SUBRESULTANTS, SYLVESTER SUMS AND THE RATIONAL INTERPOLATION PROBLEM

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Abstract. We present a solution for the classical univariate rational interpolation problem by means of (univariate) subresultants. In the case of Cauchy interpolation (interpolation without multiplicities), we recover explicit formulas for the solution in terms of symmetric functions of the input data, hence generalizing the well-known formulas for Lagrange interpolation. In the case of the osculatory rational interpolation (interpolation with multiplicities), we get determinantal expressions in terms of the input data.

1. Cauchy and osculatory interpolation

The *Cauchy interpolation problem* or rational interpolation problem considered already in [Cau1841, Ros1845, Pred1953] is the following:

> Let K be a field, and $a, b \in \mathbb{N}_0$. Set $\ell = a + b$. Given a set $\{x_0, \ldots, x_\ell\}$ of $\ell + 1$ points in K, and $y_0, \ldots, y_\ell \in K$, determine –if possible– polynomials A, $B \in K[x]$ such that $\deg(A) \leq a$, $\deg(B) \leq b$ and

$$\frac{A}{B}(x_i) = y_i,\ 0 \leq i \leq \ell. \tag{1}$$

This might be considered as a generalization of the classical Lagrange interpolation problem for polynomials, where $b = 0$ and $a = \ell$. In contrast with that case, there is not always a solution to this problem, since for instance by setting $y_0 = \cdots = y_a = 0$, the numerator A is forced to be identically zero, and therefore the remaining y_{a+k}, $1 \leq k \leq \ell - a$, have to be zero as well.

The obvious generalization of the Cauchy interpolation problem receives the name *osculatory rational interpolation problem* or rational Hermite interpolation problem:

> Let K be a field, and $a, b \in \mathbb{N}_0$. Set $\ell = a + b$. Given a set $\{x_0, \ldots, x_k\}$ of $k + 1$ points in K, $a_0, \ldots, a_k \in \mathbb{N}$ such that $a_0 + \cdots + a_k = \ell + 1$, and $y_{i,j} \in K$, $0 \leq i \leq k$, $0 \leq j < a_i$, not all of them equal to zero, determine –if possible– polynomials

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A, $B \in K[x]$ such that $\deg(A) \le a$, $\deg(B) \le b$ and

$$\left(\frac{A}{B}\right)^{(j)}(x_i) = j!\, y_{i,j},\ 0 \le i \le k,\ 0 \le j < a_i. \tag{2}$$

This problem has also been extensively studied from both an algorithmic and theoretical point of view, see for instance [Sal1962, Kah1969, Wuy1975, TF2000] and the references therein. In all these previous works, the solution of the problem comes in some kind of recursive continuous fraction, and it is not actually explicit. A unified framework, called *rational function reconstruction*, is presented in the book [vzGG2003] for both the Cauchy interpolation and the osculatory rational interpolation problems. It uses remainder sequences in the Extended Euclidean Algorithm.

Yet an explicit general formula in terms of the input data is not found in the literature. In the classical (Cauchy) case, an explicit formula can be derived from the results on symmetric operators in a suitable ring of polynomials presented in [Las2003], as shown in [Las]. In this paper, we get the explicit formulas for the Cauchy interpolants by expressing the subresultants in terms of the Sylvester sums introduced by Sylvester in [Syl1853]. Furthermore, we give determinantal expressions for the solution of the osculatory rational interpolation problem in terms of the input data. These latter expressions are given as quotients of determinants of matrices. Note that there are also explicit formulas for the Hermite interpolants in terms of the input data, see for instance [DKS2012] for a presentation of them. But yet we do not have a closed formula for subresultants in roots in the multiple case, so a generalization of Theorem 1.1 is yet not a forward application of Sylvester's multiple sums, and some more work on the subject must be done in order to shed light to the general problem.

We first present the result for the Cauchy interpolation problem. For $U, V \subset K$, we set $R(U,V) := \prod_{u\in U, v\in V}(u-v)$.

Theorem 1.1. *Given* (a,b), $X := \{x_0,\dots,x_\ell\}$ *and* $y_0,\dots,y_\ell$ *as above. Let* d *be maximal such that* $0 \le a \le d$ *and*

$$A := \sum_{X'\subset X, |X'|=d} R(x,X')\Big(\prod_{x_j\notin X'} y_j\Big)/R(X\setminus X', X') \ \in K[x]$$

is not identically zero. Then a solution for the Cauchy interpolation problem (1) *exists if and only if for*

$$B := \sum_{X''\subset X, |X''|=\ell-d} R(X'',x)\Big(\prod_{x_j\in X''} y_j\Big)/R(X'', X\setminus X'') \ \in K[x],$$

one has $B(x_i) \ne 0$ *for all* $0 \le i \le \ell$. *In that case the (essentially unique) solution is given by* A/B.

In particular, when $a = \ell$, Theorem 1.1 will specialize to the well-known *Lagrange interpolation polynomial*, associated to the data $\{(x_i,y_i)\}_{0\le i\le \ell}$, which is equal to

$$\sum_{0\le i\le\ell} y_i \frac{\prod_{j\ne i}(x-x_j)}{\prod_{j\ne i}(x_i-x_j)} = \sum_{0\le i\le\ell} y_i \frac{R(x, X\setminus\{x_i\})}{R(x_i, X\setminus\{x_i\})}.$$

Before stating our main result for the osculatory rational interpolation problem, we need to set a notation:

Notation 1.2. *Set* $a, b \in \mathbb{N}$ *such that* $a + b = \ell$, $a_0, \ldots, a_k \in \mathbb{N}$ *such that* $a_0 + \cdots + a_k = \ell + 1$, *as in* (2). *We define*

- $\overline{X} := \big((x_0, a_0); \ldots; (x_k, a_k)\big)$ *an array of pairs in* $K \times \mathbb{N}$ *and* $Y := \big(y_{i,j};\ 0 \le i \le k,\ 0 \le j < a_j\big)$. *We call these the* input data *for the osculatory rational interpolation problem.*
- *Set* $u \in \mathbb{N}$. *The* generalized Vandermonde *or* confluent matrix *(c.f.* [Kal1984]*) of size* $u + 1$ *associated to* $\overline{X}$ *is the (non-necessarily square) matrix* $V_{u+1}(\overline{X}) \in K^{(u+1)\times(\ell+1)}$ *defined by*

$$V_{u+1}(\overline{X}) := \overbrace{\left|\begin{array}{c|c|c} V_{u+1}(x_0, a_0) & \ldots & V_{u+1}(x_k, a_k) \end{array}\right|}^{\ell+1}\ {\scriptstyle u+1}\ ,$$

where for any t, $V_{u+1}(x_i, t+1) \in K^{(u+1)\times(t+1)}$ *is defined by*

$$V_{u+1}(x_i, t+1) := \overbrace{\left|\begin{array}{ccccc} 1 & 0 & 0 & \ldots & 0 \\ x_i & 1 & 0 & \ldots & 0 \\ x_i^2 & 2x_i & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x_i^u & ux_i^{u-1} & \binom{u}{2}x_i^{u-2} & \ldots & \binom{u}{t}x_i^{u-t} \end{array}\right|}^{t+1}\ {\scriptstyle u+1}$$

- *We define the matrix* $U_{u+1}(\overline{X}, Y) \in K^{(u+1)\times(\ell+1)}$ *associated to* $\overline{X}$ *and* Y *as:*

$$U_{u+1}(\overline{X}, Y) := \overbrace{\left|\begin{array}{c|c|c} U_{u+1}(x_0; y_{0,0}, \ldots, y_{0,a_0-1}) & \ldots & U_{u+1}(x_k; y_{k,0}, \ldots, y_{k,a_k-1}) \end{array}\right|}^{\ell+1}\ {\scriptstyle u+1}\ ,$$

where for any t, $U_{u+1}(x_i, y_{i,0}, \ldots, y_{i,t}) \in K^{(u+1)\times(t+1)}$ *is defined by*

$$\overbrace{\left|\begin{array}{cccc} y_{i,0} & y_{i,1} & \cdots & y_{i,t} \\ x_i y_{i,0} & x_i y_{i,1} + y_{i,0} & \cdots & x_i y_{i,t} + y_{i,t-1} \\ \vdots & \vdots & & \vdots \\ x_i^u y_{i,0} & x_i^u y_{i,1} + ux_i^{u-2} y_{i,0} & \cdots & \sum_{j=0}^{t} \binom{u}{j} x_i^{u-j} y_{i,t-j} \end{array}\right|}^{t+1}\ {\scriptstyle u+1}\ ,$$

with the convention that when $u < j$, $\binom{u}{j} x_i^{u-j} = 0$.

The next result expresses the solution of the osculatory rational interpolation problem in terms of the input data as follows:

Theorem 1.3. *Under the notation above, let d be maximal such that $0 \leq d \leq a$ and*

$$A := -\det \begin{array}{c|c|c|l} \multicolumn{1}{c}{\scriptstyle \ell+1} & \multicolumn{1}{c}{\scriptstyle 1} & \multicolumn{1}{c}{} & \\ \cline{1-2} V_{d+1}(\overline{X}) & \begin{matrix}1\\ \vdots \\ x^d\end{matrix} & \scriptstyle d+1 & \\ \cline{1-2} U_{\ell-d+1}(\overline{X},Y) & \mathbf{0} & \scriptstyle \ell-d+1 & \\ \cline{1-2} \end{array} \in K[x]$$

is not identically zero. Then a solution for the osculatory rational interpolation problem (2) *exists if and only if for*

$$B := \det \begin{array}{c|c|c|l} \multicolumn{1}{c}{\scriptstyle \ell+1} & \multicolumn{1}{c}{\scriptstyle 1} & \multicolumn{1}{c}{} & \\ \cline{1-2} V_{d+1}(\overline{X}) & \mathbf{0} & \scriptstyle d+1 & \\ \cline{1-2} U_{\ell-d+1}(\overline{X},Y) & \begin{matrix}1\\ \vdots \\ x^{\ell-d}\end{matrix} & \scriptstyle \ell-d+1 & \\ \cline{1-2} \end{array} \in K[x],$$

one has $B(x_i) \neq 0$ for all $0 \leq i \leq k$. In that case the (essentially unique) solution is given by A/B.

2. Proof of the results

Let us start by showing that a solution for the rational interpolation problem is essentially unique.

Proposition 2.1. *If the osculatory rational interpolation problem* (2) *has a solution, then there is a unique solution A/B with* $\gcd(A, B) = 1$ *and A monic.*

Proof. If there is a solution, then, cleaning common factors and dividing by the leading coefficient of A, there is a solution satisfying the same degree bounds with $\gcd(A, B) = 1$ and A monic. Assume A_1/B_1 and A_2/B_2 are both solutions of the same type. Then, $(A_1/B_1)^{(j)}(x_i) = (A_2/B_2)^{(j)}(x_i)$ implies

$$(A_1B_2 - A_2B_1/B_1B_2)^{(j)}(x_i) = 0 \text{ for } \ 0 \leq i \leq k,\, 0 \leq j < a_i,$$

which inductively implies that $(A_1B_2 - A_2B_1)^{(j)}(x_i) = 0$ for the $\ell + 1$ conditions. But $A_1B_2 - A_2B_1$ is a polynomial of degree at most ℓ, and therefore $A_1B_2 = A_2B_1$. Therefore, $A_1 = cA_2$ and $B_1 = cB_2$ with $c \in K \setminus \{0\}$, as both A_1 and A_2 are monic, then $c = 1$ and the claim follows. □

To prove our results, we first connect the solutions of the osculatory rational interpolation problem (and the Cauchy interpolation problem as a special case) to subresultants of the following two polynomials:

- $f := \prod_{j=0}^{k}(x - x_j)^{a_j}$, which we write $f = \sum_{i=0}^{\ell+1} f_i x^i$, where $f_{\ell+1} = 1$.

- $g \in K[x]$ the unique polynomial of degree less than or equal to ℓ such that
$$g^{(j)}(x_i) = j!\, y_{i,j},\ 0 \le i \le k,\ \ 0 \le j < a_i,$$
which we expand as $g = \sum_{i=0}^{\ell} g_i x^i$, where $g_i = 0$ for $\deg(g) < i \le \ell$. This polynomial is the so-called *Hermite interpolation polynomial* associated to the data
$$\{(x_0, a_0), \dots, (x_k, a_k)\},\ \{y_{i,j}\}_{0 \le i \le k,\, 0 \le j < a_i}.$$
By assumption $g \ne 0$ since we assumed that some of the $y_{i,j}$ is non-zero.

For $d \le \ell$, consider the d-th subresultant polynomial $\mathrm{Sres}_d(f, g)$ of f and g, defined as

$$\mathrm{Sres}_d(f,g) := \det \overbrace{\begin{array}{|cccccc|}\hline f_{\ell+1} & \cdots & & \cdots & f_{d+1-(\ell-d-1)} & x^{\ell-d-1}f(x) \\ & \ddots & & & \vdots & \vdots \\ & & f_{\ell+1} & \cdots & f_{d+1} & x^0 f(x) \\ \hline g_\ell & \cdots & & \cdots & g_{d+1-(\ell-d)} & x^{\ell-d}g(x) \\ & \ddots & & & \vdots & \vdots \\ & & g_\ell & \cdots & g_{d+1} & x^0 g(x) \\ \hline \end{array}}^{2\ell+1-2d} \begin{array}{l} \\ \ell-d \\ \\ \\ \ell+1-d \\ \\ \end{array} .$$

Note that the previous definition makes sense even if $\deg(g) = m < \ell$, and agrees for $d \le m$ with the usual definition of subresultant of f and g given by the matrix of the right size $\ell + 1 + m - 2d$, since f is monic. For $m < d < \ell$ we have, according to the definition above, that $\mathrm{Sres}_d(f, g) = 0$, and for $d = \ell$, $\mathrm{Sres}_\ell(f, g) = g = \mathrm{Sres}_m(f, g)$.

We have the universal subresultant Bézout identity

$$\mathrm{Sres}_d(f, g) = F_d\, f + G_d\, g, \tag{3}$$

where

$$F_d := \det \overbrace{\begin{array}{|cccccc|}\hline f_{\ell+1} & \cdots & & \cdots & f_{d+1-(\ell-d-1)} & x^{\ell-d-1} \\ & \ddots & & & \vdots & \vdots \\ & & f_{\ell+1} & \cdots & f_{d+1} & x^0 \\ \hline g_\ell & \cdots & & \cdots & g_{d+1-(\ell-d)} & 0 \\ & \ddots & & & \vdots & \vdots \\ & & g_\ell & \cdots & g_{d+1} & 0 \\ \hline \end{array}}^{2\ell+1-2d} \begin{array}{l} \\ \ell-d \\ \\ \\ \ell+1-d \\ \\ \end{array} ,$$

and

$$G_d := \det \overbrace{\begin{array}{|cccccc|}\hline f_{\ell+1} & \cdots & & \cdots & f_{d+1-(\ell-d-1)} & 0 \\ & \ddots & & & \vdots & \vdots \\ & & f_{\ell+1} & \cdots & f_{d+1} & 0 \\ \hline g_\ell & \cdots & & \cdots & g_{d+1-(\ell-d)} & x^{\ell-d} \\ & \ddots & & & \vdots & \vdots \\ & & g_\ell & \cdots & g_{d+1} & x^0 \\ \hline \end{array}}^{2\ell+1-2d} \begin{array}{l} \\ \ell-d \\ \\ \\ \ell+1-d \\ \\ \end{array} .$$

Observe that $\deg(G_d) \leq \ell - d$ when $G_d \neq 0$.

The result below, strongly related to [vzGG2003, Theorem 5.16], expresses the existence and uniqueness of the solution of the osculatory rational interpolation problem in terms of the subresultant sequence of f and g.

Proposition 2.2. *Let (a, b) be given, and set $d \leq a$ to be maximal such that $\mathrm{Sres}_d(f, g) \neq 0$. Then the osculatory rational interpolation problem* (2) *has a solution if and only if $G_d(x_i) \neq 0$ for all $0 \leq i \leq k$. In this case, the (essentially unique) solution is given by*

$$\frac{A}{B} = \frac{Sres_d(f, g)}{G_d},$$

and moreover $\gcd\big(\mathrm{Sres}_d(f, g), G_d\big) = 1$.

Proof. We use [vzGG2003, Th. 5.16]: Problem (2) has a solution if and only if the minimal row $r_j = s_j f + t_j g$ in the Extended Euclidean Algorithm such that $d_j := \deg(r_j) \leq a$ satisfies that $\gcd(r_j, t_j) = 1$, in which case r_j/t_j is the canonical solution. Note that for $d_i := \deg(r_i)$, we have $d_j \leq a < d_{j-1}$ by the definition of r_j.
Then, by the Fundamental Theorem of PRS (c.f. [Coll1967, BT1971] or [GCL1996, Th.7.4]), $\mathrm{Sres}_{d_j}(f, g)$ and $\mathrm{Sres}_{d_{j-1}-1}(f, g)$ are (non-zero) constant multiples of r_j, and $\mathrm{Sres}_{d'}(f, g) = 0$ for $d_j < d' < d_{j-1} - 1$. In any case, if $d \leq a$ is the largest such that $\mathrm{Res}_d(f, g) \neq 0$, one has $\mathrm{Res}_d(f, g) = c\, r_j$ for some $c \neq 0$ in K.
But $\mathrm{Res}_d(f, g) = F_d f + G_d g$ with $\deg(\mathrm{Res}_d(f, g)) + \deg(G_d) < \deg(f)$ implies by [vzGG2003, Lemma 5.15] that $F_d = c\, s_j$ and $G_d = c\, t_j$ too. Moreover, by [vzGG2003, Lemma 3.15 (v)], $\gcd(F_d, G_d) = 1$, which implies that $\gcd\big(\mathrm{Sres}_d(f, g), G_d\big) = 1 \;\Leftrightarrow\; G_d(x_i) \neq 0$ for all $0 \leq i \leq k$, by the definition of f. Therefore, there is a solution to the problem if and only if $G_d(x_i) \neq 0$, for all $0 \leq i \leq k$, and the essentially unique solution is given by $\mathrm{Sres}_d(f, g)/G_d$. □

Proposition 2.2 has the advantage that it can be applied to produce explicit formulae for the rational interpolation problem in terms of the input data. We first prove our result on the Cauchy interpolation problem, when f has simple roots.

Proof of Theorem 1.1. Set $X = \{x_0, \ldots, x_\ell\}$ for the set of roots of $f = \prod_{i=0}^{\ell}(x - x_i)$ and Z for the set of m roots of g, the unique polynomial of degree bounded by ℓ which satisfies $g(x_i) = y_i$ for $0 \leq i \leq \ell$.
Using Proposition 2.2 above, it is sufficient to prove that $Sres_d(f, g) = A$ and $G_d = B$. We use Sylvester's single-sum formula in roots for Sres_d (see for instance the original paper of Sylvester [Syl1853, Art. 21] or the many other references on the topic), and also for G_d ([Syl1853, Art. 29], or [KS2012], Remark after Lemma 6):

$$\begin{aligned}
\mathrm{Sres}_d(f,g) &= \sum_{|X'|=d} R(x,X')\,\frac{R(X\setminus X',Z)}{R(X\setminus X',X')},\\
&= \sum_{|X'|=d} R(x,X')\,\frac{\prod_{x_i\notin X'} g(x_i)}{R(X\setminus X',X')},\\
G_d &= (-1)^{\ell-d}\sum_{|X''|=\ell-d} R(x,X'')\,\frac{R(X'',Z)}{R(X'',X\setminus X'')}\\
&= \sum_{|X''|=\ell-d} R(X'',x)\,\frac{R(X'',Z)}{R(X'',X\setminus X'')}\\
&= \sum_{|X''|=\ell-d} R(X'',x)\frac{\prod_{x_i\in X''} g(x_i)}{R(X'',X\setminus X'')}.
\end{aligned}$$

where both X', $X''\subset X$. We get that $A = Sres_d(f,g)$ and $B = G_d$ if we substitute $g(x_i) = y_i$ for all i above. □

Observe that in the previous theorem, if $a = \ell$ then $\mathrm{Sres}_\ell(f,g) = g \neq 0$ and the solution is given by the Lagrange interpolation polynomial, as mentioned in the introduction.

To prove Theorem 1.3 for the osculatory rational interpolation problem we need the following notation, that we already used in [DKS2012].

Notation 2.3. *Recall* $\overline{X} = \big((x_0,a_0);\ldots;(x_k,a_k)\big)$. *Given a polynomial* $h(z)$ *and* $u\in\mathbb{N}$, *the* generalized Wronskian *of size* $u+1$ *associated to* $\overline{X}$ *is the (non-necessarily square) matrix* $W_{h,u+1}(\overline{X})\in K^{(u+1)\times(\ell+1)}$ *defined by*

$$W_{h,u+1}(\overline{X}) := \overbrace{\boxed{\begin{array}{c|c|c} W_{h,u+1}(x_0,a_0) & \ldots & W_{h,u+1}(x_k,a_k)\end{array}}}^{\ell+1}\ {\scriptstyle u+1}\ ,$$

where for any t, $W_{h,u+1}(x_i,t+1)\in K^{(u+1)\times(t+1)}$ *is defined by*

$$W_{h,u+1}(x_i,t+1) := \overbrace{\boxed{\begin{array}{cccc} h(x_i) & h'(x_i) & \ldots & \frac{h^{(t)}(x_i)}{t!}\\ (zh)(x_i) & (zh)'(x_i) & \ldots & \frac{(zh)^{(t)}(x_i)}{t!}\\ \vdots & \vdots & & \vdots\\ (z^uh)(x_i) & (z^uh)'(x_i) & \ldots & \frac{(z^uh)^{(t)}(x_i)}{t!}\end{array}}}^{t+1}\ {\scriptstyle u+1}\ ,$$

with the convention that when $k<j$, $\binom{k}{j}x_i^{k-j} = 0$.

First we prove the following lemma:

Lemma 2.4. *Using the notation above, if there is a solution for the osculatory rational interpolation problem, then the essentially unique solution is* *given by*

$$\frac{A}{B} = -\frac{\det \begin{array}{|c|c|}\multicolumn{1}{c}{\scriptstyle \ell+1} & \multicolumn{1}{c}{\scriptstyle 1} \\ \hline V_{d+1}(\overline{X}) & \begin{matrix}1\\ \vdots \\ x^d\end{matrix} \\ \hline W_{g,\ell-d+1}(\overline{X}) & \mathbf{0} \\ \hline \end{array} \begin{matrix}\scriptstyle d+1 \\ \\ \scriptstyle \ell-d+1\end{matrix}}{\det \begin{array}{|c|c|}\multicolumn{1}{c}{\scriptstyle \ell+1} & \multicolumn{1}{c}{\scriptstyle 1} \\ \hline V_{d+1}(\overline{X}) & \mathbf{0} \\ \hline W_{g,\ell-d+1}(\overline{X}) & \begin{matrix}1\\ \vdots \\ x^{\ell-d}\end{matrix} \\ \hline \end{array} \begin{matrix}\scriptstyle d+1 \\ \\ \scriptstyle \ell-d+1\end{matrix}}$$

where $d \le a$ is maximum such that the numerator does not vanish.

Proof. We apply Proposition 2.2 for the maximum $d \le a$ such that $\mathrm{Sres}_d(f,g) \neq 0$. In [DKS2012, Theorem 2.5] we proved that

$$\mathrm{Sres}_d(f,g) = (-1)^{\ell+1-d} \det\big(V_{\ell+1}(\overline{X})\big)^{-1} \det \begin{array}{|c|c|}\multicolumn{1}{c}{\scriptstyle \ell+1} & \multicolumn{1}{c}{\scriptstyle 1} \\ \hline V_{d+1}(\overline{X}) & \begin{matrix}1\\ \vdots \\ x^d\end{matrix} \\ \hline W_{g,\ell-d+1}(\overline{X}) & \mathbf{0} \\ \hline \end{array} \begin{matrix}\scriptstyle d+1 \\ \\ \scriptstyle \ell-d+1\end{matrix} .$$

To prove a similar statement about G_d consider the following matrices:

$$M_f := \begin{array}{|cccccc|c|}\multicolumn{6}{c}{\scriptstyle 2\ell-d+2} & \multicolumn{1}{c}{} \\ \hline f_0 & \dots & f_{\ell+1} & & & & 0 \\ & \ddots & & & \ddots & & \vdots \\ & & f_0 & & \dots & f_{\ell+1} & 0 \\ \hline \end{array} \ {\scriptstyle \ell-d}\ , \quad M_g := \begin{array}{|cccccc|c|}\multicolumn{6}{c}{\scriptstyle 2\ell-d+2} & \multicolumn{1}{c}{} \\ \hline g_0 & \dots & g_{\ell} & & & & x^0 \\ & \ddots & & & \ddots & & \vdots \\ & & g_0 & & \dots & g_{\ell} & x^{\ell-d} \\ \hline \end{array} \ {\scriptstyle \ell+1-d}$$

and

$$U_d := \begin{array}{|c|}\multicolumn{1}{c}{\scriptstyle 2\ell-d+2} \\ \hline I_{d+1} \\ \hline M_f \\ \hline M_g \\ \hline \end{array} \begin{matrix}\scriptstyle d+1 \\ \scriptstyle \ell-d \\ \scriptstyle \ell+1-d\end{matrix} ,$$

where I_{d+1} is the $(d+1) \times (2\ell - d + 2)$ matrix with the identity matrix on the left and zero otherwise. Then from the definition of G_d we have that

$$G_d = \det(U_d).$$

Also, similarly as in [DKS2012, Theorem 2.5], we have

$$
\begin{array}{r}
\scriptstyle d+1\\ \scriptstyle \ell-d\\ \scriptstyle \ell+1-d
\end{array}
\overset{\scriptstyle 2\ell-d+2}{\begin{array}{|c|}\hline I_{d+1}\\ \hline M_f\\ \hline M_g\\ \hline\end{array}}
\;
\begin{array}{|c|c|}
\multicolumn{1}{c}{\scriptstyle \ell+1} & \multicolumn{1}{c}{\scriptstyle \ell-d+1}\\ \hline
V_{2\ell-d+1}(\overline{X}) & \mathbf{0}\\ \hline
0 & \mathrm{Id}_{\ell-d+1}\\ \hline
\end{array}
\begin{array}{l}
\\ \scriptstyle \ell+1\\ \scriptstyle \ell-d+1
\end{array}
=
\begin{array}{|c|c|c|}
\multicolumn{1}{c}{\scriptstyle \ell+1} & \multicolumn{1}{c}{\scriptstyle \ell-d} & \multicolumn{1}{c}{\scriptstyle 1}\\ \hline
V_{d+1}(\overline{X}) & * & 0\\ \hline
\mathbf{0} & M'_f & 0\\ \hline
W_{g,\ell+1-d}(\overline{X}) & * & \begin{matrix}1\\ \vdots\\ x^{\ell-d}\end{matrix}\\ \hline
\end{array}
\begin{array}{l}
\\ \scriptstyle d+1\\ \scriptstyle \ell-d\\ \scriptstyle \ell+1-d
\end{array},
$$

where M'_f is a triangular matrix with $f_{\ell+1}=1$ in its diagonal. This implies that

$$
G_d = (-1)^{\ell-d}\det\left(V_{\ell+1}(\overline{X})\right)^{-1}\det
\begin{array}{|c|c|}
\multicolumn{1}{c}{\scriptstyle \ell+1} & \multicolumn{1}{c}{\scriptstyle 1}\\ \hline
V_{d+1}(\overline{X}) & \mathbf{0}\\ \hline
W_{g,\ell-d+1}(\overline{X}) & \begin{matrix}1\\ \vdots\\ x^{\ell-d}\end{matrix}\\ \hline
\end{array}
\begin{array}{l}
\\ \scriptstyle d+1\\ \scriptstyle \ell+1-d
\end{array}.
$$

□

Now we can easily prove Theorem 1.3:

Proof of Theorem 1.3. We simply compute in the previous lemma the entries of the matrix $W_{g,\ell-d+1}(\overline{X})$, applying Leibniz rule and the fact that $g^{(t-j)}(x_i) = (t-j)!\,y_{i,j}$ for $0 \le i \le k$ and $0 \le j < a_i$:

$$\frac{(z^u g)^{(t)}(x_i)}{t!} = \sum_{j=0}^{t} \binom{u}{j} x_i^{u-j} y_{i,t-j}.$$

□

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