Sylvester's Double Sums: the general case

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Abstract

In 1853 Sylvester introduced a *family* of double sum expressions for two finite sets of indeterminates and showed that some members of the family are essentially the polynomial subresultants of the monic polynomials associated with these sets. A question naturally arises: What are the *other* members of the family? This paper provides a complete answer to this question. The technique that we developed to answer the question turns out to be general enough to characterise *all* members of the family, providing a uniform method.

Key words: Subresultants, double sums, Vandermonde determinants.

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1. Introduction

Let A and B be finite lists (ordered sets) of distinct indeterminates. In Sylvester (1853), Sylvester introduced for each $0 \le p \le |A|, 0 \le q \le |B|$ the following *double-sum* expression in A and B:

$$\mathrm{Sylv}^{p,q}(A,B;x) := \sum_{\substack{A' \subset A,\, B' \subset B \\ |A'|=p,\, |B'|=q}} R(x,A')\,R(x,B')\,\frac{R(A',B')\,R(A\backslash A',B\backslash B')}{R(A',A\backslash A')\,R(B',B\backslash B')},$$

where

$$R(Y,Z) := \prod_{y\in Y, z\in Z}(y-z), \qquad R(y,Z) := \prod_{z\in Z}(y-Z).$$

Let now f, g be univariate polynomials such that

$$\begin{aligned} f &:= \textstyle\prod_{\alpha\in A}(x-\alpha) = x^m + a_{m-1}x^{m-1} + \ldots + a_0 \\ g &:= \textstyle\prod_{\beta\in B}(x-\beta) = x^n + b_{n-1}x^{n-1} + \ldots + b_0, \end{aligned}$$

where $m := |A| \ge 1$ and $n := |B| \ge 1$.

Since the double sum expressions are polynomials in x and symmetric in the α's and β's, they can be expressed as polynomials in x whose coefficients are rational functions in the a_i's and the b_j's. In Sylvester (1853), the rational expression for $\mathrm{Sylv}^{p,q}(A,B;x)$ is determined for the following values of (p,q), where without loss of of generality we assume $1 \le m \le n$ and we set $d := p+q$ (see also Lascoux and Pragacz (2003)):

(1) If $0 \le d < m \le n$, then

$$\mathrm{Sylv}^{p,q}(A,B;x) = (-1)^{p(m-d)}\binom{d}{p}\mathrm{Sres}_d(f,g),$$

where $\mathrm{Sres}_d(f,g)$ is the d-th subresultant of the polynomials f and g, whose definition is recalled in Formula (1) below (cf. (Sylvester, 1853, Art. 21) and also (Lascoux and Pragacz, 2003, Theorem 0.1)).

(1') If $m = d < n$, then

$$\mathrm{Sylv}^{p,q}(A,B;x) = \binom{m}{p}f(x)$$

(cf. (Sylvester, 1853, Art. 21) and also (Lascoux and Pragacz, 2003, Proposition 2.9 (i))). In fact, the d-th subresultant is also well defined for $d = m < n$ as $\mathrm{Sres}_m(f,g) = f$. This implies that Case (1') can be seen as a special case of Case (1).

(2) If $m < d < n-1$, then $\mathrm{Sylv}^{p,q}(A,B;x) = 0$ (cf. (Sylvester, 1853, Arts. 23 & 24)).

(3) If $m < d = n-1$, then $\mathrm{Sylv}^{p,q}(A,B;x)$ is a "numerical multiplier" of $f(x)$ (cf. (Sylvester, 1853, Art. 25)), but the ratio is not established.

(4) If $m = d = n$, then

$$\mathrm{Sylv}^{p,q}(A,B;x) = \binom{m-1}{q}f(x) + \binom{m-1}{p}g(x)$$

(cf.(Sylvester, 1853, Art. 22) and also (Lascoux and Pragacz, 2003, Proposition 2.9 (ii))).

This note provides two contributions to this subject:

- **Completion**: Note that the above cases do not completely cover all possible values of p and q such that $0 \le p \le m$ and $0 \le q \le n$: values when $n \le p+q \le m+n$ (except if $p+q=m=n$) are not covered. In Main Theorem below we provide expressions for *all* the possible values of p and q, finally completing the previous efforts.
- **Uniformity**: Sylvester (1853) and Lascoux and Pragacz (2003) gave different proofs for each of the cases listed above. In Section 2, we provide a *uniform* technique that can be applied to *all* the possible cases. We obtained this technique by generalizing the matrix formulation, used in D'Andrea et al. (2007) for dealing with the cases (1) and (2), into a "global" matrix formulation. Approaches to double sum expressions via matrix constructions have already been used in Borchardt (1860, 1878) (see also Apéry and Jouanolou (2006)).

In order to state our main result we recall that for $0 \le k \le m < n$ or $0 \le k < m = n$, the k-th subresultant of the polynomials f and g is defined as

$$\operatorname{Sres}_k(f,g) := \det \overbrace{\left|\begin{array}{cccc} a_m & \cdots \quad \cdots & a_{k+1-(n-k-1)} & x^{n-k-1}f(x) \\ & \ddots & \vdots & \vdots \\ & a_m \quad \cdots & a_{k+1} & x^0 f(x) \\ \hline b_n & \cdots \quad \cdots & b_{k+1-(m-k-1)} & x^{m-k-1}g(x) \\ & \ddots & \vdots & \vdots \\ & b_n \quad \cdots & b_{k+1} & x^0 g(x) \end{array}\right|}^{m+n-2k} \begin{array}{l} \\ n-k \\ \\ \\ m-k \\ \\ \end{array} \tag{1}$$

with $a_\ell = b_\ell = 0$ for $\ell < 0$.

Expanding the determinant by the last column gives an expression

$$\operatorname{Sres}_k(f,g) = F_k(x)f(x) + G_k(x)g(x) \tag{2}$$

where the polynomials F_k and G_k (satisfying $\deg F_k \le n-k-1$ and $\deg G_k \le m-k-1$) are given by:

$$F_k := \det\left|\begin{array}{cccc} a_m & \cdots \quad \cdots & a_{k+1-(n-k-1)} & x^{n-k-1} \\ & \ddots & \vdots & \vdots \\ & a_m \quad \cdots & a_{k+1} & x^0 \\ \hline b_n & \cdots \quad \cdots & b_{k+1-(m-k-1)} & 0 \\ & \ddots & \vdots & \vdots \\ & b_n \quad \cdots & b_{k+1} & 0 \end{array}\right|, \quad G_k := \det\left|\begin{array}{cccc} a_m & \cdots \quad \cdots & a_{k+1-(n-k-1)} & 0 \\ & \ddots & \vdots & \vdots \\ & a_m \quad \cdots & a_{k+1} & 0 \\ \hline b_n & \cdots \quad \cdots & b_{k+1-(m-k-1)} & x^{m-k-1} \\ & \ddots & \vdots & \vdots \\ & b_n \quad \cdots & b_{k+1} & x^0 \end{array}\right|.$$

Now we are ready to state the main result that will be proven in the next section.

Main Theorem. *Let* $1 \le m \le n$, $0 \le p \le m$, $0 \le q \le n$, *and set* $d := p+q$, $k := m+n-d-1$, $\sigma := q(m-p)+n(d-m)+d+n-q-1$. *Then*

$$\mathrm{Sylv}^{p,q}(A,B;x) = \begin{cases} (-1)^{p(m-d)}\binom{d}{p} Sres_d(f,g) & \text{for } 0 \le d < m \text{ or } m = d < n \\ 0 & \text{for } m < d < n-1 \\ (-1)^{(m+q)(p+1)}\binom{m}{p} f & \text{for } m < d = n-1 \\ (-1)^{\sigma}\Big(\binom{k}{m-p}F_k\ f - \binom{k}{n-q}G_k\ g\Big) & \text{for } n \le d \le m+n-1 \\ \mathrm{Res}(f,g)\ fg & \text{for } d = m+n. \end{cases}$$

We note that the previous Case (4) is covered here by the case $n \le d \le m+n-1$, where indeed, for $m = d = n$, $F_{m-1} = -1$ and $G_{m-1} = 1$.

Finally, let us add two remarks kindly pointed out by one of the referees. First, there are some avenues for further extensions: such as more general types of summations considered by Sylvester himself (see for instance Sylvester (1973)), and the cases when $A \cap B \neq \emptyset$, or when A and/or B has repeated elements. Second, the results concerning Sylvester's sums can be viewed as generalizations of interpolation formulas (the $m = n-1$, $p = 0$, $q = m$ case giving the Lagrange interpolation formula), and thus an approach using specialization instead of linear algebra may also be possible for proving such equalities.

2. Proof

As in D'Andrea et al. (2007), we define for a polynomial $h(t)$, a finite list $\Gamma := (\gamma_1, \dots, \gamma_u)$ of scalars and a non-negative integer v the (non necessarily square) matrix of size $v \times u$:

$$\langle h(t), \Gamma\rangle_v := \overbrace{\begin{array}{|ccc|} \gamma_1^0 h(\gamma_1) & \dots & \gamma_u^0 h(\gamma_u) \\ \vdots & & \vdots \\ \gamma_1^{v-1} h(\gamma_1) & \dots & \gamma_u^{v-1} h(\gamma_u) \end{array}}^{u}{}_{v}\,.$$

For instance, under this notation,

$$\langle x-t, \Gamma\rangle_v = (\gamma_j^{i-1}x - \gamma_j^i)_{1\le i\le v, 1\le j\le u},$$

and for $v = u$ we have the following equality for the Vandermonde determinant $\mathcal{V}(\Gamma)$ associated to Γ:

$$\mathcal{V}(\Gamma) := \det\big(\gamma_j^{i-1}\big)_{1\le i,j\le u} = \det\big(\langle 1, \Gamma\rangle_u\big).$$

For the rest of the paper, $d \in \mathbb{N}$, $0 \le d \le m+n$ and $d' := m+n-d$. We take a new variable T

and we denote by $U_d(x,T)$ the following square matrix of size $m+n=d'+d$:

$$U_d(x,T) := \begin{array}{|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \\ \hline \langle 1,B\rangle_{d'} & \langle T,A\rangle_{d'} & d' \\ \hline \langle x-t,B\rangle_d & \langle x-t,A\rangle_d & d \\ \hline \end{array},$$

where $\langle T,A\rangle_{d'} = \left(T\alpha^j\right)_{\alpha\in A,\,0\le j\le d'-1}$. Finally we denote by $u_d(x,T)$ its determinant, that we develop in the powers of T:

$$u_d(x,T) := \det\left(U_d(x,T)\right) = u_{d,0}(x)T^m + \cdots + u_{d,m-1}(x)T + u_{d,m}(x). \tag{3}$$

We are now ready to state our first result, that relates $\mathrm{Sylv}^{p,d-p}(A,B;x)$ to the coefficient $u_{d,p}(x)$:

Theorem 1. *Let $0\le d\le m+n$, $0\le p\le m$ and define $q := d-p$. Following Notation (3), we have that if $0\le q\le n$ then*

$$u_{d,p}(x) = (-1)^{q(m-p)}\mathcal{V}(A)\,\mathcal{V}(B)\ \mathrm{Sylv}^{p,q}(A,B;x)$$

while otherwise $u_{d,p}(x)=0$.

Proof. For any set $A'\subset A$ (resp. $B'\subset B$) we will denote with A'' (resp. B'') its complementary set, i.e. $A'' := A\setminus A'$ (resp. $B'' := B\setminus B'$).

We perform a Laplace expansion of the determinant of the matrix $U_d(x,T)$ on the last d rows and we get the following expression:

$$u_d(x,T) = \sum_{\substack{A'\subset A,\, B'\subset B\\ |A'|+|B'|=d}} \sigma(B''\cup A'', B\cup A)T^{m-|A'|}\mathcal{V}(B''\cup A'')R(x,B')R(x,A')\mathcal{V}(B'\cup A'),$$

where, as in D'Andrea et al. (2007), "$\cup$" stands for list concatenation, "$\backslash$" means list subtraction and, for $S\subseteq T$ finite lists, $\sigma(S,T) := (-1)^j$, j being the number of transpositions needed to take T to $S\cup(T\backslash S)$.

We write $u_d(x,T)$ in powers of T, with $0\le p\le m$ and $0\le q := d-p\le n$ implying $\max\{0,d-n\}\le p\le\min\{d,m\}$:

$$u_d(x,T) = \sum_{p=\max\{0,d-n\}}^{\min\{d,m\}}\left(\sum_{\substack{A'\subset A,\,B'\subset B\\ |A'|=p,\,|B'|=q}} \sigma(B''\cup A'', B\cup A)R(x,B')R(x,A')\mathcal{V}(B'\cup A')\mathcal{V}(B''\cup A'')\right)T^{m-p}.$$

We recall the elementary fact that transposing a block of j columns with an adjacent block of i columns produces in the determinant a change of sign of order $(-1)^{ij}$. Hence, for $|A'|=p$ and $|B'|=q$,

$$\sigma(B''\cup A'', B\cup A) = \sigma(A'',A)\sigma(B'',B)(-1)^{q(m-p)},$$

and we have for $\max\{0, d-n\} \leq p \leq \min\{d, m\}$:

$$u_{d,p}(x) = (-1)^{q(m-p)} \sum_{\substack{A' \subset A,\, B' \subset B \\ |A'| = p,\ |B'| = q}} \sigma(A'', A)\sigma(B'', B)R(x, A')R(x, B')\mathcal{V}(B' \cup A')\mathcal{V}(B'' \cup A'').$$

Now we apply repeatedly the elementary fact that

$$\mathcal{V}(X \cup Y) = \mathcal{V}(X)\, \mathcal{V}(Y)\, R(Y, X)$$

for any pair of finite lists X, Y:

$$\begin{aligned}
\mathcal{V}(B' \cup A')\, \mathcal{V}(B'' \cup A'') &= \mathcal{V}(A')\mathcal{V}(B')R(A', B')\mathcal{V}(A'')\mathcal{V}(B'')R(A'', B'') \\
&= \frac{\mathcal{V}(A'' \cup A')}{R(A', A'')} \frac{\mathcal{V}(B'' \cup B')}{R(B', B'')} R(A', B')R(A'', B'') \\
&= \sigma(A'', A)\sigma(B'', B)\mathcal{V}(A)\mathcal{V}(B) \frac{R(A', B')R(A'', B'')}{R(A', A'')R(B', B'')}.
\end{aligned}$$

We finally obtain that

$$\begin{aligned}
u_{d,p}(x) &= (-1)^{q(m-p)}\mathcal{V}(A)\mathcal{V}(B) \sum_{\substack{A' \subset A,\, B' \subset B \\ |A'| = p,\ |B'| = q}} R(x, A')R(x, B') \frac{R(A', B')R(A'', B'')}{R(A', A'')R(B', B'')} \\
&= (-1)^{q(m-p)}\mathcal{V}(A)\mathcal{V}(B)\, \mathrm{Sylv}^{p,q}(A, B; x).
\end{aligned}$$

□

In view of Theorem 1, in order to produce a rational expression for $\mathrm{Sylv}^{p,q}(A, B; x)$ it is enough to give a rational expression for $u_{d,p}(x)$. To this aim we first observe the following straightforward factorization formula for $U_d(x, T)$ as a product of two rectangular matrices of sizes $(m+n) \times (m+n+1)$ and $(m+n+1) \times (m+n)$ respectively.

Lemma 2.

$$U_d(x,T) = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle d'} & \multicolumn{1}{c}{\scriptstyle d+1} \\ \cline{2-3} \scriptstyle d' & \mathbb{I}_{d'} & \mathbf{0} \\ \cline{2-3} \scriptstyle d & \mathbf{0} & \begin{matrix} x & -1 & 0 & \dots & 0 & 0 \\ 0 & x & -1 & \dots & 0 & 0 \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & x & -1 \end{matrix} \\ \cline{2-3} \end{array} \; \begin{array}{|c|c|c} \multicolumn{1}{c}{\scriptstyle n} & \multicolumn{1}{c}{\scriptstyle m} & \\ \hline \langle 1, B\rangle_{d'} & \langle T, A\rangle_{d'} & \scriptstyle d' \\ \hline \langle 1, B\rangle_{d+1} & \langle 1, A\rangle_{d+1} & \scriptstyle d+1 \\ \hline \end{array}, \tag{4}$$

where $\mathbb{I}_{d'}$ denotes the identity matrix of size d'.

The previous factorization of $U_d(x,T)$ immediately yields

Proposition 3 (Arts. 23 & 24 Sylvester (1853))**.** *If $m < d < n-1$, then $u_d(x,T) = 0$.*

Proof. The assumption implies $\max\{d', d+1\} < n$. Then the first n columns of the matrix at the right of (4) have deficient rank since all $n \times n$ minors vanish. A Binet-Cauchy expansion of $u_d(x,T)$ therefore implies that $u_d(x,T)$ vanishes as well. □

Our goal now is to provide a factorization like in (4), but with square matrices, that allows to recover $u_d(x,T)$.

Theorem 4. *Let $1 \le m \le n$. If $0 \le d \le m$ or $n-1 \le d \le m+n$, then there exist polynomials $P(x) := P_0 + \cdots + P_d x^d$ and $Q(x) := Q_0 + \ldots + Q_{d'-1}x^{d'-1}$ with $P \neq 0$, $k := \deg P \le d$ and $\deg Q \le d'-1$ if $d' \neq 0$ such that we have the following matrix identity:*

$$
\begin{array}{c|c|c|}
 & d' & d+1 \\
\hline
d' & \mathbb{I}_{d'} & \mathbf{0} \\
\hline
d & \mathbf{0} & \begin{matrix} x & -1 & & 0 \\ & \ddots & \ddots & \\ 0 & \ldots & x & -1 \end{matrix} \\
\hline
1 & Q_0 \ \ldots \ Q_{d'-1} & P_0 \ \ldots \ldots \ P_d \\
\hline
\end{array}
\
\begin{array}{|c|c|c|c}
\multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\
\hline
\langle 1,B\rangle_{d'} & \langle T,A\rangle_{d'} & \mathbf{0} & d' \\
\hline
\langle 1,B\rangle_{d+1} & \langle 1,A\rangle_{d+1} & \mathbf{e}_k & d+1 \\
\hline
\end{array}
=
\begin{array}{|c|c|c}
\multicolumn{1}{c}{m+n} & \multicolumn{1}{c}{1} & \\
\hline
U_d(x,T) & * & m+n \\
\hline
\mathbf{0} & P_k & 1 \\
\hline
\end{array},
$$

where $\mathbf{e}_k$ is defined as the vertical vector of size $d+1$ with a single non-zero entry 1 in position $k+1$ and P_k is the leading coefficient of P.

Moreover, $P(x)$ can be defined as

$$
P(x) = \begin{cases}
\mathrm{Sres}_d(f,g) & \text{for} \quad 0 \le d < m \quad \text{or} \quad d = m < n \\
f & \text{for} \quad m < d = n-1 \\
F_{d'-1}f + TG_{d'-1}g & \text{for} \quad n \le d < m+n \\
fg & \text{for} \quad d = m+n,
\end{cases}
$$

where $F_{d'-1}, G_{d'-1}$ are as in Identity (2) for $k = d'-1$.

Remark 5. We note that $P(x) = F_{d'-1}f + TG_{d'-1}g$ is the determinant of a matrix similar to the matrix (1) that defines $\mathrm{Sres}_{d'}(f,g)$: we simply need to replace $g(x)$ by $Tg(x)$ in the last column of the matrix (1).

Proof. To get the factorization stated in Theorem 4, we only need to look at the equations that can be read from the lower row of the matrix on the right. These are

$$
\begin{aligned}
(Q_0 + \ldots + Q_{d'-1}\beta^{d'-1}) + (P_0 + \ldots + P_d\beta^d) &= 0 \\
(TQ_0 + \ldots + TQ_{d'-1}\alpha^{d'-1}) + (P_0 + \ldots + P_d\alpha^d) &= 0
\end{aligned}
$$

for all $\beta \in B$, $\alpha \in A$. In order to solve these equations, it is enough to produce polynomials $P(x) := P_0 + \cdots + P_d x^d$ and $Q(x) := Q_0 + \cdots + Q_{d'-1} x^{d'-1}$ with $P \neq 0$, $\deg P \leq d$ and $\deg Q \leq d'-1$ if $d' \neq 0$ such that the following $m+n$ equations are satisfied:

$$\begin{cases} Q(\beta) + P(\beta) = 0, & \forall \beta \in B \\ TQ(\alpha) + P(\alpha) = 0, & \forall \alpha \in A. \end{cases} \tag{5}$$

For $0 \leq d \leq m$ if $m < n$ and $0 \leq d < m$ if $m = n$, we define

$$\begin{cases} P(x) := \mathrm{Sres}_d(f,g) = F_d(x)f(x) + G_d(x)g(x) \\ Q(x) := -F_d(x)f(x) - \frac{1}{T} G_d(x)g(x) \end{cases}$$

where F_d, G_d are as in Identity (2) for $k := d$. Thus $\deg P = \deg \mathrm{Sres}_d(f,g) = d$ and $\deg_x Q \leq \max\{\deg(F_d f), \deg(G_d g)\} \leq d' - 1$. We look at Condition (5):

$$\begin{cases} Q(\beta) + P(\beta) = (1 - \frac{1}{T}) G_d(\beta) g(\beta) = 0, & \forall \beta \in B \\ TQ(\alpha) + P(\alpha) = (1 - T) F_d(\alpha) f(\alpha) = 0, & \forall \alpha \in A. \end{cases}$$

For $m < d = n - 1$, we define

$$P(x) := f(x) \quad \text{and} \quad Q(x) := -f(x).$$

We have $\deg P = m < d$ and $\deg Q = m = m + n - d - 1 = d' - 1$ in this case. Condition (5) is trivially satisfied.

For $n \leq d < m + n$, we observe that $0 \leq d' - 1 \leq m - 1$. Thus $\mathrm{Sres}_{d'-1}(f,g)$ is well defined and we define

$$\begin{cases} Q(x) := -\mathrm{Sres}_{d'-1}(f,g) = -F_{d'-1}(x)f(x) - G_{d'-1}(x)g(x) \\ P(x) := F_{d'-1}(x)f(x) + TG_{d'-1}(x)g(x) \end{cases}$$

where $F_{d'-1}, G_{d'-1}$ are as in Identity (2) for $k := d' - 1$. Thus $\deg Q = \deg \mathrm{Sres}_{d'-1}(f,g) = d' - 1$ and $\deg_x P \leq \max\{\deg(F_{d'-1} f), \deg(G_{d'-1} g)\} \leq \max\{m+n-(d'-1)-1, n+m-(d'-1)-1\} = d$. Also $P \neq 0$ since the leading terms can not cancel each other. We look again at Condition (5):

$$\begin{cases} Q(\beta) + P(\beta) = (T - 1) G_{d'-1}(\beta) g(\beta) = 0, & \forall \beta \in B \\ TQ(\alpha) + P(\alpha) = (1 - T) F_{d'-1}(\alpha) f(\alpha) = 0, & \forall \alpha \in A. \end{cases}$$

For $d = m + n$, since $d' = 0$ in this case, we define $P(x) = f(x)g(x)$, which is of degree d, to satisfy Condition (5). $\square$

Theorem 4 immediately implies that $u_d(x, T)$ can be computed as the determinant of two square matrices for the values of $d \leq m$ and $n - 1 \leq d$. Our next goal is to compute P_k in each case, as well as the determinants of these square matrices.
To this aim, for $0 \leq d \leq m < n$ or $0 \leq d < m = n$, we set $\Delta_k(f,g)$ for the leading coefficient of $\mathrm{Sres}_k(f,g)$, i.e. $\Delta_k(f,g)$ is the k-th scalar subresultant of f, g.

For $k = m = n$, we define for the coherence of the next results

$$\Delta_m(f,g) := 1.$$

Lemma 6. *Let $1 \leq m \leq n$. Following the notations of Theorem 4, we have*

$$\begin{cases} \deg P = d \quad \textit{and } P_k = \Delta_d(f,g) & \textit{for} \quad 0 \leq d \leq m < n \quad \textit{or} \quad d < m = n \\ \deg P = m \quad \textit{and } P_k = 1 & \textit{for} \quad m < d = n-1 \\ \deg P = d \quad \textit{and } P_k = (-1)^{d-n}\Delta_{d'}(f,g)(T-1) & \textit{for} \quad m \leq n \leq d < m+n \\ \deg P = d \quad \textit{and } P_k = 1 & \textit{for} \quad d = m+n. \end{cases}$$

Proof. The first two cases and the last case are straightforward from the definition of P_k.
For $m \leq n \leq d < m+n$, we have that $P(x) = F_{d'-1}(x)f(x) + TG_{d'-1}(x)g(x)$. Thus $\deg_x P = \max\{\deg_x(F_{d'-1}f), \deg_x(G_{d'-1}g)\}$ since the leading terms can not cancel each other. A direct computation on the matrix in (1) that defines $\mathrm{Sres}_k(f,g)$ shows that —since for $k := d'-1 < m$, $n-k > 1$ and $m-k > 1$ hold— then $\deg_x F_k = n-k-1 = d-m$ and $\deg_x G_k = m-k-1 = d-n$. Therefore $\deg_x P = \max\{m+n-(d'-1)-1, n+m-(d'-1)-1\} = d$.
Finally, since f and g are monic, the leading coefficient of $F_k(x)$ equals

$$(-1)^{m+n-2k+1}(-1)^{n-k+1}\Delta_{k+1}(f,g) = (-1)^{d-n+1}\Delta_{d'}(f,g)$$

and the leading coefficient of $G_k(x)$ equals

$$(-1)^{m+n-2k+n-k+1}\Delta_{k+1}(f,g) = (-1)^{d-n}\Delta_{d'}(f,g).$$

Therefore $P_d = (-1)^{d-n}\Delta_{d'}(f,g)(T-1)$. □

Lemma 7.

$$\det \begin{array}{c|c|c|l} \multicolumn{1}{c}{} & \multicolumn{1}{c}{d'} & \multicolumn{1}{c}{d+1} & \\ \cline{2-3} & \mathbb{I}_{d'} & \mathbf{0} & d' \\ \cline{2-3} & \mathbf{0} & \begin{matrix} x & -1 & & 0 \\ & \ddots & \ddots & \\ 0 & \dots & x & -1 \end{matrix} & d \\ \cline{2-3} & Q_0 \ \dots \ Q_{d'-1} & P_0 \ \dots \ \dots \ P_d & 1 \\ \cline{2-3} \end{array} = P_0 + \cdots + P_d x^d = P(x).$$

Proof. Because of the block triangular structure, this determinant equals

$$\det \begin{array}{|cccc|l} \multicolumn{4}{c}{d+1} \\ \hline x & -1 & & 0 & \\ & \ddots & \ddots & & d \\ 0 & \dots & x & -1 & \\ \hline P_0 & \dots & \dots & P_d & 1 \\ \hline \end{array} .$$

We can permute the first d-block with the last row and expand the determinant by this new first row. We get

$$(-1)^d \left(P_0(-1)^d - P_1 x(-1)^d + \ldots + (-1)^d P_d x^d \right).$$

□

Lemma 8. *Let $1 \le m \le n$. Then*

$$\det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \hline \langle 1, B\rangle_{d'} & \langle T, A\rangle_{d'} & \mathbf{0} & d' \\ \hline \langle 1, B\rangle_{d+1} & \langle 1, A\rangle_{d+1} & \mathbf{e}_d & d+1 \\ \hline \end{array} =$$

$$= \begin{cases} (-1)^{dm} \mathcal{V}(A)\, \mathcal{V}(B)\, \Delta_d(f,g)\, T^{m-d} (T-1)^d & \textit{for} \quad 0 \le d \le m \\ (-1)^{m(d-1)+d} \mathcal{V}(A) \mathcal{V}(B) (T-1)^m & \textit{for} \quad m < d = n-1 \\ (-1)^{d'n} \mathcal{V}(A)\, \mathcal{V}(B)\, \Delta_{d'}(f,g)\, (T-1)^{d'} & \textit{for} \quad n \le d < m+n \\ \mathcal{V}(A)\, \mathcal{V}(B)\, \mathrm{Res}(f,g) & \textit{for} \quad d = m+n. \end{cases}$$

Proof. First, let us recall (D'Andrea et al., 2007, Lemma 2):

$$\mathrm{Sres}_k(f,g) \mathcal{V}(A) = \det \begin{array}{|c|l} \multicolumn{1}{c}{m} & \\ \hline \langle x - t, A\rangle_k & k \\ \hline \langle g(t), A\rangle_{m-k} & m-k \\ \hline \end{array} ,$$

which implies that its leading coefficient satisfies

$$\Delta_k(f,g)\mathcal{V}(A) = \det \begin{array}{c} \scriptstyle m \\ \left[\begin{array}{c} \langle 1, A\rangle_k \\ \hline \langle g(t), A\rangle_{m-k} \end{array}\right] \end{array} \begin{array}{l} \scriptstyle k \\ \scriptstyle m-k \end{array}. \tag{6}$$

To simplify the notation of the proof, we will denote the matrix on the left side of the claim of the Lemma by M_d.
In case $0 \le d \le m$ or $n \le d < m+n$, $\deg P(x) = d$ by Lemma 6 and $\mathbf{e}_d := (0,\ldots,0,1)^t$. Therefore

$$\det(M_d) = \det \begin{array}{cc} \scriptstyle n & \scriptstyle m \\ \multicolumn{2}{c}{\left[\begin{array}{c|c} \langle 1, B\rangle_{d'} & \langle T, A\rangle_{d'} \\ \hline \langle 1, B\rangle_d & \langle 1, A\rangle_d \end{array}\right]} \end{array} \begin{array}{l} \scriptstyle d' \\ \scriptstyle d \end{array}.$$

For $d \le m$, we have that $d' \ge n \ge m \ge d$ holds and therefore row operations yield

$$\det(M_d) = \det \begin{array}{cc} \scriptstyle n & \scriptstyle m \\ \multicolumn{2}{c}{\left[\begin{array}{c|c} \langle 1, B\rangle_{d'} & \langle T, A\rangle_{d'} \\ \hline \mathbf{0} & \langle 1-T, A\rangle_d \end{array}\right]} \end{array} \begin{array}{l} \scriptstyle d' \\ \scriptstyle d \end{array}$$

$$= \det \begin{array}{cc} \scriptstyle n & \scriptstyle m \\ \multicolumn{2}{c}{\left[\begin{array}{c|c} \langle 1, B\rangle_n & \langle T, A\rangle_n \\ \hline \mathbf{0} & \langle Tg(t), A\rangle_{m-d} \\ \hline \mathbf{0} & \langle 1-T, A\rangle_d \end{array}\right]} \end{array} \begin{array}{l} \scriptstyle n \\ \scriptstyle m-d \\ \scriptstyle d \end{array} \quad \text{since for all } \beta \in B, g(\beta) = 0$$

$$= \mathcal{V}(B) T^{m-d}(1-T)^d \det \begin{array}{c} \scriptstyle m \\ \left[\begin{array}{c} \langle g(t), A\rangle_{m-d} \\ \hline \langle 1, A\rangle_d \end{array}\right] \end{array} \begin{array}{l} \scriptstyle m-d \\ \scriptstyle d \end{array}$$

$$\begin{aligned} &= \mathcal{V}(B)\, T^{m-d}(1-T)^d(-1)^{d(m-d)}\, \mathcal{V}(A)\, \Delta_d(f,g) \qquad \text{by (6)} \\ &= (-1)^{d(m-d+1)}\mathcal{V}(A)\, \mathcal{V}(B)\, \Delta_d(f,g)\, T^{m-d}(T-1)^d \\ &= (-1)^{dm}\mathcal{V}(A)\, \mathcal{V}(B)\, \Delta_d(f,g)\, T^{m-d}(T-1)^d. \end{aligned}$$

In case $d \ge n$, we have that $d' \le m \le d$ holds and therefore row operations yield

$$\det(M_d) = \det \begin{array}{|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \\ \cline{1-2} \mathbf{0} & \langle T-1, A\rangle_{d'} & d' \\ \cline{1-2} \langle 1, B\rangle_d & \langle 1, A\rangle_d & d \\ \cline{1-2} \end{array}$$

$$= \det \begin{array}{|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \\ \cline{1-2} \mathbf{0} & \langle T-1, A\rangle_{d'} & d' \\ \cline{1-2} \langle 1, B\rangle_n & \langle 1, A\rangle_n & n \\ \cline{1-2} \mathbf{0} & \langle g(t), A\rangle_{d-n} & d-n \\ \cline{1-2} \end{array}$$

$$= (-1)^{d'n} \mathcal{V}(B)\,(T-1)^{d'}\ \det \begin{array}{|c|l} \multicolumn{1}{c}{n} & \\ \cline{1-1} \langle 1, A\rangle_{d'} & d' \\ \cline{1-1} \langle g(t), A\rangle_{d-n} & d-n \\ \cline{1-1} \end{array}$$

$$= (-1)^{d'n} \mathcal{V}(A)\, \mathcal{V}(B)\, \Delta_{d'}(f,g)\,(T-1)^{d'}.$$

In case $m < d = n-1$, $\deg P = m$ and $\mathbf{e}_d$ is the vertical vector with a single non-zero entry 1 in position $m+1$. Since $d+1 = n$, $d' = m+1$ and $n \geq m+1$,

$$\det(M_d) = \det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \cline{1-3} \langle 1, B\rangle_{m+1} & \langle T, A\rangle_{m+1} & \mathbf{0} & m+1 \\ \cline{1-3} \langle 1, B\rangle_n & \langle 1, A\rangle_n & \mathbf{e}_d & n \\ \cline{1-3} \end{array}$$

$$= \det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \cline{1-3} \mathbf{0} & \langle T-1, A\rangle_{m+1} & -\mathbf{e}_d & m+1 \\ \cline{1-3} \langle 1, B\rangle_n & \langle 1, A\rangle_n & \mathbf{e}_d & n \\ \cline{1-3} \end{array}$$

$$= (-1)^{(m+1)n} \det\langle 1, B\rangle_n\ \det \begin{array}{|c|c|l} \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \cline{1-2} \langle T-1, A\rangle_{m+1} & -\mathbf{e}_d & m+1 \\ \cline{1-2} \end{array}$$

$$= -(-1)^{(m+1)n} \det\langle 1, B\rangle_n\ \det\langle 1, A\rangle_m (T-1)^m$$

$$= (-1)^{m(d-1)+d} \mathcal{V}(A)\mathcal{V}(B)(T-1)^m.$$

Finally the case $d = m+n$ is straightforward since

$$\det(M_d) = \mathcal{V}(B \cup A) = \mathcal{V}(A)\mathcal{V}(B)\mathrm{Res}(f,g).$$

□

We are ready now to compute $u_d(x,T)$ for all values of d, $0 \le d \le m+n$, and to deduce $\mathrm{Sylv}^{p,q}(A,B;x)$ for all possible values of p and q.

Theorem 9. *Let $1 \le m \le n$. Then*

$$u_d(x,T) = \begin{cases} (-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\mathit{Sres}_d(f,g)\,T^{m-d}(T-1)^d & \text{for } 0 \le d < m \text{ or } m = d < n \\ 0 & \text{for } m < d < n-1 \\ (-1)^{\sigma}\mathcal{V}(A)\,\mathcal{V}(B)\,f(x)\,(T-1)^m & \text{for } m < d = n-1 \\ (-1)^{\sigma}\mathcal{V}(A)\,\mathcal{V}(B)\,\left(F_{d'-1}(x)f(x)+TG_{d'-1}(x)g(x)\right)\,(T-1)^{d'-1} & \text{for } n \le d < m+n \\ \mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Res}(f,g)\,f(x)\,g(x) & \text{for } d = m+n, \end{cases}$$

where $\sigma = (d'-1)n+d$, and $F_{d'-1}, G_{d'-1}$ are defined as in Identity (2) for $k := d'-1$.

Proof. If $m < d < n-1$ then by Proposition 3 we have that $u_d(x,T) = 0$. For the other cases of $0 \le d \le m+n$, we apply Theorem 4 and Lemma 7. Using the notation of Theorem 4 we get

$$u_d(x,T)\cdot P_k = P(x)\cdot\det \begin{array}{|c|c|c|l} \multicolumn{1}{c}{n} & \multicolumn{1}{c}{m} & \multicolumn{1}{c}{1} & \\ \hline \langle 1,B\rangle_{d'} & \langle T,A\rangle_{d'} & \mathbf{0} & d' \\ \hline \langle 1,B\rangle_{d+1} & \langle 1,A\rangle_{d+1} & \mathbf{e}_k & d+1 \\ \hline \end{array}.$$

Now for each of the following cases we also apply Lemmas 6 and 8:
For $0 \le d < m$ or $d = m$ if $m < n$, $P(x) = \mathrm{Sres}_d(f,g)$, $k = d$ and $P_k = \Delta_d(f,g)$, therefore

$$\begin{aligned} u_d(x,T) &= \frac{1}{\Delta_d(f,g)}\left(\mathrm{Sres}_d(f,g)(-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\Delta_d(f,g)\,T^{m-d}(T-1)^d\right) \\ &= (-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Sres}_d(f,g)\,T^{m-d}(T-1)^d. \end{aligned}$$

For $m < d = n-1$ we have that $P(x) = f(x)$ $k = m$ and $P_k = 1$, then

$$u_d(x,T) = f(x)(-1)^{m(d-1)+d}\mathcal{V}(A)\mathcal{V}(B)(T-1)^m,$$

and to get the sign $(-1)^{\sigma}$ as in the claim, we note that in this case $m = d'-1$ and thus $m(d-1)+d \equiv (d'-1)n+d \pmod 2$.

For $n \le d < m+n$ we have that $P(x) = F_{d'-1}(x)f(x)+TG_{d'-1}(x)g(x)$, $k = d'-1$ and $P_k = (-1)^{d-n}\Delta_{d'}(f,g)(T-1)$. We conclude

$$\begin{aligned} u_d(x,t) &= \frac{(F_{d'-1}(x)f(x)+TG_{d'-1}(x)g(x))\,(-1)^{nd'}\mathcal{V}(A)\,\mathcal{V}(B)\,\Delta_{d'}(f,g)\,(T-1)^{d'}}{(-1)^{d-n}(T-1)\Delta_{d'}(f,g)} \\ &= (-1)^{n(d'-1)+d}\mathcal{V}(A)\,\mathcal{V}(B)\,\left(F_{d'-1}(x)f(x)+TG_{d'-1}(x)g(x)\right)\,(T-1)^{d'-1}. \end{aligned}$$

The last case, $d = m+n$, is straightforward. We note that in this case $u_x(x,T)$ is equal to $\mathcal{V}\big(A \cup B \cup \{x\}\big)$ up to a sign. $\square$

Proof. (Main Theorem.)
By Theorem 1 we have that

$$u_d(x,T) = (-1)^{q(m-p)}\mathcal{V}(A)\,\mathcal{V}(B)\ \mathrm{Sylv}^{p,q}(A,B;x). \tag{7}$$

For $0 \le d := p+q \le m < n$ or for $0 \le d < m = n$, we have by Theorem 9

$$u_d(x,T) = \sum_{p=0}^{d} u_{d,p}(x)T^{m-p} = (-1)^{dm}\mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Sres}_d(f,g)\,T^{m-d}(T-1)^d,$$

which implies that

$$u_{d,p}(x) = (-1)^{dm}(-1)^p\binom{d}{d-p}\,\mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Sres}_d(f,g).$$

Therefore, using (7),

$$\begin{aligned}\mathrm{Sylv}^{p,q}(A,B;x) &= (-1)^{dm+p-q(m-p)}\binom{d}{p}\mathrm{Sres}_d(f,g)\\ &= (-1)^{p(m-d)}\binom{d}{p}\mathrm{Sres}_d(f,g)\end{aligned}$$

since

$$dm+p-q(m-p) = pm+p+qp = p(m-d)+p(d+1+q) \equiv p(m-d)+p(p+1) \pmod 2.$$

For $m < d < n-1$, $\mathrm{Sylv}^{p,q}(A,B;x) = 0$ since $u_d(x,T) = 0$.
For $m < d := p+q = n-1$,

$$u_d(x,T) = \sum_{p=0}^{m} u_{d,p}(x)T^{m-p} = (-1)^{(d'-1)n+d}\mathcal{V}(A)\mathcal{V}(B)(T-1)^m f(x)$$

which implies that

$$u_{d,p}(x,T) = (-1)^{(d'-1)n+q}\binom{m}{p}\mathcal{V}(A)\mathcal{V}(B)f(x).$$

Therefore, using (7), we get

$$\mathrm{Sylv}^{p,q}(A,B;x) = (-1)^{(m+q)(p+1)}\binom{m}{p}f(x),$$

since

$$(d'-1)n+q-q(m-p) = m(p+q-1)+q-qm+qp \equiv (m+q)(p+1) \pmod 2.$$

For $m \leq n \leq d := p+q < m+n$,

$$u_d(x,T) = \sum_{p=d-n}^{m} u_{d,p}(x) T^{m-p} = (-1)^{n(d-m)+d} \mathcal{V}(A)\,\mathcal{V}(B)\, \left(F_{d'-1}(x)f(x) + TG_{d'-1}(x)g(x)\right)\,(T-1)^{d'-1},$$

which implies that for $d-n > p$, i.e. $d > p+n$, we have $u_{d,p}(x) = 0$, while for $d-n \leq p < m$ or $d-n < p \leq m$,

$$\begin{aligned} u_{d,p}(x) &= (-1)^{n(d-m)+d}\Big((-1)^{n-q-1}\tbinom{d'-1}{m-p}F_{d'-1}(x)f(x) + (-1)^{n-q}\tbinom{d'-1}{m-p-1}G_{d'-1}(x)g(x)\Big)\,\mathcal{V}(A)\,V(B) \\ &= (-1)^{n(d-m)+d+n-q-1}\Big(\tbinom{d'-1}{m-p}F_{d'-1}(x)f(x) - \tbinom{d'-1}{m-p-1}G_{d'-1}(x)g(x)\Big)\,\mathcal{V}(A)\,V(B) \end{aligned}$$

Therefore, by (7), for $n \leq d \leq m+n-1$ we have

$$\mathrm{Sylv}^{p,q}(A,B;x) = (-1)^{q(m-p)+n(d-m)+d+n-q-1}\Big(\tbinom{d'-1}{m-p}F_{d'-1}(x)f(x) - \tbinom{d'-1}{n-q}G_{d'-1}(x)g(x)\Big),$$

since

$$\binom{d'-1}{m-p-1} = \binom{d'-1}{d'-m+p} = \binom{d'-1}{n-q}.$$

Finally for $d = m+n$, i.e. $p = m, q = n$ we have

$$u_d(x,T) = \mathcal{V}(A)\,\mathcal{V}(B)\ \mathrm{Sylv}^{m,n}(A,B;x) = \mathcal{V}(A)\,\mathcal{V}(B)\,\mathrm{Res}(f,g)f(x)g(x)$$

which implies the claim. The main theorem has been proved. $\square$

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