $e + (d-m)n + m + n \equiv (d-m)(n-q) + d - n \pmod{2}$. □

The following proposition is another application of the Exchange Lemma 3.1, and gives simple identities for the polynomials F_k and G_k in the Bezout Identity (2) in terms of the roots, which enable us to make a further connection with Schur polynomials.

Proposition 3.12. *Let $0 \leq k \leq \min\{m-1, n-1\}$. Then*

$$\begin{aligned} F_k &= (-1)^{k(m-k)} \sum_{C' \subset A \cup \{x\}, |C'|=k+1} \frac{R((A \cup \{x\}) \setminus C', B)}{R(C', (A \cup \{x\}) \setminus C')}, \\ G_k &= (-1)^{m(n-k)} \sum_{D' \subset B \cup \{x\}, |D'|=k+1} \frac{R((B \cup \{x\}) \setminus D', A)}{R(D', (B \cup \{x\}) \setminus D')}. \end{aligned}$$

Proof. These identities are also an immediate consequence of Exchange Corollary 3.2, applied to $r = 0$ variables. We set $C := A \cup \{x\}$, $D := B \cup \{x\}$.

$$\begin{aligned}
 F_k(f, g) &= (-1)^{m-k} \sum_{B' \subset B, |B'|=k+1} \frac{R(C, B \setminus B')}{R(B', B \setminus B')} \\
 &= (-1)^{(m-k)(n-k)} \sum_{B' \subset B, |B'|=k+1} \frac{R(B \setminus B', C)}{R(B \setminus B', B')} \\
 &= (-1)^{(m-k)(n-k)} \sum_{C' \subset C, |C'|=k+1} \frac{R(B, C \setminus C')}{R(C', C \setminus C')} \\
 &= (-1)^{k(m-k)} \sum_{C' \subset C, |C'|=k+1} \frac{R(C \setminus C', B)}{R(C', C \setminus C')}, \\
 G_k(f, g) &= (-1)^{(m-k)(n-k)} F_k(g, f) \\
 &= (-1)^{m(n-k)} \sum_{D' \subset D, |D'|=k+1} \frac{R(D \setminus D', A)}{R(D', D \setminus D')}.
 \end{aligned}$$

□

As a corollary, we can express $F_k(f, g)$ and $G_k(f, g)$ as symmetric polynomials in the variables $A \cup \{x\}$ and $B \cup \{x\}$, respectively.

Corollary 3.13. *Let $0 \leq k \leq \min\{m-1, n-1\}$. Then we have the following expressions for $F_k(f, g)$ and $G_k(f, g)$ as symmetric polynomials in $A \cup \{x\}$ and $B \cup \{x\}$, respectively:*

$$\det \begin{array}{c|ccc}
 & \overset{m+1}{g(\alpha_1)\alpha_1^{m-k-1}} & \dots & g(\alpha_m)\alpha_m^{m-k-1} & g(x)x^{m-k-1} \\
 & \vdots & & \vdots & \\
 & g(\alpha_1) & \dots & g(\alpha_m) & g(x) \\
 \hline
 & \alpha_1^k & \dots & \alpha_m^k & x^k \\
 & \vdots & & \vdots & \\
 & 1 & \dots & 1 & 1
 \end{array} \begin{array}{l} m-k \\ k+1 \end{array}$$

$$F_k(f, g) = (-1)^{m-k} \frac{\det(V(A \cup \{x\}))}{\det(V(A \cup \{x\}))}.$$

and

$$\det \begin{array}{c|ccc}
 & \overset{n+1}{f(\beta_1)\beta_1^{n-k-1}} & \dots & f(\beta_n)\beta_n^{n-k-1} & f(x)x^{n-k-1} \\
 & \vdots & & \vdots & \\
 & f(\beta_1) & \dots & f(\beta_n) & f(x) \\
 \hline
 & \beta_1^k & \dots & \beta_n^k & x^k \\
 & \vdots & & \vdots & \\
 & 1 & \dots & 1 & 1
 \end{array} \begin{array}{l} n-k \\ k+1 \end{array}$$

$$G_k(f, g) = (-1)^{(m-k-1)(n-k)} \frac{\det(V(B \cup \{x\}))}{\det(V(B \cup \{x\}))},$$

Proof. We prove the claim for $F_k(f, g)$, the claim for $G_k(f, g)$ follows from $G_k(f, g) = (-1)^{(m-k)(n-k)} F_k(g, f)$.

First we verify that the expression in the right-hand side is a polynomial: the denominator divides the numerator since making $\alpha_i = \alpha_j$ or $\alpha_i = x$ yields the vanishing of the numerator. It is symmetric in $A \cup \{x\}$ because both the numerator and the denominator are alternate. We set $C := A \cup \{x\}$. We expand the determinant

in the numerator of the right-hand side by the first $m - k$ rows, and get that the ratio of determinants equals

$$\begin{aligned} & \frac{1}{V(C)} \sum_{C'' \subset C, |C''|=m-k} \text{sg}(C'', C) V(C \setminus C'') V(C'') \prod_{c \in C''} g(C'') \\ &= \sum_{C'' \subset C, |C''|=m-k} \frac{\prod_{c \in C''} g(C'')}{R(C'', C \setminus C'')} \\ &= \sum_{C' \subset C, |C'|=k+1} \frac{\mathcal{R}(C \setminus C', B)}{\mathcal{R}(C \setminus C', C')} = (-1)^{m-k} F_k(f, g), \end{aligned}$$

using for the first equality that

$$\det(V(C)) = \text{sg}(C'', C) \det(V(C'')) \det(V(C \setminus C'')) \mathcal{R}(C'', C \setminus C''),$$

where $\text{sg}(C'', C)$ is the sign needed to bring the columns of C'' into $\{1, \dots, m - k\}$. \square

We close the paper by pointing to a simple connection of the previous corollary to Schur polynomials. Recall that the Schur polynomial $s_\lambda(X)$ corresponding to an ℓ -tuple of indeterminates $X = (x_1, \dots, x_\ell)$ and a partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0$, is defined as

$$s_\lambda(X) := \frac{\det \begin{array}{ccc} & & \ell \\ x_1^{\lambda_1 + \ell - 1} & \dots & x_\ell^{\lambda_1 + \ell - 1} \\ x_1^{\lambda_2 + \ell - 2} & \dots & x_\ell^{\lambda_2 + \ell - 2} \\ \vdots & & \vdots \\ x_1^{\lambda_{\ell-1} + 1} & \dots & x_\ell^{\lambda_{\ell-1} + 1} \\ x_1^{\lambda_\ell} & \dots & x_\ell^{\lambda_\ell} \end{array}}{\det(V(X))},$$

or equivalently, it is the determinant of the submatrix of the Vandermonde matrix $V_{\lambda_1 + \ell}(X)$ corresponding to the rows indexed by the exponents $\lambda_1 + \ell - 1, \lambda_2 + \ell - 2, \dots, \lambda_\ell$, divided by the usual Vandermonde determinant $\det(V(X))$.

To see the connection between Corollary 3.13 and Schur polynomials, assume first that $g = x^n$. Then we have

$$\det \begin{array}{ccc} & & m+1 \\ \alpha_1^{n+m-k-1} & \dots & \alpha_m^{n+m-k-1} & x^{n+m-k-1} \\ \vdots & & \vdots & \\ \alpha_1^n & \dots & \alpha_m^n & x^n \\ \hline \alpha_1^k & \dots & \alpha_m^k & x^k \\ \vdots & & \vdots & \\ 1 & \dots & 1 & 1 \end{array} \begin{array}{l} m-k \\ k+1 \end{array}$$

$$F_k(f, x^n) = (-1)^{m-k} \frac{\det(V(A \cup \{x\}))}{\det(V(A \cup \{x\}))}.$$

Thus, by definition, we immediately get that

$$F_k(f, x^n) = (-1)^{m-k} s_\lambda(A \cup \{x\}),$$

where $\lambda := (n - k - 1, \dots, n - k - 1, 0, \dots, 0) = ((n - k - 1)^{m-k}; 0^{k+1}) \in \mathbb{Z}_{\geq 0}^{m+1}$. Similar expressions can be obtained for $G_k(x^m, g)$ in terms of the Schur polynomial on $B \cup \{x\}$ with respect to the partition $((m - k - 1)^{n-k}; 0^{k+1})$.

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