AN APPROACH TO QUILLEN'S CONJECTURE VIA CENTRALIZERS OF SIMPLE GROUPS

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ABSTRACT. For any given subgroup H of a finite group G, the Quillen poset $\mathcal{A}_p(G)$ of nontrivial elementary abelian p-subgroups, is obtained from $\mathcal{A}_p(H)$ by attaching elements via their centralizers in H. We exploit this idea to study Quillen's conjecture, which asserts that if $\mathcal{A}_p(G)$ is contractible then G has a nontrivial normal p-subgroup. We prove that the original conjecture is equivalent to the $\mathbb{Z}$-acyclic version of the conjecture (obtained by replacing contractible by $\mathbb{Z}$-acyclic). We also work with the $\mathbb{Q}$-acyclic (strong) version of the conjecture, reducing its study to extensions of direct products of simple groups of p-rank at least 2. This allows us to extend results of Aschbacher-Smith and to establish the strong conjecture for groups of p-rank at most 4.

1. INTRODUCTION

Given a finite group G and a prime number p, let $\mathcal{A}_p(G)$ be the Quillen poset of nontrivial elementary abelian p-subgroups of G. We can study the homotopical properties of $\mathcal{A}_p(G)$ by means of its order complex. In [16], Quillen proved that $\mathcal{A}_p(G)$ is contractible if G has a nontrivial normal p-subgroup. He also conjectured the converse, giving rise to the well-known *Quillen conjecture*. That is, if $\mathcal{A}_p(G)$ is contractible then G has a nontrivial normal p-subgroup. Equivalently, if G has no nontrivial normal p-subgroup, then $\mathcal{A}_p(G)$ is not contractible. This conjecture has been widely studied during the past decades and various cases have been proved, but the full conjecture remains open in general.

In this article, we consider the following versions of the conjecture. Let $O_p(G)$ be the largest normal p-subgroup of G, and $\tilde{H}_*(X,R)$ the reduced homology of a finite poset X (which is the homology of its order complex $\mathcal{K}(X)$), with coefficients in the ring R.

(QC) If $O_p(G)=1$ then $\mathcal{A}_p(G)$ is not contractible.
(R-QC) If $O_p(G)=1$ then $\tilde{H}_*(\mathcal{A}_p(G),R)\neq 0$.

We will usually take $R=\mathbb{Z}$ or $\mathbb{Q}$. Hence we have:

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($\mathbb{Z}$-QC) If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Z}) \neq 0$.
($\mathbb{Q}$-QC) If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.

Note that ($\mathbb{Q}$-QC) implies ($\mathbb{Z}$-QC), which implies the original conjecture (QC).

The most important advances on the conjecture were achieved on the stronger version ($\mathbb{Q}$-QC). Quillen established ($\mathbb{Q}$-QC) for solvable groups, groups of p-rank at most 2 and some families of groups of Lie type [16]. Later, various authors dealt with the p-solvable case (see [1, 7] and [20, Ch.8]) and in [4] Aschbacher and Kleidman showed ($\mathbb{Q}$-QC) for almost-simple groups. In [5], Aschbacher and Smith proved that a group G satisfies ($\mathbb{Q}$-QC) if $p > 5$ and whenever G has a unitary component $\mathrm{U}_n(q)$ with $q \equiv -1 \pmod{p}$ and q odd, then $(\mathcal{QD})_p$ holds for all p-extensions of $\mathrm{U}_m(q^{p^e})$ with $m \leq n$ and $e \in \mathbb{Z}$ (see Definition 3.2). In a joint work with Sadofschi Costa and Viruel [14], we proved new cases of the conjecture, not included in the previously mentioned results. In [14], we worked with the integer version of the conjecture ($\mathbb{Z}$-QC) and proved that it holds if $\mathcal{K}(\mathcal{S}_p(G))$ is homotopy equivalent to a 2-dimensional and G-invariant subcomplex. Recall that $\mathcal{S}_p(G)$ is the Brown poset of nontrivial p-subgroups of G and that $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is a homotopy equivalence (see [16, Proposition 2.1]). In particular, the integer version holds for groups of p-rank at most 3. Recall that the p-rank of G is the dimension of $\mathcal{K}(\mathcal{A}_p(G))$ plus one.

Further applications and results concerning the homotopy type of the p-subgroup complexes can be found in [6, 9, 15, 20]. In [10, (1.4)], the authors considered a version of the conjecture even stronger than the rational one ($\mathbb{Q}$-QC): if $O_p(G) = 1$ then the Euler characteristic of $\mathcal{A}_p(G)$ is not 1. We will not work with this version.

In this article, we approach the study of Quillen's conjecture via the examination of the centralizers of the elementary abelian p-subgroups on suitable subgroups. Roughly, if H is a nontrivial subgroup of G, $\mathcal{A}_p(G)$ can be obtained first by passing from $\mathcal{A}_p(H)$ to the homotopy equivalent subposet $\mathcal{N}(H)$, consisting of members $E \in \mathcal{A}_p(G)$ with $E \cap H \neq 1$, and then from $\mathcal{N}(H)$ to $\mathcal{A}_p(G)$ by attaching the remaining subgroups along their links in $\mathcal{N}(H)$. If $E \in \mathcal{A}_p(G)$ and $E \cap H = 1$, its link in $\mathcal{N}(H)$ is $\mathcal{A}_p(C_H(E))$, where $C_H(E)$ is the centralizer of E in H. We can understand the homotopy type of $\mathcal{A}_p(G)$ from that of $\mathcal{A}_p(H)$ and the structure of these centralizers. In some cases, we extract points $E \in \mathcal{A}_p(G)$ with contractible link in $\mathcal{N}(H)$, and this is guaranteed precisely when $O_p(C_H(E)) \neq 1$. In this way, we can work with smaller subposets and apply inductive arguments. This approach has its roots in the previous work with E.G. Minian on the fundamental group of these complexes [12]. The poset $\mathcal{N}(H)$ was also considered by Segev and Webb [18, 19]. In this article we put more emphasis on the attachment process, the behaviour of these centralizers as links, and the extraction of points. This viewpoint seems to have been barely exploited, and we hope that the techniques and results of this article can shed more light on future methods for studying the topology of the p-subgroup posets and consequences, beyond Quillen's conjecture.

We will study ($\mathbb{Z}$-QC) and ($\mathbb{Q}$-QC) by using the idea described above and working under the following *inductive* assumption. Let $R = \mathbb{Z}$ or $\mathbb{Q}$.

$(\mathrm{H1})_R$ Proper subgroups and proper central quotients of G satisfy (R-QC).

By a proper central quotient of G we mean a quotient of G by a nontrivial central subgroup $Z \leq Z(G)$ (here $Z(G)$ denotes the center of G). The hypothesis of the central quotients is motivated by the fact that if $Z \leq Z(G)$ is a p'-group, then $\mathcal{A}_p(G)$ is naturally isomorphic to $\mathcal{A}_p(G/Z)$ (see Proposition 2.4).

The inductive assumption $(\mathrm{H1})_R$ is valid in the context of a counterexample of minimal order to the conjecture. That is, if H satisfies (R-QC) for all $|H| < |G|$, then $(\mathrm{H1})_R$ holds for G. Therefore, we may replace the content of $(\mathrm{H1})_R$ by this stronger inductive requirement.

The following theorem summarizes some of the main results of this article. It shows (R-QC) under $(\mathrm{H1})_R$ and some extra hypothesis on the group. We follow the notation of [5, 8] for the simple groups.

Theorem 1. *Let G be a finite group and p a prime number. Let $R = \mathbb{Z}$ or $\mathbb{Q}$. Suppose that $(H1)_R$ holds for G and that one of the following holds:*

(1) $O_{p'}(G) \neq 1$;
(2) $\mathcal{A}_p(G)$ is not simply connected;
(3) $p = 3$ and G has a component L such that $L/Z(L) \cong \mathrm{U}_3(2^3)$;
(4) G has a component L such that $L/Z(L)$ has p-rank 1.

Then G satisfies (R-QC).

In particular, a counterexample of minimal order G to (R-QC) fails (1)-(4), and hence it is an extension of a direct product of non-abelian simple groups of p-rank at least 2.

Our theorem has no restriction on the prime p (except for item (3)) and it is stated for both versions of the conjecture ($\mathbb{Z}$-QC) and ($\mathbb{Q}$-QC). For example, item (1) is a generalization to every prime p of the analogous result [5, Proposition 1.6] stated for $p > 5$, and in our proof we use the classification of simple groups to a much lesser extent. Item (2) of the above theorem is based on our previous work on the fundamental group [11]. The more technical hypotheses of items (3) and (4) are focused on extending [5, Main Theorem] to every odd prime p.

We describe now some consequences of Theorem 1.

First, it allows us to conclude that the original conjecture (QC) and the integer homology version ($\mathbb{Z}$-QC) are equivalent for every prime p.

Theorem 2. *The original Quillen's conjecture and the integer Quillen's conjecture ($\mathbb{Z}$-QC) are equivalent. That is, (QC) holds for all finite groups if and only if ($\mathbb{Z}$-QC) holds for all finite groups.*

This result strongly depends on the integer version of items (1) and (2) of Theorem 1. It does not follow from [5, Proposition 1.6], which is stated for rational homology and $p > 5$.

On the other hand, items (3) and (4) of Theorem 1 allow us to handle the groups containing a component isomorphic to $\mathrm{L}_2(2^3)$ ($p = 3$), $\mathrm{U}_3(2^3)$ ($p = 3$) or $\mathrm{Sz}(2^5)$ ($p = 5$). Aschbacher and Smith excluded the groups containing these components during their analysis of the conjecture for odd p, mainly because the centralizers of their field automorphisms of order p have nontrivial normal p-subgroups (see Section 5 for a more detailed discussion). Nevertheless, our theorem shows that we can suppose that G does not contain these components if we are aiming to prove Quillen's conjecture. This allows us to extend the main result of Aschbacher-Smith to $p = 5$.

Corollary 3. *Theorem* [5, Main Theorem] *also holds for $p = 5$.*

The extension of [5, Main Theorem] to $p = 3$ is not immediate since its proof depends on [5, Theorem 5.3], which is stated for $p \geq 5$. [1]

Finally, we combine Theorem 1 with the classification of the simple groups of low p-rank, the structure of their centralizers and the classification of groups with a strongly p-embedded subgroup (i.e. with disconnected Quillen's complex), to yield the p-rank 4 case of the conjecture.

Theorem 4. *If G has p-rank at most 4, then it satisfies ($\mathbb{Q}$-QC).*

Theorem 1 follows from Theorems 4.1, 5.1 and 6.1, and Corollary 4.4, and Remark 2.1. The proof of Theorem 2 is given in Section 4, and the proof of Corollary 3 can be found below the statement of Theorem 5.1. Finally, Theorem 4 is proved in Section 7. We also include an Appendix containing basic properties of the almost-simple groups with a strongly p-embedded subgroup.

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2. Preliminary results

In this section we establish the main definitions and tools that we will use throughout the paper. We refer to [3] for more details on finite group theory.

All the groups considered here are finite. By a simple group we will mean a non-abelian simple group. We adopt the conventions of [8] for the names of the simple groups and their automorphisms. We denote by C_n, D_n, $\mathbb{S}_n$ and $\mathbb{A}_n$ the cyclic group of order n, the dihedral group of order n, the symmetric group on n letters and the alternating group on n letters, respectively.

Let G be a finite group and p a prime number. Denote by $Z(G)$ the center of G. Let $O_p(G)$ be the largest normal p-subgroup of G and $O_{p'}(G)$ the largest normal p'-subgroup of G. The Fitting subgroup $F(G)$ of G is the largest normal nilpotent subgroup of G, and it is the direct product of the subgroups $O_q(G)$, for q prime dividing the order of G. For a fixed prime p, let $\Omega_1(G) := \langle x \in G : x^p = 1\rangle$. The p-rank of G is

$$m_p(G) := 1 + \dim \mathcal{K}(\mathcal{A}_p(G)) = \max\{\log_p(|A|) \ : \ A \in \mathcal{A}_p(G) \cup \{1\}\}.$$

If $H, K \leq G$ are subgroups of G, then $N_H(K)$ denotes the normalizer of K in H and $C_H(K)$ the centralizer of K in H. Denote by $[H, K]$ the subgroup generated by the commutators between elements of H and K. If $g \in G$, write $H^g = g^{-1}Hg$.

Recall that $E(G)$, the layer of G, is the (central) product of the components of G, and that the generalized Fitting subgroup $F^*(G)$ is the central product of $E(G)$ and $F(G)$. We refer to [3, Chapter 11] for the main properties of these subgroups.

Let $\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$ denote the group of outer automorphisms of H. If H is a group of Lie-type, $\mathrm{Inn\ diag}(H)$ denotes the subgroup of inner-diagonal automorphisms of H, and $\mathrm{Out\ diag}(H) = \mathrm{Inn\ diag}(H)/\mathrm{Inn}(H)$.

[1] In a forthcoming work based on the results of this article, we also extend [5, Main Theorem] to $p = 3$, and hence to every odd prime.

Remark 2.1. If $O_p(G) = 1 = O_{p'}(G)$, then $F(G) \leq O_p(G)O_{p'}(G) = 1$ and $F^*(G) = E(G)$ has trivial center. Therefore, $F^*(G) = L_1 \dots L_n$ is the direct product of the components $\{L_1, \dots, L_n\}$ of G, which are non-abelian simple groups of order divisible by p. Since $F^*(G)$ is self-centralizing, $F^*(G) \leq G \leq \mathrm{Aut}(F^*(G))$. Moreover, $\mathrm{Aut}(F^*(G))$ can be easily described by using the fact that if L is a simple group, then $\mathrm{Aut}(L^n) \cong \mathrm{Aut}(L) \wr \mathbb{S}_n$ and that $\mathrm{Aut}(L \times K) \cong \mathrm{Aut}(L) \times \mathrm{Aut}(K)$ if L and K are non-isomorphic simple groups.

If X is a finite poset, we can study its homotopy properties by means of its associated order complex $\mathcal{K}(X)$, whose simplices are the nonempty chains of X. If $x \in X$ and $Y \subseteq X$ is a subposet, let $Y_{\geq x} = \{y \in Y : y \geq x\}$. Define analogously $Y_{>x}$, $Y_{\leq x}$ and $Y_{<x}$. The link of x in Y is $Y_{<x} \cup Y_{>x}$.

Recall that if $f, g : X \to Y$ are two order preserving maps between finite posets X and Y such that $f \leq g$ (i.e. $f(x) \leq g(x)$ for all $x \in X$), then f and g are homotopic when regarded as simplicial maps. Write $X \simeq Y$ if $\mathcal{K}(X) \simeq \mathcal{K}(Y)$. Note that this is not the usual convention that we employed in the previous articles [11, 13].

We recall below Quillen's fiber lemma for finite posets. If Y, X are sets, $X - Y$ denotes the complement of Y in X.

Proposition 2.2 ([16, Proposition 1.6])**.** *Let $f : X \to Y$ be an order preserving map between finite posets. If $f^{-1}(Y_{\leq y})$ is contractible for all $y \in Y$ (resp. $f^{-1}(Y_{\geq y})$ is contractible for all $y \in Y$), then f is a homotopy equivalence. In particular, if $X \subseteq X_0$ and $X_{>x}$ is contractible for all $x \in X_0 - X$ (resp. $X_{<x}$ is contractible for all $x \in X_0 - X$) then $X \hookrightarrow X_0$ is a homotopy equivalence.*

The *Brown poset* $\mathcal{S}_p(G)$ is the poset of nontrivial p-subgroups of G. The inclusion $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is a homotopy equivalence by [16, Proposition 2.1], and if $O_p(G) \neq 1$ then $\mathcal{A}_p(G)$ is contractible (see [16, Proposition 2.4]).

Quillen related the direct product of groups with the join of their p-subgroup posets. The join of two posets $X * Y$ is the poset whose underlying set is the disjoint union of X and Y, keeping the given ordering within X and Y, and setting $x < y$ for each $x \in X$ and $y \in Y$. Moreover, $\mathcal{K}(X * Y)$ equals the join of simplicial complexes $\mathcal{K}(X) * \mathcal{K}(Y)$, and this is homeomorphic to the topological join of $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ (see [16, Proposition 1.9]). If $Y \subseteq X$ are finite posets and $x \in X$, then note that the link of x in Y is the join $Y_{<x} * Y_{>x}$.

Proposition 2.3 ([16, Proposition 2.6])**.** $\mathcal{A}_p(G_1 \times G_2) \simeq \mathcal{A}_p(G_1) * \mathcal{A}_p(G_2)$.

The following two results show that, in some sense, it is enough to study the homotopical properties of $\mathcal{A}_p(G)$ when $Z(G)$ is the trivial group. Let $R = \mathbb{Z}$ or $\mathbb{Q}$.

Proposition 2.4. *Let $Z \leq Z(G)$. The following hold.*

1. *If Z is a nontrivial p-group, then $\mathcal{A}_p(G)$ is contractible.*
2. *If Z is a p'-group then the induced map $\mathcal{A}_p(G) \to \mathcal{A}_p(G/Z)$ is an isomorphism of posets. Moreover, $O_p(G/Z) \cong O_p(G)$.*
3. *In particular, if G satisfies (H1)$_R$ and $Z \neq 1$ is a p'-group, then G satisfies (R-QC), where $R = \mathbb{Z}$ or $\mathbb{Q}$. Therefore, we may assume that $Z(G) = 1$ under (H1)$_R$ to study (R-QC).*

Proof. Part (1) follows easily since $O_p(Z) \leq O_p(G)$. Part (2) follows directly from the isomorphism theorems and Sylow's theorems. For the "Moreover" part of (2), note that if $H \leq G$ then $O_p(HZ/Z) \cong O_p(HZ)$ since Z is a central p'-subgroup of G. Finally, part (3) is a consequence of the definition of the $(\mathrm{H1})_R$ hypothesis and part (2). □

Below we give an immediate consequence of the $(\mathrm{H1})_R$ hypothesis.

Lemma 2.5. *If G satisfies $(H1)_R$ and either $Z(G) \neq 1$ or $\Omega_1(G) < G$, then G satisfies (R-QC).*

Therefore, under $(H1)_R$, we may assume that $Z(G) = 1$ and $\Omega_1(G) = G$ to study (R-QC).

Proof. Suppose that $O_p(G) = 1$. If $Z(G) \neq 1$ then $\mathcal{A}_p(G) = \mathcal{A}_p(G/Z(G))$ by the above lemma. Since $G/Z(G)$ is a proper central quotient of G, it satisfies (R-QC) by $(\mathrm{H1})_R$. Moreover, we also have that $O_p(G/Z(G)) = O_p(G) = 1$, so $\tilde{H}_*(\mathcal{A}_p(G), R) = \tilde{H}_*(\mathcal{A}_p(G/Z(G)), R) \neq 0$.

If $\Omega_1(G) < G$, then $\Omega_1(G)$ satisfies (R-QC) by $(\mathrm{H1})_R$. Note that $O_p(\Omega_1(G)) = 1$ since $\Omega_1(G)$ is normal in G. Since $\mathcal{A}_p(G) = \mathcal{A}_p(\Omega_1(G))$, $\tilde{H}_*(\mathcal{A}_p(G), R) = \tilde{H}_*(\mathcal{A}_p(\Omega_1(G)), R) \neq 0$. □

In the lemmas below, we recall some results that will play a fundamental role in the proof of our main theorems. For a given subgroup $H \leq G$, we "inflate" the subposet $\mathcal{A}_p(H)$ and then we show that the remaining points of $\mathcal{A}_p(G)$ are attached to this inflated subposet throughout their centralizers in H.

Definition 2.6. For $H \leq G$, let

$$\mathcal{N}(H) := \{E \in \mathcal{A}_p(G) \ : \ E \cap H \neq 1\}.$$

We sometimes abbreviate $\mathcal{N}_H = \mathcal{N}(H)$.

We can also regard the poset $\mathcal{N}(H)$ as the "neighbourhood" of $\mathcal{A}_p(H)$, and $\mathcal{N}(H) - \mathcal{A}_p(H)$ as the "boundary" of this neighbourhood. We give below some consequences of this definition, which were used for computing the examples given in [14] (cf. [18, 19]).

Lemma 2.7. *If $H \leq G$ then $\mathcal{A}_p(H) \hookrightarrow \mathcal{N}(H)$ is a strong deformation retract.*

Proof. Let $i : \mathcal{A}_p(H) \hookrightarrow \mathcal{N}(H)$ be the inclusion and $\varphi : \mathcal{N}(H) \to \mathcal{A}_p(H)$ the map defined by $\varphi(E) = E \cap H$. Then i and φ are order preserving maps with $i\varphi \leq \mathrm{Id}_{\mathcal{N}(H)}$ and $\varphi i = \mathrm{Id}_{\mathcal{A}_p(H)}$. □

Next, we show that the elements outside $\mathcal{N}(H)$ attach to it via their centralizers in H.

Lemma 2.8. *Let $H \leq G$ be a subgroup and let $E \in \mathcal{A}_p(G)$ be such that $E \cap H = 1$. Then $\mathcal{N}(H)_{>E}$ is homotopy equivalent to $\mathcal{A}_p(C_H(E))$.*

Proof. Let $f : \mathcal{A}_p(C_H(E)) \to \mathcal{N}(H)_{>E}$ and $g : \mathcal{N}(H)_{>E} \to \mathcal{A}_p(C_H(E))$ be the maps defined by $f(A) = AE$ and $g(A) = A \cap H$. Then $fg(A) = (A \cap H)E \leq A$ and $gf(A) = (AE) \cap H = A$ (by modular law). Hence $fg \leq \mathrm{Id}_{\mathcal{N}(H)_{>E}}$ and $gf = \mathrm{Id}_{\mathcal{A}_p(C_H(E))}$. □

We can rebuild $\mathcal{A}_p(G)$ from $\mathcal{N}(H)$ by attaching points in the following way. Take a linear extension of the complement $\mathcal{A}_p(G) - \mathcal{N}(H) = \{E_1, \ldots, E_r\}$ such that $E_i \leq E_j$ implies $i \leq j$. For each $0 \leq i \leq r$, consider the subposet $X_i = \mathcal{N}(H) \cup \{E_1, \dots, E_i\}$. This gives rise to a filtration

$$\mathcal{N}(H) = X_0 \subseteq X_1 \subseteq \dots \subseteq X_r = \mathcal{A}_p(G),$$

where $X_i = X_{i-1} \cup \{E_i\}$ and the link of E_i in X_{i-1} is

$$(\mathcal{A}_p(E_i) - \{E_i\}) * \mathcal{N}(H)_{>E_i} \simeq \left(\bigvee_{l=1}^{k_i} \mathbb{S}^{m_p(E_i)-2} \right) * \mathcal{A}_p(C_H(E_i)),$$

with $k_i = p^{\binom{m_p(E_i)}{2}}$. Recall that if E is an elementary abelian p-group, $\mathcal{A}_p(E) - \{E\}$ is homotopy equivalent to a wedge of $p^{\binom{m_p(E)}{2}}$ spheres of dimension $m_p(E) - 2$ (cf. [16, p.58]).

This provides a useful way to study the homotopy type of $\mathcal{A}_p(G)$ if we select a convenient subgroup $H \leq G$ for which we understand the structure of these centralizers. In particular, if they are contractible, the homotopy type of $\mathcal{A}_p(H)$ does not change.

Lemma 2.9 (cf. [14, Lemma 4.3])**.** *Let G be a finite group and let $H \leq G$. In addition, suppose that $O_p(C_H(E)) \neq 1$ for each $E \in \mathcal{A}_p(G)$ with $E \cap H = 1$. Then $\mathcal{A}_p(G) \simeq \mathcal{A}_p(H)$.*

Proof. Let $E \in \mathcal{A}_p(G) - \mathcal{N}(H)$. By Lemma 2.8 $\mathcal{N}(H)_{>E} \simeq \mathcal{A}_p(C_H(E))$, which is contractible by hypothesis. Finally, by Proposition 2.2 and Lemma 2.7, $\mathcal{A}_p(G) \simeq \mathcal{N}(H) \simeq \mathcal{A}_p(H)$. □

For example, we will usually take H to be $LC_G(L)$, where L is a simple component of G. Note that $F^*(G) \leq H$. If $E \in \mathcal{A}_p(N_G(L))$ and $E \cap H = 1$ then $C_H(E) = C_L(E)C_G(LE)$ and $\mathcal{A}_p(C_H(E)) \simeq \mathcal{A}_p(C_L(E)) * \mathcal{A}_p(C_G(LE))$. The group $C_L(E)$ is the centralizer of an elementary abelian p-group acting on the simple group L, which can be described by using the classification of the finite simple groups. We may also apply inductive arguments on $C_G(LE)$.

3. The homology propagation lemma

The aim of this section is to propose a generalization of the Homology Propagation Lemma [5, Lemma 0.27], stated in Lemma 3.14. Both lemmas allow us to propagate non-zero (free) homology from proper subposets to the whole Quillen poset. These tools will be very useful to establish Quillen's conjecture when we have an extra inductive assumption such as $(\mathrm{H1})_R$.

Our Lemma 3.14 shares the spirit of [5, Lemma 0.27] but with two extra features: it works for homology with coefficients in $\mathbb{Z}$, and it can be applied to suitable proper subposets $X \subset \mathcal{A}_p(G)$. This subposet X will be typically chosen to be homotopy equivalent to $\mathcal{A}_p(G)$ but better behaved, in certain sense, than the Quillen poset. In many cases, we will see that X satisfies the hypotheses of Lemma 3.14 while $\mathcal{A}_p(G)$ does not.

Before proceeding with the proof of this lemma, we need some definitions and generalizations of the results of [5]. From now on, we suppress the coefficient notation on the homology and suppose that they are taken in the ring $R = \mathbb{Z}$ or $\mathbb{Q}$. The definitions given below do not depend on the coefficient ring.

Let X be a finite poset. Recall that a chain of X is a subset $a \subseteq X$ whose elements are pairwise comparable. We usually write $a = (x_0 < x_1 < \dots < x_n)$

to emphasize the order of its elements. Denote by $\max a$ and $\min a$ the maximum and minimal element of a respectively, if a is nonempty. Let X' be the poset of nonempty chains of X. Equivalently, X' is the face poset of $\mathcal{K}(X)$.

Denote by $\tilde{C}_*(X)$ the augmented chain complex of X with coefficients in R. Recall that $\tilde{C}_n(X)$ is freely generated by the chains $(x_0 < x_1 < \ldots < x_n)$ in X. Write $\tilde{Z}_n(X)$ for the subgroup of n-cycles and $\tilde{H}_*(X)$ for the reduced homology of X. We say that a chain $a \in X'$ is an addend of $\alpha \in \tilde{C}_n(X)$, and we write $a \in \alpha$, if a appears with non-zero coefficient in the sum decomposition of α in the canonical basis of $\tilde{C}_n(X)$.

Definition 3.1. Let X be a finite poset. A chain $a \in X'$ is *full* if for every $x \in X$ such that $\{x\} \cup a$ is a chain we have that $x \in a$ or $x \geq \max a$. A chain b containing a is called *a-initial chain* if for every $x \in b - a$, we have $x > \max a$.

The following property was introduced by Aschbacher and Smith in [5].

Definition 3.2. We say that G satisfies the *Quillen dimension property at p*, $(\mathcal{QD})_p$ for short, if $\tilde{H}_{m_p(G)-1}(\mathcal{A}_p(G)) \neq 0$. That is, $\mathcal{A}_p(G)$ has non-zero homology in the highest possible degree.

Observe that the top integer homology group of $\mathcal{A}_p(G)$ is always free, so this definition does not depend on the chosen coefficient ring $\mathbb{Z}$ or $\mathbb{Q}$. It is worth noting that finite groups may not satisfy $(\mathcal{QD})_p$ in general. This had been already observed by Quillen in [16].

Definition 3.3. Suppose that G satisfies $(\mathcal{QD})_p$ and let $m = m_p(G) - 1$. If $\alpha \in \tilde{H}_m(\mathcal{A}_p(G)) = \tilde{Z}_m(\mathcal{A}_p(G))$ is a non-zero cycle and $a \in \alpha$ is an addend of α, we say that a or $\max a$ *exhibits* $(\mathcal{QD})_p$ for G. Note that a is a full chain.

Next, we recall a special configuration of the p-solvable case of the conjecture. Its proof depends on the classification of the finite simple groups and can be found in [20, Theorem 8.2.12]. See also [1, 5, 7].

Theorem 3.4. *If $G = O_{p'}(G)A$, where A is an elementary abelian p-group acting faithfully on $O_{p'}(G)$, then G satisfies $(\mathcal{QD})_p$ exhibited by A.*

Now we set up the proper context that we need to culminate in the proof of Lemma 3.14. We shall work under [5, Hypothesis 0.15], which we state below.

Hypothesis 3.5 (Central product)**.** $H \leq G$ and $K \leq C_G(H)$ with $H \cap K$ a p'-group.

Hypothesis 3.5 implies that $[H, K] = 1$ and $H \cap K \leq Z(H) \cap Z(K)$. Moreover, we have that

$$\mathcal{A}_p(HK) \simeq \mathcal{A}_p(H/H \cap K) * \mathcal{A}_p(K/H \cap K)$$

since HK is a central product and the shared central subgroup $H \cap K$ is a p'-group (see [5, Lemma 0.11]).

Under appropriate circumstances, there will be non-zero cycles α and β in the homology of $\mathcal{A}_p(H)$ and $\mathcal{A}_p(K)$ respectively, and they will give rise to a non-zero cycle $\alpha \times \beta$ (the shuffle product) in the homology of $\mathcal{A}_p(HK)$. The final goal is to show that this product cycle produces non-zero homology for $\mathcal{A}_p(G)$. In order to do that, we will ask for some subgroup $A \in \mathcal{A}_p(H)$ involved in α to satisfy suitable strong hypotheses. With these hypotheses, if $\alpha \times \beta$ is the zero cycle (i.e. a boundary) in $\tilde{H}_*(\mathcal{A}_p(G))$, we will reduce to a calculation in $\tilde{H}_*(\mathcal{A}_p(K))$ and then arrive to a contradiction.

The idea of this section is to perform the above homology computations in a typically proper subposet X, which, in general, will be constructed to be homotopy equivalent to $\mathcal{A}_p(G)$. Hence, showing that $\alpha \times \beta$ is a non-zero cycle in $\tilde{H}_*(X)$ will lead to non-zero homology in $\mathcal{A}_p(G)$, as desired.

The reduction described above is in fact carried out inside the subposet $\mathcal{N}(K)$, lying over $\mathcal{A}_p(K)$. We shall take X containing $\mathcal{N}(K)$.

Definition 3.6. Under Hypothesis 3.5, we call a subposet $X \subseteq \mathcal{A}_p(G)$ an $\mathcal{N}_K$-*superset* if $\mathcal{N}(K) \subseteq X$.

We proceed now to generalize the definitions and results coming after [5, Hypothesis 0.15].

Definition 3.7 ([5, Definition 0.19])**.** Assume Hypothesis 3.5. Let $a = (A_0 < \ldots < A_m)$ and $b = (B_0 < \ldots < B_n)$ be a chains of $\mathcal{A}_p(H)$ and $\mathcal{A}_p(K)$ respectively. We define the following chain in $\mathcal{A}_p(HK)$:

$$a * b := (A_0 < \ldots < A_m < B_0A_m < \ldots < B_nA_m).$$

Let $c = (0, 1, 2, \ldots, m + n + 1)$. A *shuffle* is a permutation σ of the set $\{0, 1, 2, \ldots, m + n + 1\}$ such that $\sigma(i) < \sigma(j)$ if $i < j \leq m$ or $m + 1 \leq i < j$. Let $\sigma(c) := (\sigma(0), \sigma(1), \ldots, \sigma(m + n + 1))$.

With the notation of the above definition, let $C_j = A_j$ if $j \leq m$ or $B_{j-(m+1)}$ if $j \geq m + 1$. For a shuffle σ, define $(a \times b)_\sigma$ to be the chain whose i-th element is $C_{\sigma(0)}C_{\sigma(1)} \ldots C_{\sigma(i)}$.

Definition 3.8 ([5, Definition 0.21])**.** Assume Hypothesis 3.5. The shuffle product of a and b is

$$a \times b := \sum_{\sigma \text{ shuffle}} (-1)^\sigma (a \times b)_\sigma \in \tilde{C}_{m+n+1}(\mathcal{A}_p(HK)).$$

Extend this definition by linearity to the tensor product of the chain complexes $\tilde{C}_*(\mathcal{A}_p(H))$ and $\tilde{C}_*(\mathcal{A}_p(K))$.

Recall that we are aiming to apply later Lemma 3.14 which, in contrast to [5, Lemma 0.27], works with a potentially proper $\mathcal{N}_K$-superset X of $\mathcal{A}_p(G)$. In many situation it will be the case where $\mathcal{A}_p(H) \subseteq X$, hence $\tilde{C}_*(\mathcal{A}_p(H)) \subseteq \tilde{C}_*(X)$ and the following lemmas are automatic. However, our overall arguments do not require this assumption, so we supply the lemmas below to also cover these cases.

Lemma 3.9. *Assume Hypothesis 3.5 and let X be an $\mathcal{N}_K$-superset.*

(i) If $a \in X' \cap \mathcal{A}_p(H)'$, $b \in \mathcal{A}_p(K)'$ and σ is a shuffle, then $(a \times b)_\sigma \in X'$.
(ii) If $\alpha \in \tilde{C}_*(X) \cap \tilde{C}_*(\mathcal{A}_p(H))$ and $\beta \in \tilde{C}_*(\mathcal{A}_p(K))$ then $\alpha \times \beta \in \tilde{C}_*(X) \cap \tilde{C}_*(\mathcal{A}_p(HK))$.

Proof. If $C \in (a \times b)_\sigma$, then either $C \in a \subseteq X$ or else C contains some subgroup $B \in b$. In the latter case, $C \cap K \geq B \neq 1$, so $C \in \mathcal{N}(K) \subseteq X$ since X is an $\mathcal{N}_K$-superset. This proves part (i). Part (ii) follows from (i), by $\tilde{C}_*(X) \cap \tilde{C}_*(\mathcal{A}_p(H)) = \tilde{C}_*(X \cap \mathcal{A}_p(H))$ and a linearity argument. □

Proposition 3.10 (cf. [5, Corollary 0.23])**.** *Under Hypothesis 3.5, if $\alpha \in \tilde{Z}_m(\mathcal{A}_p(H))$ and $\beta \in \tilde{Z}_n(\mathcal{A}_p(K))$ then $\alpha \times \beta \in \tilde{Z}_{m+n+1}(\mathcal{A}_p(HK))$. In addition, if X is an $\mathcal{N}_K$-superset and $\alpha \in \tilde{C}_*(X)$ then $\alpha \times \beta \in \tilde{Z}_{m+n+1}(X)$.*

Proof. The first part is [5, Corollary 0.23], and the second part follows from Lemma 3.9. □

Remark 3.11. Let X be a finite poset and $a \in X'$. Denote by $\tilde{C}_*(X)_a$ the subgroup of a-initial chains and by $\tilde{C}_*(X)_{\neg a}$ the subgroup of non-a-initial chains. Clearly we have a decomposition

$$\tilde{C}_*(X) = \tilde{C}_*(X)_a \oplus \tilde{C}_*(X)_{\neg a}.$$

Moreover, if ∂ denotes the boundary map of its chain complex, then

$$\partial(\tilde{C}_*(X)_{\neg a}) \subseteq \tilde{C}_*(X)_{\neg a}.$$

If $\gamma \in \tilde{C}_*(X)$ then $\gamma = \gamma_a + \gamma_{\neg a}$, where γ_a corresponds to the a-initial part of γ, and

$$\partial\gamma = \partial(\gamma_a) + \partial(\gamma_{\neg a}) = (\partial(\gamma_a))_a + (\partial(\gamma_a))_{\neg a} + \partial(\gamma_{\neg a}).$$

This remark yields the following lemma.

Lemma 3.12 (cf. [5, Lemma 0.24])**.** *If $a \in X'$ is a full chain then $(\partial\gamma)_a = (\partial\gamma_a)_a$.*

The following lemma generalizes [5, Lemma 0.25(i)] to $\mathcal{N}_K$-supersets.

Lemma 3.13 (cf. [5, Lemma 0.25])**.** *Assume Hypothesis 3.5.*

- *(i) If $a \in \mathcal{A}_p(H)'$ and $b \in \mathcal{A}_p(K)'$ then $(a \times b)_a = (a \times b)_{\sigma=id} = a * b$ in $\tilde{C}_*(\mathcal{A}_p(HK)) \subseteq \tilde{C}_*(\mathcal{A}_p(G))$;*
- *(ii) In addition, if X is an $\mathcal{N}_K$-superset and $a \in X'$, then the conclusion of (i) remains true in $\tilde{C}_*(X)$.*

Proof. Let σ be a shuffle and $C \in (a \times b)_\sigma$ with $C \notin a$. Then $C \geq B$ for some $B \in b$, so $C \in \mathcal{N}(K)$ (see the proof of Lemma 3.9). Since $H \cap K$ is a p'-group, $C \notin \mathcal{A}_p(H)$. Therefore, $(a \times b)_\sigma$ is a-initial if and only if $\sigma = \mathrm{id}$, proving item (i). Item (ii) follows from Lemma 3.9. □

We prove now the mentioned generalization of the fundamental Homology Propagation Lemma [5, Lemma 0.27]. Recall that we are working with coefficients in $R = \mathbb{Z}$ or $\mathbb{Q}$.

Lemma 3.14. *Let G be a finite group. Let $H, K \leq G$ and $X \subseteq \mathcal{A}_p(G)$ be such that:*

- *(i) H and K satisfy Hypothesis 3.5;*
- *(ii) X is an $\mathcal{N}_K$-superset;*
- *(iii) There exist a chain $a \in \mathcal{A}_p(H)' \cap X'$ and a cycle $\alpha \in \tilde{C}_m(\mathcal{A}_p(H)) \cap \tilde{C}_m(X)$ such that the coefficient of a in α is invertible and $\alpha \neq 0$ in $\tilde{H}_m(\mathcal{A}_p(H))$ (for some $m \geq -1$);*
- *(iv) In addition, such a is a full chain in X and $X_{>\max a} \subseteq \mathcal{N}(K)$;*
- *(v) $\tilde{H}_*(\mathcal{A}_p(K)) \neq 0$.*

Then $\tilde{H}_*(X) \neq 0$.

In particular, under (i), hypotheses (ii), (iii) and (iv) hold if:

- *(a) coefficients are taken in $\mathbb{Q}$,*
- *(b) $X = \mathcal{A}_p(G)$, and*
- *(c) H has $(\mathcal{QD})_p$ exhibited by $A \in \mathcal{A}_p(H)$ such that $\mathcal{A}_p(G)_{>A} \subseteq A \times K$*

(This is the hypothesis in [5, Lemma 0.27]*, so the present result is indeed a generalization).*

Proof. We essentially carry out the original proof of [5, Lemma 0.27] inside $\tilde{C}_*(X)$ since X is an $\mathcal{N}_K$-superset.

By hypothesis (v), there exists a cycle $\beta \in \tilde{C}_n(\mathcal{A}_p(K))$ which is not a boundary in $\tilde{C}_*(\mathcal{A}_p(K))$. Choose a chain a and a cycle α as in the hypothesis (iii). Then $\alpha \times \beta \in \tilde{Z}_{m+n+1}(X)$ by hypotheses (i), (ii), (iii) and Proposition 3.10. We show that $\alpha \times \beta$ gives a non-zero cycle in the homology of X.

Suppose by way of contradiction that for some chain $\gamma \in \tilde{C}_{m+n+2}(X)$ we have that

$$\alpha \times \beta = \partial\gamma. \tag{3.1}$$

Write $\beta = \sum_i q_i(B_0^i < \ldots < B_n^i)$ and $\gamma = \sum_{j\in J} p_j(C_0^j < \ldots < C_{m+n+2}^j)$. Now take a-initial parts in both sides of the expression of (3.1).

$$(\alpha \times \beta)_a = (\partial\gamma)_a. \tag{3.2}$$

Note that no intermediate group lying in X can be added within a due to hypothesis (iv). Let $A = \max a$. By item (ii) of Lemma 3.13, the left-hand-side of (3.2) becomes

$$(\alpha \times \beta)_a = q(a \times \beta) = q\sum_i q_i \; a \cup (AB_0^i < \ldots < AB_n^i), \tag{3.3}$$

where by hypothesis (iii), $q \neq 0$ is the coefficient of a in α, and it is invertible. The expression in (3.3) is then equal to the right-hand-side of (3.2), which using Lemma 3.12 is

$$\begin{aligned}(\partial\gamma)_a &= \sum_{j\in J'} p_j \sum_{k=m+1}^{m+n+2} (-1)^k \; a \cup (C_{m+1}^j < \ldots < \hat{C}_k^j < \ldots < C_{m+n+2}) \\ &= \sum_{j\in J'} p_j (-1)^{m+1} \sum_{k=0}^{n+1} (-1)^k a \cup (C_{m+1}^j < \ldots < \hat{C}_{k+m+1}^j < \ldots < C_{m+n+2}).\end{aligned} \tag{3.4}$$

The hat notation $\hat{C}_k^j$ means that this term does not appear in the chain, and $J' = \{j \in J \,:\, a \subseteq (C_0^j < \ldots < C_{m+n+2}^j)\}$. For $0 \leq k \leq n+1$, set $D_k^j := C_{k+m+1}^j$. Since $C_{k+m+1}^j > A$, we have $D_k^j \in \mathcal{N}(K)$ by hypothesis (iv), so that $D_k^j \cap K \neq 1$.

We use now the $\sim$ operation, which maps an a-initial chain to its subchain beginning just after $\max a$. Apply the $\sim$ operation on both sides of the equation of a-initial chains (3.2) using the expressions of (3.3) and (3.4) respectively. The left-hand-side of (3.2) becomes $q\tilde{\beta}$, where

$$\tilde{\beta} = \sum_i q_i(AB_0^i < \ldots < AB_n^i) \in \tilde{C}_*(\mathcal{A}_p(\mathcal{N}(K))).$$

The right-hand-side of (3.2) becomes $\partial\tilde{\gamma}$, with

$$\tilde{\gamma} = \sum_{j\in J'} p_j(-1)^{m+1}(D_0^j < \ldots < D_{n+1}^j) \in \tilde{C}_*(\mathcal{A}_p(\mathcal{N}(K))).$$

Now we reduce the above homology computation in $\mathcal{N}(K)$ to a calculation in $\tilde{H}_*(\mathcal{A}_p(K))$. Consider the homotopy equivalence given by the poset map $\varphi : \mathcal{N}(K) \to \mathcal{A}_p(K)$ of Lemma 2.7. Denoting by φ_* the induced map in the chain complexes, we get the following equalities in $\tilde{C}_*(\mathcal{A}_p(K))$:

$$q\beta = \varphi_*(q\tilde{\beta}) = \varphi_*(\partial(\tilde{\gamma})) = \partial(\varphi_*(\tilde{\gamma})), \tag{3.5}$$

where $\varphi_*(\tilde{\gamma}) \in \tilde{C}_{n+1}(\mathcal{A}_p(K))$. Since q is invertible, we have found that β is a boundary in the chain complex $\tilde{C}_*(\mathcal{A}_p(K))$, contradicting our initial assumption on β. $\square$

Note that Lemma 3.14 does not require H to have $(\mathcal{QD})_p$, which by contrast was fundamental in [5, Lemma 0.27]. This assumption is relaxed within the statements of (iii) and (iv).

Remark 3.15. If coefficients are taken in $\mathbb{Q}$, then the coefficient requirement in hypothesis (iii) is automatically guaranteed if $a \in \alpha$. If they are taken in $\mathbb{Z}$, then hypothesis (iii) implies that the coefficient of $a \in \alpha$ is ± 1. We may eliminate this restriction in hypothesis (iii) if we can take $\beta \in \tilde{H}_*(\mathcal{A}_p(K), \mathbb{Z})$ of order prime to the coefficient q of a in α by (3.5) (or if it is not a torsion element).

Remark 3.16. Indeed, getting hypothesis (v) in the above lemma is in general the hard part. In [5], this hypothesis is frequently obtained by applying [5, Theorem 2.4], which has certain restrictions on the prime p. One of our goals is to try to avoid restrictions on p, so we investigate other methods to get this hypothesis.

4. The reduction $O_{p'}(G) = 1$

In this section we show that if G satisfies $(\mathrm{H1})_R$ and $O_{p'}(G) \neq 1$ then G satisfies (R-QC), with $R = \mathbb{Z}$ or $\mathbb{Q}$. This reduces the study of (R-QC) to finite groups G with $O_{p'}(G) = 1$.

This is motivated by the original result [5, Proposition 1.6], stated for $p > 5$ and $R = \mathbb{Q}$. We prove a more general version of this fact by using Lemma 3.14, without those restrictions on the prime p and for $R = \mathbb{Z}$ or $\mathbb{Q}$. In the proof, we will construct a subposet X of $\mathcal{A}_p(G)$ satisfying the hypotheses of Lemma 3.14. This route is comparatively elementary in contrast with [5, Proposition 1.6], which quotes the strongly CFSG-dependent result [5, Theorem 2.4]. Our proof does not depend on that result and we will only need to quote Theorem 3.4, which uses only easy facts about coprime automorphisms of the simple groups (whereas [5, Theorem 2.4] requires much deeper details about the structure of the simple groups).

Theorem 4.1. *Suppose that G satisfies $(H1)_R$ and that $O_{p'}(G) \neq 1$. Then G satisfies (R-QC).*

Proof. Suppose that G satisfies $(\mathrm{H1})_R$ and that $O_{p'}(G) \neq 1$. Assume that $O_p(G) = 1$. Our goal is to show that $\tilde{H}_*(\mathcal{A}_p(G), R) \neq 0$, with coefficients in $R = \mathbb{Z}$ or $\mathbb{Q}$.

First, note that if $H < G$ is a proper normal subgroup such that $\mathcal{A}_p(H) \simeq \mathcal{A}_p(G)$, then $O_p(H) \leq O_p(G) = 1$ and hence, by $(\mathrm{H1})_R$, $0 \neq \tilde{H}_*(\mathcal{A}_p(H), R) \cong \tilde{H}_*(\mathcal{A}_p(G), R)$. Therefore, we can further assume that no such subgroup exists:

(H2) If $H < G$ is a proper normal subgroup then $\mathcal{A}_p(H) \not\simeq \mathcal{A}_p(G)$.

Now we head to the construction of a homotopy equivalent subposet X of $\mathcal{A}_p(G)$ to apply Lemma 3.14 and get our goal. To achieve this, we are going to deduce a series of properties on our group G which will lead to the choice of convenient subgroups H and K and the definition of X satisfying the hypotheses of Lemma 3.14. In view of Lemma 2.5, we can suppose that:

(H3) $Z(G) = 1$ and $\Omega_1(G) = G$.

Let $L := O_{p'}(G)$, which is nontrivial by hypothesis. The following claim holds by (H3).

Claim 1. $C_G(L) < G$, and hence some $A \in \mathcal{A}_p(G)$ acts faithfully on L.

Recall that for $A \in \mathcal{A}_p(G)$, we have that $O_p(LA) = C_A(L)$ since A is a Sylow p-subgroup of LA. Moreover, A acts faithfully on L if and only if $C_A(L) = 1$. Let

$$\mathcal{F} = \{A \in \mathcal{A}_p(G) \,:\, A \text{ acts faithfully on } L\},$$
$$\mathcal{N} = \{A \in \mathcal{A}_p(G) \,:\, A \text{ acts non-faithfully on } L\}.$$

The following assertion is immediate from these definitions.

Claim 2. $\mathcal{F}$ and $\mathcal{N}$ are disjoint, $\mathcal{A}_p(G) = \mathcal{F} \cup \mathcal{N}$, $\mathcal{F}$ is nonempty by Claim 1 and $\mathcal{N} = \mathcal{N}(C_G(L))$.

With an eye on the notation of Lemma 3.14, for $A \in \mathcal{F}$ let $H_A := LA$ and $K_A := C_G(H_A)$. The following claim gives some properties of H_A and K_A.

Claim 3. H_A and K_A satisfy Hypothesis 3.5, $K_A = C_{C_G(L)}(A)$ and $\mathcal{N}_{>A} = \mathcal{N}(K_A)_{>A}$.

Proof. Note that $C_G(LA) = C_{C_G(L)}(A)$ and that $H_A \cap K_A \leq Z(H_A) = Z(LA)$ is a p'-group since $O_p(LA) = C_A(L) = 1$. Therefore H_A and K_A satisfy Hypothesis 3.5.

Since $C_G(LA) \leq C_G(L)$, we have that $\mathcal{N}(K_A) \subseteq \mathcal{N}(C_G(L)) = \mathcal{N}$. Hence $\mathcal{N}(K_A)_{>A} \subseteq \mathcal{N}_{>A}$. Conversely, if $B \in \mathcal{N}_{>A}$ then $B \leq C_G(A)$ and $C_B(L) \neq 1$. Hence, $1 \neq C_B(L) = C_G(A) \cap B \cap C_G(L) = B \cap C_G(LA) = B \cap K_A$. □

Now we will see how the configuration of Lemma 3.14 brings new ideas beyond the analogous result of [5]. We show next how to get hypothesis (v) of this lemma (see Remark 3.16).

Claim 4. There is $A \in \mathcal{F}$ with $O_p(K_A) = 1$.

Proof. If for all $A \in \mathcal{F}$ we have $O_p(C_{C_G(L)}(A)) = O_p(K_A) \neq 1$, then, by Claim 2 and Lemma 2.9, we get $\mathcal{A}_p(G) \simeq \mathcal{A}_p(C_G(L))$. This contradicts (H2) since $C_G(L)$ is normal in G. □

Next, we define the subposet X by removing points of $\mathcal{A}_p(G)$ with contractible link, so we preserve the homotopy type. By Claim 4, we can take $A \in \mathcal{F}$ of maximal p-rank subject to $O_p(K_A) = 1$. Let $X = \mathcal{A}_p(G) - \mathcal{F}_{>A} = \mathcal{N} \cup (\mathcal{F} - \mathcal{F}_{>A})$. Note that $\mathcal{N} \cap (\mathcal{F} - \mathcal{F}_{>A}) = \emptyset$ by Claim 2.

Claim 5. If $B \in \mathcal{A}_p(G) - X$ then $X_{>B} = \mathcal{N}_{>B}$ is contractible. In particular, $X \simeq \mathcal{A}_p(G)$.

Proof. If $B \in \mathcal{A}_p(G) - X = \mathcal{F}_{>A}$ then, by the consequences above, $X_{>B} = \mathcal{N}_{>B}$ since $\mathcal{F}_{>B} \subseteq \mathcal{F}_{>A}$ and $\mathcal{A}_p(G)_{>B} = \mathcal{F}_{>B} \cup \mathcal{N}_{>B}$. It follows from Lemma 2.8 that

$$X_{>B} = \mathcal{N}_{>B} \simeq \mathcal{A}_p(C_{C_G(L)}(B)) = \mathcal{A}_p(K_B) \simeq *$$

since $O_p(K_B) \neq 1$. Finally, by Proposition 2.2, $X \simeq \mathcal{A}_p(G)$. □

Claim 6. $\mathcal{A}_p(LA) \subseteq X$.

Proof. Let $B \in \mathcal{A}_p(LA)$. Since A is a Sylow p-subgroup of LA, there exists $g \in L$ such that $B \leq A^g$, so $C_B(L) \leq C_{A^g}(L) = (C_A(L))^g = 1$. Therefore B is faithful on L, that is, $B \in \mathcal{F}$, and $|B| \leq |A|$. Hence $B \notin \mathcal{F}_{>A}$, which means that $B \in X$. □

We check the hypotheses of Lemma 3.14 with $H = H_A = LA$, $K = K_A = C_G(LA)$ and the subposet X defined above.

(i) It holds by Claim 2.
(ii) If $B \notin X$, then B acts faithfully on L, so $1 = C_B(L) \geq C_B(LA) = B \cap K$. In consequence, $\mathcal{N}(K) \subseteq X$ and X is an $\mathcal{N}_K$-superset.
(iii) By Theorem 3.4 applied to $H = LA$, we can pick a non-zero element $\alpha \in \tilde{H}_m(\mathcal{A}_p(H), R)$, where $m = m_p(A) - 1$. Since $\tilde{Z}_m(\mathcal{A}_p(H)) = \tilde{H}_m(\mathcal{A}_p(H), R)$, α is actually a cycle, and by a dimension argument, it involves a full chain a. Since A is a Sylow p-subgroup of H, after conjugating α, we may suppose that $A \in a$. Moreover, by Claim 6 $\tilde{C}_*(\mathcal{A}_p(H)) \subseteq \tilde{C}_*(X)$.
The coefficient of a in α is invertible if $R = \mathbb{Q}$. For $R = \mathbb{Z}$, this is also true, but it is less immediate and depends on the results of [7]. See below for further details.
(iv) By (iii) a is a full chain, $A = \max a$ and $X_{>A} = \mathcal{N}(K_A)_{>A}$ by Claims 2 and 3 and the consequences above.
(v) It holds by $(\mathrm{H1})_R$ since $O_p(K_A) = 1$ by the choice of A satisfying Claim 4, and $K_A = C_G(LA) \leq C_G(L) < G$ by Claim 1.

By Claim 5 and Lemma 3.14, $\tilde{H}_*(\mathcal{A}_p(G), R) \cong \tilde{H}_*(X, R) \neq 0$.

We explain now how to obtain the invertible coefficient for a in α if $R = \mathbb{Z}$, in order to fulfil hypothesis (iii) of Lemma 3.14. In the proof of (iii) above, we begin by fixing the cycle α, and then we choose a chain $a \in \alpha$, which is always a full chain since α lies in the top degree chain group. Hence, we need to show that for some $a \in \alpha$ its coefficient is equal to ± 1 (see Remark 3.15). This is possible by using the explicit description of a nontrivial cycle that Díaz Ramos gave for the p-solvable case in [7]. It follows from the proofs of [7, Theorems 5.1, 5.3 & 6.6].

This concludes the proof of the theorem for both versions of the conjecture. □

We give below some applications of Theorem 4.1. The following results depend on the results obtained on the fundamental group of the p-subgroup posets [12] and the almost-simple case of the conjecture [4].

Theorem 4.2 (cf. [12, Theorem 5.2])**.** *If G is not an almost-simple group and $O_{p'}(G) = 1$, then $\pi_1(\mathcal{A}_p(G))$ is a free group.*

Theorem 4.3 ([4])**.** *If G is an almost-simple group, then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.*

Corollary 4.4. *Suppose that G satisfies $(H1)_R$, with $R = \mathbb{Z}$ or $\mathbb{Q}$, and that $\mathcal{A}_p(G)$ is not simply connected. Then G satisfies (R-QC).*

Proof. Suppose that $O_p(G) = 1$. The result clearly holds if $\mathcal{A}_p(G)$ is not connected, since $\tilde{H}_0(\mathcal{A}_p(G), \mathbb{Z})$ is a non-zero free group in that case. If $O_{p'}(G) \neq 1$ then G satisfies (R-QC) by Theorem 4.1. On the other hand, if G is almost-simple then we are done by Theorem 4.3.

Therefore we can assume that $\mathcal{A}_p(G)$ is connected, $O_{p'}(G) = 1$ and that G is not almost-simple. By Theorem 4.2, $\pi_1(\mathcal{A}_p(G))$ is a free group, and since $\mathcal{A}_p(G)$ is connected but not simply connected, we also have that $\pi_1(\mathcal{A}_p(G)) \neq 1$. Finally, by the Hurewicz isomorphism in the first homology group, we conclude that $\tilde{H}_1(\mathcal{A}_p(G), \mathbb{Z})$ is a non-zero free group. Hence, G satisfies (R-QC). □

Now we prove Theorem 2.

Proof of Theorem 2. We only need to prove that if the original conjecture (QC) holds for all finite groups, then the integer homology version ($\mathbb{Z}$-QC) holds.

Let G be a group with $O_p(G) = 1$. We shall prove that $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Z}) \neq 0$. By induction, we can assume that ($\mathbb{Z}$-QC) holds for every group H with $|H| < |G|$, so G satisfies $(\mathrm{H1})_{\mathbb{Z}}$. By Corollary 4.4 we can also suppose that $\mathcal{A}_p(G)$ is simply connected. Since G satisfies (QC), some of its homotopy groups are nontrivial, so by the Hurewicz theorem $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Z}) \neq 0$. □

We close this section by using the rational version of Theorem 4.1 to extend some results, of [14] on the integer conjecture ($\mathbb{Z}$-QC), to the rational conjecture ($\mathbb{Q}$-QC). Below we recall some of the main results of [14].

Theorem 4.5 ([14, Corollary 3.3])**.** *Suppose that $\mathcal{K}(\mathcal{S}_p(G))$ is homotopy equivalent to a 2-dimensional and G-invariant subcomplex. Then G satisfies ($\mathbb{Z}$-QC).*

Corollary 4.6 ([14, Corollary 3.4])**.** *The integer Quillen conjecture ($\mathbb{Z}$-QC) holds for groups of p-rank at most 3.*

Corollary 4.7. *Suppose that G satisfies $(H1)_{\mathbb{Q}}$ and that $\mathcal{K}(\mathcal{S}_p(G))$ is homotopy equivalent to a 2-dimensional G-invariant subcomplex. Then G satisfies ($\mathbb{Q}$-QC).*

Proof. Suppose that $O_p(G) = 1$. We show that $\mathcal{A}_p(G)$ is not $\mathbb{Q}$-acyclic. By Lemma 2.5 and Theorems 4.1 and 4.3, we may further assume that $Z(G) = 1$, $O_{p'}(G) = 1$ and that G is not an almost-simple group.

On the other hand, by Theorem 4.5, $\mathcal{A}_p(G)$ is not $\mathbb{Z}$-acyclic. In order to prove that it is not $\mathbb{Q}$-acyclic, it is enough to show that $\mathcal{A}_p(G)$ has free abelian homology. Since it has the homotopy type of a 2-dimensional complex $K \subseteq \mathcal{K}(\mathcal{S}_p(G))$, we only need to verify that $H_n(\mathcal{A}_p(G), \mathbb{Z}) \cong H_n(K, \mathbb{Z})$ is a free abelian group for $n = 0, 1, 2$.

Clearly $H_2(K, \mathbb{Z})$ and $H_0(K, \mathbb{Z})$ are free abelian groups. Finally, by Theorem 4.2, $\pi_1(\mathcal{A}_p(G))$ is a free group, so its abelianization $H_1(K, \mathbb{Z})$ is a free abelian group. This completes the proof. □

We can extend Corollary 4.6 on the p-rank 3 case of ($\mathbb{Z}$-QC), to the rational version ($\mathbb{Q}$-QC).

Corollary 4.8. *The rational Quillen's conjecture ($\mathbb{Q}$-QC) holds for groups of p-rank at most 3.*

5. Particular cases

In this section we prove Theorem 5.1, which allows us to eliminate components isomorphic to $\mathrm{L}_2(2^3)$ ($p = 3$), $\mathrm{U}_3(2^3)$ ($p = 3$) and $\mathrm{Sz}(2^5)$ ($p = 5$) in a minimal counterexample to Quillen's conjecture. These simple groups were excluded as possible components during the analysis of the conjecture of Aschbacher-Smith [5], due to the particular structure of the centralizers of their field automorphisms of order p. Namely, these centralizers have nontrivial normal p-subgroups. However, we will see that, because of this property, we can establish Quillen's conjecture for groups containing these types of components.

And since $\mathrm{Sz}(2^5)$ was the only obstruction in [5] for $p = 5$, in particular we get their Main Theorem also for $p = 5$. For that purpose, we will use the structure of these centralizers, which can be found in [8]. Recall that $R = \mathbb{Z}$ or $\mathbb{Q}$.

Theorem 5.1. *Suppose that G satisfies $(H1)_R$ and that G contains a component L such that $L/Z(L)$ is isomorphic to $\mathrm{L}_2(2^3)$ ($p = 3$), $\mathrm{U}_3(2^3)$ ($p = 3$) or $\mathrm{Sz}(2^5)$ ($p = 5$). Then G satisfies (R-QC).*

We summarize next the scheme of the proof of [5, Main Theorem] to see why these cases were excluded and to expose the key points to extend their theorem to every odd prime p. In particular, we include the proof of Corollary 3 during the discussion.

Let p be an odd prime, and G a group of minimal order subject to failing ($\mathbb{Q}$-QC). Assume further that if $\mathrm{U}_n(q)$ is a component of G, with $q \equiv -1 \pmod p$ and q odd, then the p-extensions of $\mathrm{U}_m(q^{p^e})$ satisfy $(\mathcal{QD})_p$ for all $m \leq n$ and $e \in \mathbb{Z}$. In particular G satisfies $(\mathrm{H1})_{\mathbb{Q}}$. The proof splits into three steps.

Step 1. We get $O_{p'}(G) = 1$ (this is [5, Proposition 1.6]).

To prove $O_{p'}(G) = 1$ in [5], Theorems 2.3 and 2.4 there are invoked. However, these theorems, stated for p odd, require that G does not contain components isomorphic to $\mathrm{L}_2(2^3)$, $\mathrm{U}_3(2^3)$ or $\mathrm{Sz}(2^5)$ with $p = 3, 3, 5$ respectively. By Theorem 4.1, we can get the same reduction over G since $(\mathrm{H1})_{\mathbb{Q}}$ holds, without those restrictions on p and the components.

Step 2. If L is a component of G, then some p-extension of L fails $(\mathcal{QD})_p$ (this is [5, Proposition 1.7]).

Similarly as in Step 1, Theorems 2.3 and 2.4 of [5] are invoked here. By Theorem 5.1, G does not contain components isomorphic to $\mathrm{L}_2(2^3)$, $\mathrm{U}_3(2^3)$ or $\mathrm{Sz}(2^5)$ with $p = 3, 3, 5$ respectively. Therefore, we can invoke Theorems 2.3 and 2.4 of [5] without those restrictions on p, so this step of the proof also extends to $p \geq 3$. The hypothesis on the unitary components also allows us to conclude that G does not contain unitary components $\mathrm{U}_n(q)$ with $p \mid q + 1$ and q odd.

Step 3. $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$ since $\tilde{\chi}(\mathcal{A}_p(G)^g) \neq 0$ for some $g \in G$.

First, note that if $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) = 0$ then $\tilde{\chi}(\mathcal{A}_p(G)^g) = 0$ for all $g \in G$, by the Lefschetz fixed point theorem. This step consists then on deriving a contradiction by showing that $\tilde{\chi}(\mathcal{A}_p(G)^g) \neq 0$ for some $g \in G$ (see [5, p.490]). This is Robinson's method [17]. The idea is to look for a subgroup $Q = \langle g \rangle \times O_2(Q) \leq G$ with $O_2(Q) \neq 1$, for which $\tilde{\chi}(\mathcal{A}_p(G)^g) \equiv \tilde{\chi}(\mathcal{A}_p(G)^Q) \pmod 2$ is non-zero. To construct such subgroup Q, [5, Theorem 5.3] is invoked, which is stated for $p \geq 5$. Therefore, we can extend this step to $p = 5$ and, consequently, [5, Main Theorem] to $p \geq 5$, but not to $p \geq 3$. This proves Corollary 3. See also the comment below [5, Main Theorem] and the remark at the bottom of [5, p.493].

The extension of [5, Main Theorem] to $p = 3$ will be treated in a forthcoming work. Roughly, if we want to construct such subgroup Q for $p = 3$ in Step 3, we need first to eliminate some possible components of G, as we are doing in Theorem 5.1.

The problem with the components $L \cong \mathrm{L}_2(2^3)$ ($p = 3$), $\mathrm{U}_3(2^3)$ ($p = 3$) or $\mathrm{Sz}(2^5)$ ($p = 5$), is that their p-extensions do not have nonconical complements, as defined in [5, Theorem 2.3], when they contain field automorphisms of order p. This implies that we cannot invoke [5, Theorem 2.4], which is needed to get hypothesis (v) of Lemma 3.14. Moreover, a p-extension LA satisfies $(\mathcal{QD})_p$ if and only if A does not contain field automorphisms. In particular, if G does not contain field automorphisms of order p of these components, we can invoke [5, Theorem 2.4] in Step 2 of the proof of [5, Main Theorem] and eliminate these type of components. In consequence, if we want to proceed as in [5] and eliminate these components from a minimal counterexample G to Quillen's conjecture, we only need to analyse the case that G contains a p-extension LA with A including field automorphisms of order p.

The obstruction described above relies on the fact that $O_p(C_L(A)) \neq 1$ for every A inducing field automorphisms of order p on such L. Nevertheless, this special feature is exactly the ingredient we want to construct a proper subposet $X \subset \mathcal{A}_p(G)$ suitable for use as X in Lemma 3.14. This key property allows us to extract these kind of subgroups A from the poset $\mathcal{A}_p(G)$ and apply Lemma 3.14 with $H = L$.

Before we proceed with the proof of Theorem 5.1, we describe some preliminary results that will be repeatedly used from now on.

Remark 5.2. With the aim of showing that $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$, we will look for a subposet $X \subseteq \mathcal{A}_p(G)$ and subgroups H, K to fit in the hypotheses of Lemma 3.14. We will also require X homotopy equivalent to $\mathcal{A}_p(G)$. Indeed, we will construct X, H and K from a very particular component L of G. In general, we will pick $H = LA$ and $K = C_G(LA)$, where $A \leq N_G(L)$ is faithful on the component L. Our first observation is that:

(0) We will choose $A \in \mathcal{A}_p(G)$ such that $A \cap L \neq 1$, so $A \leq N_G(L)$ if $Z(L)$ is a p'-group.

Note that if $A \in \mathcal{A}_p(G)$ and $1 \neq A \cap L$, then $1 \neq A \cap L \leq L \cap L^a$ for all $a \in A$. When $Z(L)$ is a p'-group, this forces $L = L^a$ and $A \leq N_G(L)$.

(1) In proving ($\mathbb{Q}$-QC), we will assume $O_p(G) = 1$. Hence $O_p(L) = 1$, so $Z(L)$ is a p'-group. Therefore, when $H = L$ we get hypothesis (i) of Lemma 3.14.

(2) In proving ($\mathbb{Q}$-QC), we can also assume that $(\mathrm{H1})_{\mathbb{Q}}$, e.g. by studying a minimal counterexample.

(3) With $(\mathrm{H1})_{\mathbb{Q}}$, by Theorem 4.1 we have the reduction $O_{p'}(G) = 1$. Hence, with $O_p(G) = 1$, we get

$$Z(E(G)) \leq Z(F^*(G)) = C_G(F^*(G)) \leq F(G) \leq O_p(G)O_{p'}(G) = 1.$$

Therefore, by using (2), that is $(\mathrm{H1})_{\mathbb{Q}}$, for $H = L$ and $K = C_G(L)$ we get hypothesis (v) of Lemma 3.14. Note that in general, if L is a component of G and $F(G) = 1$, $O_p(C_G(L)) = 1$ and $Z(L) = 1$.

To conclude with this remark, we provide some possible paths to use Lemma 3.14.

Case 1. $H = L$, $K = C_G(L)$.

In particular we automatically get (i) and (v) of Lemma 3.14 by (1) and (3) above.

Subcase 1a. $X = \mathcal{A}_p(G)$.

Here we get (a) and (b) of Lemma 3.14, so we only need to establish (c) of that statement. That is, take A exhibiting $(\mathcal{QD})_p$ for L with $\mathcal{A}_p(G)_{>A} \subseteq A \times K$.

This will hold for example in Case 1 of the proof of Theorems 5.1 and 6.1. There we do get $(\mathcal{QD})_p$ for L, and "no outers" says roughly that any A maximal in L is in fact maximal-faithful on L, so every $E \in \mathcal{A}_p(G)_{>A}$ has the form $E = AC_E(L) \subseteq A \times K$.

Subcase 1b. $X \subsetneq \mathcal{A}_p(G)$.

Here we will have more work to do, e.g. by establishing conditions (ii), (iii) and (iv) of Lemma 3.14, and of course showing that X and $\mathcal{A}_p(G)$ are homotopy equivalent.

Case 2. $H = LA > L$, $K = C_G(LA)$.

In this case, we will typically begin by constructing H as LE, where E does not contain inner automorphisms of L, and later we will choose $A \in \mathcal{A}_p(LE)$ with $LA = LE$.

Denote by $\mathrm{Inn}(G)$ the subgroup of $\mathrm{Aut}(G)$ of inner automorphisms of G. If $\phi \in \mathrm{Aut}(G) - \mathrm{Inn}(G)$, we say that ϕ *induces an outer automorphism* on G. If $L \leq G$ and $E \leq N_G(L)$, then we can describe the types of automorphisms (inner or outer) induced by the action of E on L via the map $E \to \mathrm{Aut}(L)$.

Lemma 5.3. *Let $L \leq G$ and $E \leq N_G(L)$. Then*

$$E \cap (LC_G(L)) = \{x \in E \,:\, x \text{ induces an inner automorphism on } L\}.$$

In particular, $E \cap (LC_G(L)) = 1$ if and only if E acts by outer automorphisms on L.

Proof. Clearly $E \cap (LC_G(L))$ acts by inner automorphisms on L. If $x \in E$ induces an inner automorphism on L, then there exists $y \in L$ such that $z = y^{-1}x$ acts trivially on L. Therefore, $z \in C_G(L)$ and $x = yz \in LC_G(L)$. □

Remark 5.4. If $H \leq G$ and $m_p(H) = m_p(G) = m$, then we have the inclusion $\tilde{H}_{m-1}(\mathcal{A}_p(H)) \subseteq \tilde{H}_{m-1}(\mathcal{A}_p(G))$. In particular, if H has $(\mathcal{QD})_p$ then so does G.

On the other hand, if $L = L_1 \times \ldots \times L_n$ is a direct product and each L_i has $(\mathcal{QD})_p$ then L has $(\mathcal{QD})_p$. This follows from Proposition 2.3 and the homology decomposition of a join.

The following lemma follows from the p-rank 2 case of the conjecture [16, Proposition 2.10].

Lemma 5.5. *Suppose that $O_p(G) = 1$. If $m_p(G) = 1$, or $m_p(G) = 2$ with $\mathcal{A}_p(G)$ connected, then G has $(\mathcal{QD})_p$.*

Proof. By [16, Proposition 2.10], $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$. If $m_p(G) = 1$ then clearly we have $\tilde{H}_0(\mathcal{A}_p(G)) \neq 0$. If $m_p(G) = 2$ and $\mathcal{A}_p(G)$ is connected then $\tilde{H}_0(\mathcal{A}_p(G)) = 0$, so $\tilde{H}_1(\mathcal{A}_p(G)) \neq 0$. □

Now we prove Theorem 5.1.

Proof of Theorem 5.1. We prove the rational version ($\mathbb{Q}$-QC). That is, we show that $O_p(G) = 1$ implies $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$. The same proof works for the integer version ($\mathbb{Z}$-QC) by Remark 6.2.

As we are proving ($\mathbb{Q}$-QC), we have its hypothesis $O_p(G) = 1$. Let L be a component of G as in the hypotheses of the theorem. The preliminary Remark 5.2 shows then that we can suppose that $O_{p'}(G) = 1$, so $Z(L) = 1$ and $L \cong \mathrm{L}_2(2^3)$, $\mathrm{U}_3(2^3)$ or $\mathrm{Sz}(2^5)$, with $p = 3$, 3 or 5 respectively. We use the structure of the centralizers of the automorphisms of L. We refer to (7-2), (9-1), (9-3) of [8, Part I] for more details on the following assertions.

- If $L = \mathrm{L}_2(2^3)$ then $\mathrm{Aut}(L) \cong L \rtimes \langle \phi \rangle$, where ϕ induces a field automorphism of order 3 on L, and $C_L(\phi) \cong \mathrm{L}_2(2) \cong \mathbb{S}_3 \cong C_3 \rtimes C_2$. Note that $m_p(L) = 1$.
- If $L = \mathrm{U}_3(2^3)$ then $\mathrm{Out}(L) = \mathrm{Out\ diag}(L) \rtimes C_6$ and $\mathrm{Out\ diag}(L) \cong C_3$. If $\phi \in \mathrm{Aut}(L)$ is a field automorphism of order 3, $C_{\mathrm{Inn\ diag}(L)}(\phi) = C_L(\phi) \cong \mathrm{PGU}_3(2) \cong ((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$.

(A) In particular, field automorphisms and diagonal automorphisms do not commute, so any purely outer automorphism group of $\mathrm{U}_3(2^3)$ has 3-rank at most 1.

(B) Note that $m_p(L) = 2$ and $\mathcal{A}_p(L)$ is connected since it does not appear in the disconnected list of Appendix Theorem A.1.

- If $L = \mathrm{Sz}(2^5)$ then $\mathrm{Aut}(L) \cong L \rtimes \langle \phi \rangle$, where ϕ induces a field automorphism of order 5 on L, and $C_L(\phi) \cong \mathrm{Sz}(2) \cong C_5 \rtimes C_4$. Note that $m_p(L) = 1$.

In any case, L satisfies $(\mathcal{QD})_p$ by Lemma 5.5, and if $\phi \in N_G(L)$ induces a field automorphism of order p on L then $O_p(C_L(\phi)) \neq 1$. Let $\mathcal{N} = \{E \in \mathcal{A}_p(N_G(L)) : E \cap (LC_G(L)) \neq 1\}$.

We are going to use some of the claims stated in Remark 5.2. In particular, we get $O_p(C_G(L)) = 1$ by item (3) of that remark.

We split the remaining of the proof in two cases.

Case 1: $\mathcal{A}_p(N_G(L)) = \mathcal{N}$.

We are in Subcase 1a of Remark 5.2. That is, we take $X = \mathcal{A}_p(G)$, $H = L$ and $K = C_G(L)$, which satisfy hypothesis (v), (a) and (b) of Lemma 3.14.

Recall that we observed below the bullet items that $H = L$ satisfies $(\mathcal{QD})_p$, which is exhibited by some $A \in \mathcal{A}_p(L)$. The hypothesis $\mathcal{A}_p(N_G(L)) = \mathcal{N}$ of this case implies that there is no $E \in \mathcal{A}_p(G)$ inducing only outer automorphisms on L. Therefore, if $E \in \mathcal{A}_p(G)_{>A}$ then $E = AC_E(L) \subseteq A \times K$. This shows that condition (c) of Lemma 3.14 holds, and therefore we conclude that $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$ in this case.

Case 2: $\mathcal{A}_p(N_G(L)) \neq \mathcal{N}$.

Here, every $E \in \mathcal{A}_p(N_G(L)) - \mathcal{N}$ induces outer automorphisms on L (see Lemma 5.3). In conjunction with item (A) of our above remark on the unitary group $\mathrm{U}_3(2^3)$, we conclude that $|E| = p$.

Next, we show that if E induces field automorphisms on L then $\mathcal{A}_p(G)_{>E}$ is contractible. Recall that the existence of field automorphisms of order p in G for these types of componentes is the real obstruction in [5] to handle these components. The idea here is to extract then these elements E from $\mathcal{A}_p(G)$ to get a homotopy equivalent subposet X which will be more suitable to apply Lemma 3.14 (see also Subcase 1b of Remark 5.2).

Claim. If $E \in \mathcal{A}_p(N_G(L))$ is a group of purely outer automorphisms of L containing field automorphisms of order p, then $\mathcal{A}_p(G)_{>E}$ is contractible.

Proof. Let E be as in the hypotheses of the claim. Therefore $|E| = p$ by the above paragraph, and also $O_p(C_L(E)) \neq 1$ by the discussion in the bullet points. We show that $\mathcal{A}_p(G)_{>E}$ is contractible by exhibiting a sequence of homotopy equivalences.

Take $M = C_L(E)C_G(LE)$. Since E induces only outer automorphisms on L and M contains only inners, we have

$$E \cap M = 1. \tag{5.1}$$

We will prove that

$$\mathcal{A}_p(G)_{>E} \simeq Y \simeq \mathcal{A}_p(M) \simeq *,$$

where

$$Y := \{B \in \mathcal{A}_p(G) : N_B(L) > E\} \subseteq \mathcal{A}_p(G)_{>E}.$$

In order to prove that $\mathcal{A}_p(G)_{>E} \simeq Y$, we show that $Y_{>B}$ is contractible for each $B \in \mathcal{A}_p(G)_{>E} - Y$, and then conclude that the inclusion $Y \hookrightarrow \mathcal{A}_p(G)_{>E}$ is a homotopy equivalence by Proposition 2.2. To that aim, we construct a homotopy equivalence $Y_{>B} \simeq \mathcal{A}_p(C_M(B))$, and then we prove that $O_p(C_M(B)) \neq 1$.

Fix $B \in \mathcal{A}_p(G)_{>E} - Y$ and take $C \in Y_{>B}$. We prove that $N_C(L) \cap M \neq 1$, and in particular $C \cap M \neq 1$. If $N_C(L)$ contains inner automorphisms of L or acts non-faithfully on L, then $N_C(L) \cap M \neq 1$ by Lemma 5.3. If $N_C(L)$ acts faithfully on L and it does not contain inner automorphisms of L, then $N_C(L)$ embeds into both $\mathrm{Aut}(L)$ and $\mathrm{Out}(L)$, and it has p-rank at least 2 since $E < N_C(L)$. However, $N_C(L)$ can only contain field automorphisms of L since diagonal and field automorphisms (of order p) do not commute (see item (A) above). On the other hand, a subgroup of $\mathrm{Out}(G)$ containing only field automorphisms of L is cyclic of order p, showing that $N_C(L)$ has p-rank at most 1. We have a well-defined homotopy equivalence $C \in Y_{>B} \mapsto C \cap M \in \mathcal{A}_p(C_M(B))$. The inverse is given by $C \in \mathcal{A}_p(C_M(B)) \mapsto CB$. Note that $CB \in Y_{>B}$ since $E \cap C \leq E \cap M = 1$ by (5.1), so

$$CB \geq N_{CB}(L) \geq EC > E.$$

Therefore, $Y_{>B} \simeq \mathcal{A}_p(C_M(B))$.

Next, we prove that $Y_{>B}$ is contractible by showing that $O_p(C_M(B)) \neq 1$. Decompose $B = E\tilde{B}$, where $\tilde{B}$ is a complement to E in B. Since $N_B(L) = E$, $\tilde{B}$ acts regularly on the set $\{L^b : b \in \tilde{B}\}$. Let $K = \langle L^b : b \in \tilde{B}\rangle$. It is not hard to see that $C_K(B) \cong C_L(E)$. Finally, observe that $C_K(B)$ is a normal subgroup of $C_M(B)$, so $O_p(C_M(B)) \neq 1$. Therefore, $Y_{>B} \simeq \mathcal{A}_p(C_M(B))$ is contractible. By Proposition 2.2, $Y \hookrightarrow \mathcal{A}_p(G)_{>E}$ is a homotopy equivalence.

By taking $B = 1$ in the above reasoning, $Y \simeq \mathcal{A}_p(M)$ is contractible since $1 \neq O_p(C_L(E)) \leq O_p(C_L(E)C_G(LE)) = O_p(M)$. In consequence, $\mathcal{A}_p(G)_{>E} \simeq Y$ is contractible. This finishes the proof of this claim. $\square$

Now we construct our subposet X homotopy equivalent to $\mathcal{A}_p(G)$, which is roughly obtained from $\mathcal{A}_p(G)$ by removing any $E \in \mathcal{A}_p(N_G(L))$ faithful on L and containing some field automorphism of order p. To this end, we will actually concatenate two homotopy equivalences. In particular, if there are no field automorphisms of order p of L in $N_G(L)$, then $X = \mathcal{A}_p(G)$.

We construct the first homotopy equivalence by extracting the purely outers that contain field automorphisms of L. That is, in combination with Proposition 2.2 and the above claim, the subposet

(5.2)
$$X_0 = \{E \in \mathcal{A}_p(G) \,:\, \text{if } |E| = p \text{ then it does not induce field automorphisms on } L\}$$

is homotopy equivalent to $\mathcal{A}_p(G)$.

For the second homotopy equivalence, we extract the remaining elementary abelian p-subgroups acting faithfully on L and containing some field automorphisms. Let $W = \{E \in \mathcal{A}_p(N_G(L)) \,:\, C_E(L) = 1$ and E contains some field automorphism of $L\}$ and

$$X = \mathcal{A}_p(G) - W.$$

If $E \in X_0 - X$, then E is faithful on L and it contains some field automorphisms of order p of L, but $|E| > p$ since $E \in X_0$. On the other hand, since field and diagonal automorphisms do not commute in the unitary case by item (A) above, we see that E contains inner automorphisms of L. That is, $1 \neq E \cap (LC_G(L))$ by Lemma 5.3, and, moreover, $|E : E \cap (LC_G(L))| = p$. Hence, $X_{<E} = \mathcal{A}_p(E \cap (LC_G(L))) \simeq *$. By Proposition 2.2, $X \simeq X_0 \simeq \mathcal{A}_p(G)$.

Note that if we do not have field automorphisms of order p in $N_G(L)$ but we are in Case 2, then we have no points to remove from $\mathcal{A}_p(G)$ and $X = \mathcal{A}_p(G)$. Moreover, the only possibility here is $L = \mathrm{U}_3(2^3)$ and $p = 3$, with $N_G(L)$ containing diagonal automorphisms of order 3 since $\mathcal{A}_p(N_G(L)) \neq \mathcal{N}$. Since both Inn diag($\mathrm{U}_3(2^3)$) and $\mathrm{U}_3(2^3)$ have 3-rank 2 and the latter satisfies $(\mathcal{QD})_p$, by Remark 5.4 Inn diag($\mathrm{U}_3(2^3)$) also satisfies $(\mathcal{QD})_p$, exhibited by cycles of $\mathcal{A}_3(\mathrm{U}_3(2^3))$.

In any scenario, to conclude the proof of Case 2, apply Lemma 3.14 with the subposet X obtained above, $H = L$, $K = C_G(L)$, and $a \in \mathcal{A}_p(L)'$ any chain exhibiting $(\mathcal{QD})_p$ for L. Note that we get the inclusion $X_{>\max a} \subseteq \mathcal{N}_K$ since we removed the elements of $\mathcal{A}_p(G)$ that contain field automorphisms of L, and $\max a \leq L$ has p-rank 2 and it is maximal-faithful on L. □

6. Components of p-rank 1

In this section we show that, under $(\mathrm{H1})_R$, if G has a component of p-rank 1 then G satisfies (R-QC), with $R = \mathbb{Z}$ or $\mathbb{Q}$ (see Theorem 6.1 below). We refer to (7-13) of [8, Part I] for the main properties on simple groups of p-rank 1. Recall that there are no simple groups of 2-rank 1.

Theorem 6.1. *Suppose that G satisfies $(H1)_R$ and that it contains a component L such that $L/Z(L)$ has p-rank 1. Then G satisfies (R-QC).*

Proof. We proceed similarly to Theorem 5.1, so following Remark 5.2 we can suppose that $O_p(G) = 1 = O_{p'}(G)$, and we get $O_p(C_G(L)) = 1$ and $Z(L) = 1$. Hence L is a simple group of p-rank 1 and p is odd. Recall also that if $B \in \mathcal{A}_p(G)$ and $B \cap L \neq 1$ then $B \leq N_G(L)$.

Let $\mathcal{N} = \{E \in \mathcal{A}_p(N_G(L)) : E \cap (LC_G(L)) \neq 1\}$. We split the proof in two cases.

Case 1. $\mathcal{A}_p(N_G(L)) = \mathcal{N}$.

In this case, there are no outer automorphisms of order p of L inside G, and $\Omega_1(N_G(L)) \cong L \times \Omega_1(C_G(L))$. Let $A \in \mathcal{A}_p(L)$. Since L has p-rank 1, A is maximal faithful on L and it represents a connected component of $\mathcal{A}_p(L)$ exhibiting $(\mathcal{QD})_p$ for L (see Lemma 5.5). Moreover, by item (0) in Remark 5.2 we get $\mathcal{A}_p(G)_{>A} \subseteq A \times C_G(L)$. The hypotheses of Lemma 3.14 are verified with $H = L$, $K = C_G(L)$, $a = (A) \in \alpha \in \tilde{C}_0(\mathcal{A}_p(L))$ and $X = \mathcal{A}_p(G)$. This finishes the proof of Case 1.

In following case, we will see that in some subcases we take $H = LA > L$ and we do not remove points, so we apply Lemma 3.14 with $X = \mathcal{A}_p(G)$.

Case 2. $\mathcal{A}_p(N_G(L)) \neq \mathcal{N}$.

Note that every $E \in \mathcal{A}_p(N_G(L)) - \mathcal{N}$ induces outer automorphisms on L and has order p since $m_p(L) = 1$ (see Table 1 in Appendix A). By Theorem 5.1, we can suppose that L is not isomorphic to $\mathrm{L}_2(2^3)$ ($p = 3$) nor to $\mathrm{Sz}(2^5)$ ($p = 5$). Therefore, $m_p(LE) = 2$ and LE has $(\mathcal{QD})_p$ (i.e. it is connected, see Table 1 in Appendix A).

Case 2a. There exists $E \in \mathcal{A}_p(N_G(L)) - \mathcal{N}$ with $O_p(C_G(LE)) = 1$.

Take such element E and let $K = C_G(LE)$. Note that hypotheses (v), (a) and (b) of Lemma 3.14 hold with $X = \mathcal{A}_p(G)$, $H = LE$ and $K = C_G(LE)$. We show how to get hypothesis (c).

Pick $A \in \mathcal{A}_p(LE)$ of order p^2 exhibiting $(\mathcal{QD})_p$ for LE, and observe that $LA = LE$ and $|A \cap L| = p$. If $B \in \mathcal{A}_p(G)_{>A}$ then $B \cap L \neq 1$ and hence $B \in \mathcal{A}_p(N_G(L))$. Moreover, $C_B(L) \neq 1$ since $A \leq B/C_B(L) \leq \mathrm{Aut}(L)$, and the latter has p-rank 2. In consequence, $B = AC_B(L)$. This shows that $\mathcal{A}_p(G)_{>A} \subseteq A \times K$, and for some 1-cycle α and chain $a \in \mathcal{A}_p(LA)'$, we have $A \in a \in \alpha \in \tilde{C}_1(\mathcal{A}_p(LA))$ exhibiting $(\mathcal{QD})_p$ for H. Hence hypothesis (c) of Lemma 3.14 holds and we get $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$. This concludes the proof of Case 2a.

Case 2b. For all $E \in \mathcal{A}_p(N_G(L)) - \mathcal{N}$ we have $O_p(C_G(LE)) \neq 1$.

Here we show that the poset X defined in (6.1) below is a homotopy equivalent and proper subposet of $\mathcal{A}_p(G)$ that fits in the hypotheses of Lemma 3.14 with $H = L$ and $K = C_G(L)$.

Consider the subposets $\mathcal{F}_0 = \mathcal{A}_p(N_G(L)) - \mathcal{N}$, $\mathcal{F}_1 = \{E \in \mathcal{A}_p(N_G(L)) : |E| = p^2, C_E(L) = 1, E \cap L \neq 1\}$ and

$$X := \mathcal{A}_p(G) - \mathcal{F}_1. \tag{6.1}$$

We show that $X \simeq \mathcal{A}_p(G)$. If $E \in \mathcal{F}_1$, then $E = (E \cap L)E_0$, where $E_0 \cap (LC_G(L)) = 1$. Hence $E_0 \in \mathcal{F}_0$ and $O_p(C_G(LE)) = O_p(C_G(LE_0)) \neq 1$. Let $B \in X_{>E}$. Since $m_p(\mathrm{Aut}(L)) = 2$ and E acts faithfully on L with $B \cap L \geq E \cap L \neq 1$, we have that $B \leq N_G(L)$, $C_B(L) \neq 1$ and $B = EC_B(L)$. Therefore, $X_{>E} \simeq \mathcal{A}_p(C_G(LE))$, where the homotopy equivalence is given by $B \mapsto C_B(L)$ with inverse $C \mapsto CE$. By Proposition 2.2, $X \simeq \mathcal{A}_p(G)$.

Finally, we appeal to Lemma 3.14 on this subposet X, with $H = L$, $K = C_G(L)$ and $a = (A) \in \alpha \in \tilde{C}_0(\mathcal{A}_p(L))$, where $A \in \mathcal{A}_p(L)$. This concludes the proof of Case 2b.

In any case, we have shown that G satisfies ($\mathbb{Q}$-QC). For the integer version ($\mathbb{Z}$-QC) of this case, see Remark 6.2 below. $\square$

Remark 6.2. In the proofs of Theorems 5.1 and 6.1, we invoked Lemma 3.14 with some cycle α containing an arbitrary full chain a. Note that we could have chosen first α and then $a \in \alpha$ in these proofs. Since α is an element of either $\tilde{C}_1(X)$ or $\tilde{C}_0(X)$, it can be taken to have coefficients equal to ± 1. For example, if $\alpha \in \tilde{C}_1(X)$, then pick α to be a simple cycle (that is, a cycle in the 1-skeleton of the simplicial complex that does not self-intersect). By Remark 3.15, Theorems 5.1 and 6.1 extend to the integer version of the conjecture ($\mathbb{Z}$-QC).

7. The p-rank 4 case of Quillen's conjecture

In this section we prove Theorem 4, establishing ($\mathbb{Q}$-QC) for groups of p-rank at most 4. We use the results of the previous sections together with the classification of finite groups with a strongly p-embedded subgroup. We provide in Appendix A further details of this classification, as well as some properties of these groups. We will see that the structure of the centralizers of the simple groups of low p-rank plays a fundamental role in the proof of this theorem.

The following elementary remark will be useful in the proof of Theorem 4.

Remark 7.1. Suppose that L is a normal subgroup of G such that $Z(L)$ is a p'-group (e.g. when L is a normal component of G). If every order p element of G induces an inner automorphism on L, then $\Omega_1(G) \leq LC_G(L)$ by Lemma 5.3. In particular, $\mathcal{A}_p(G) = \mathcal{A}_p(\Omega_1(G)) \simeq \mathcal{A}_p(L) * \mathcal{A}_p(C_G(L))$ by Propositions 2.3 and 2.4.

Proof of Theorem 4. Let G be a group of p-rank at most 4 and suppose that $O_p(G) = 1$. We prove that $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$. Without loss of generality, we can assume that G satisfies $(\mathrm{H1})_{\mathbb{Q}}$, and hence that the following conditions hold:

(1) $G = \Omega_1(G)$ and $Z(G) = 1$ (by Lemma 2.5);
(2) $m_p(G) = 4$ (by Corollary 4.8);
(3) $O_{p'}(G) = 1$ (by Theorem 4.1);
(4) $F^*(G) = L_1 \ldots L_n$ is the direct product of simple components L_i of order divisible by p (by Remark 2.1);
(5) G is not an almost-simple group (by Theorem 4.3);
(6) Every component of G has p-rank at least 2 (by Theorem 6.1).

Since G is not almost-simple, $n \geq 2$. By (6), $m_p(L_i) \geq 2$ for all i, and since $4 \geq m_p(F^*(G)) \geq 2n$, we conclude that $n = 2$, $m_p(L_1) = 2 = m_p(L_2)$ and $m_p(F^*(G)) = 4$.

If both $\mathcal{A}_p(L_1)$ and $\mathcal{A}_p(L_2)$ are connected, then L_1 and L_2 have $(\mathcal{QD})_p$ by Lemma 5.5, and so does $F^*(G)$ and G by Remark 5.4. Indeed, more generally, this argument shows the following case.

Case 0. If G contains a direct product of distinct subgroups G_1, G_2 of p-rank 2 for which $\mathcal{A}_p(G_1)$ and $\mathcal{A}_p(G_2)$ are connected with $O_p(G_i) = 1$, $i = 1, 2$, then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.

In consequence, we can suppose that $\mathcal{A}_p(L_1)$ is disconnected, i.e. that L_1 has a strongly p-embedded subgroup. The possibilities for such L_1 are described in Theorem A.1 of the Appendix. In particular, if $p = 2$ then L_1 is isomorphic either to $\mathrm{L}_2(2^2) \cong \mathbb{A}_5$ or $\mathrm{U}_3(2^2)$ by a p-rank argument (see Table 1).

On the other hand, by Remark 7.1, if G has a normal component L_i then we can suppose that G contains an outer automorphism of L_i of order p, so $p \mid |\mathrm{Out}(L_i)|$. Therefore, if p is odd, both L_i are normal in G and this forces $p = 3$ and $L_1 \cong \mathrm{L}_3(2^2)$ by Table 1.

Case 1. $p = 2$ and $L_1 \cong \mathbb{A}_5$ or $\mathrm{U}_3(2^2)$.

If f is an outer involution of $\mathbb{A}_5$ (resp. $\mathrm{U}_3(2^2)$), then $C_{\mathbb{A}_5}(f) \cong \mathbb{S}_3$ of 2-rank 1, (resp. $C_{\mathrm{U}_3(2^2)}(f) \cong \mathbb{A}_5$ of 2-rank 2). Both centralizers have disconnected Quillen poset at $p = 2$. Moreover, $\mathrm{Aut}(\mathbb{A}_5) = \mathbb{S}_5 \cong \mathbb{A}_5 \rtimes C_2$ and $\mathrm{Aut}(\mathrm{U}_3(2^2)) = \mathrm{U}_3(2^2) \rtimes C_4$, with C_4 inducing field automorphisms on $\mathrm{U}_3(2^2)$.

We split the proof in two cases: when L_1, L_2 are permuted, and when they are normal in G.

Case 1a. Some involution $x \in G$ permutes L_1 with L_2.

Then $N_G(L_1) = N_G(L_2)$ and it is a normal subgroup of G of p-rank 4 with $G = N_G(L_1)T$, where $T = \langle x \rangle$. If $N_G(L_1)$ induces no outer automorphism of order 2 on L_1, then $N_G(L_1) = L_1 \times L_2$ and hence, $G \cong L_1 \wr C_2$. In this case, $\pi_1(\mathcal{A}_2(G))$ is a nontrivial free group by [12, Theorem 5.6] and therefore $\tilde{H}_1(\mathcal{A}_2(G), \mathbb{Q}) \neq 0$.

Now assume that $N_G(L_1)$ induces some outer automorphism, say f, of order 2 on L_1. This eliminates $L_1 \cong \mathrm{U}_3(2^2)$ since $C_{L_1}(f)$ has 2-rank 2, which implies $m_2(N_G(L_1)) \geq 5$. Hence $L_1 \cong \mathbb{A}_5$. Now, $\mathbb{A}_5 \langle f \rangle \cong \mathbb{S}_5$, which has 2-rank 2 and $(\mathcal{QD})_2$ (since $\mathcal{A}_2(\mathbb{S}_5)$ is connected). By Case 0 above, $N_G(L_1) = (L_1 \times L_2) \langle f \rangle$ and f induces an outer automorphism on both L_1 and L_2. Then the subposet

$$\mathfrak{i}(\mathcal{A}_2(G)) := \{E \in \mathcal{A}_2(G) \,:\, E \text{ is the intersection of maximal elements of } \mathcal{A}_2(G)\}$$

has dimension 2 (rather than dimension 3 of $\mathcal{A}_2(G)$ itself). This can be proved by using a similar argument to that of [14, Examples 4.10 & 4.11]. Finally, Corollary 4.7 applies since $\mathcal{K}(\mathrm{i}(\mathcal{A}_2(G)))$ is a G-invariant subcomplex of $\mathcal{K}(\mathcal{S}_2(G))$ and homotopy equivalent to $\mathcal{K}(\mathcal{A}_2(G))$. This finishes the proof of Case 1a.

Case 1b. L_1 is normal in G (hence L_2 is also normal in G).
By Remark 7.1, we may assume that:

(H2) Both components L_1 and L_2 admit nontrivial outer automorphisms from G.

Let $H = L_1 C_G(L_1)$, $\mathcal{N} := \mathcal{N}(H)$ and $\mathcal{F} := \mathcal{A}_2(G) - \mathcal{N}$. By (H2), $\mathcal{F}$ is nonempty, and by Lemma 2.9, we can suppose that some $E \in \mathcal{F}$ has $1 = O_2(C_H(E)) = O_2(C_{L_1}(E))O_2(C_G(L_1E))$. Note that the elements of $\mathcal{F}$ have order 2.

Subcase 1b(i). $L_1 \cong \mathbb{A}_5$.
If $E \in \mathcal{F}$ then $L_1E \cong \mathbb{S}_5$, which has $(\mathcal{QD})_2$. Fix $E \in \mathcal{F}$ with $O_2(C_G(L_1E)) = 1$ and take $A \in \mathcal{A}_2(L_1E)$ exhibiting $(\mathcal{QD})_2$ for L_1E. Then $L_1E = L_1A$ and $O_2(C_G(L_1A)) = O_2(C_G(L_1E)) = 1$. The hypotheses of Lemma 3.14 can be checked with $X = \mathcal{A}_2(G)$, $H = L_1E$, $K = C_G(L_1E)$ and A exhibiting $(\mathcal{QD})_2$ for L_1E, so $\tilde{H}_*(\mathcal{A}_2(G), \mathbb{Q}) \neq 0$. This finishes the proof of Subcase 1b(i).

Subcase 1b(ii). $L_1 \cong \mathrm{U}_3(2^2)$.
By Subcase 1b(i), we may also suppose that $L_2 \not\cong \mathbb{A}_5$. By Case 0 and (H2), we may assume that some involution $f \in G$ induces outer automorphisms on L_1 and L_2 simultaneously. Since $C_{L_1}(f)$ has 2-rank 2, we conclude that $C_{L_2}(f)$ has 2-rank 1. This forces $L_2 \cong \mathrm{L}_2(q)$, with $q \geq 5$ odd and f inducing diagonal automorphisms on L_2, by the classification of simple groups of 2-rank 2 (see [3, Theorem 48.1]). Moreover, if $\phi \in G$ is a field automorphism of L_2 then $C_{L_2}(\phi) \cong \mathrm{L}_2(q^{1/2})$ has 2-rank 2, which leads to $m_2((L_1L_2)\langle\phi\rangle) = 5$. This contradicts our main hypothesis that $m_p(G) = 4$. In conclusion, G does not contain field automorphisms of L_2 and therefore, $G \leq \mathrm{Aut}(L_1) \times \mathrm{Inn\ diag}(L_2)$.

By Theorem A.1, $\mathcal{A}_2(L_2)$ is connected. That is, L_2 has $(\mathcal{QD})_2$ exhibited by some $A \in \mathcal{A}_2(L_2)$ (of 2-rank 2). Then $O_2(C_G(L_2A)) = O_2(C_G(L_2)) = 1$ by Remark 5.2, and if $B \in \mathcal{A}_2(G)_{>A}$ then $B/C_B(L_2) \leq \mathrm{Inn\ diag}(L_2)$, which has 2-rank 2. Hence $C_B(L_2) \neq 1$ and $B = AC_B(L_2)$. By Lemma 3.14 applied to $X = \mathcal{A}_2(G)$, $H = L_2$ and $K = C_G(L_2)$, we get $\tilde{H}_*(\mathcal{A}_2(G), \mathbb{Q}) \neq 0$. This finishes the proof of Subcase 1b(ii), and hence of Case 1b and of Case 1.

Case 2. $p = 3$ and $L_1 \cong \mathrm{L}_3(2^2)$.
Note that $\mathrm{Out}(\mathrm{L}_3(2^2)) \cong D_{12} \cong C_3 \rtimes (C_2 \times C_2)$ and $\mathrm{Inn\ diag}(\mathrm{L}_3(2^2)) \cong \mathrm{L}_3(2^2) \rtimes C_3$, so without loss of generality $G \leq \mathrm{Inn\ diag}(\mathrm{L}_3(2^2)) \times \mathrm{Aut}(L_2)$. By Proposition 2.3 and the almost-simple case of the conjecture, we can suppose that G is not a direct product of almost-simple groups. Hence, there exists $C \in \mathcal{A}_3(G) - \mathcal{A}_3(L_1L_2)$ of order 3 inducing diagonal automorphisms on $L_1 \cong \mathrm{L}_3(2^2)$. Note that $L_1C \cong (L_1L_2)C/L_2 \cong \mathrm{Inn\ diag}(\mathrm{L}_3(2^2))$.

We show that C necessarily induces outer automorphisms on L_2. If C does not induce outer automorphisms on L_2, then $C \leq L_2C_G(L_2)$ and G contains the normal subgroup $\mathrm{Inn\ diag}(L_3(2^2))$. This implies that $C_G(L_2) = \mathrm{Inn\ diag}(\mathrm{L}_3(2^2))$, so G is the direct product of $\mathrm{Inn\ diag}(\mathrm{L}_3(2^2))$ by some almost-simple group $T \leq \mathrm{Aut}(L_2)$

with $F^*(T) = L_2$. This is a contradiction to our earlier statement that G is not a direct product. Therefore C also induces outer automorphisms on L_2.

Recall that $C_{L_3(2^2)}(C) \cong \mathbb{A}_5$ or $C_7 \rtimes C_3$, both of 3-rank 1. By Table 1, the poset $\mathcal{A}_3(\mathrm{Inn\ diag}(\mathrm{L}_3(2^2)))$ is connected (not simply connected) and it has dimension 1. Therefore, there exists $D \in \mathcal{A}_3(L_1C)$ of 3-rank 2 exhibiting $(\mathcal{QD})_3$ for L_1C. Note that $L_1C = L_1D$. If $O_3(C_G(L_1C)) = 1$ for some C, then $\tilde{H}_*(\mathcal{A}_3(G), \mathbb{Q}) \neq 0$ by Lemma 3.14 with $X = \mathcal{A}_3(G)$, $H = L_1D$ and $K = C_G(L_1D) = C_G(L_1C)$. Thus we can assume that $O_3(C_G(L_1C)) \neq 1$ for all C. In this case, let $H = L_1C_G(L_1)$ and $\mathcal{N} = \mathcal{N}(H)$. The subposet $\mathcal{F} := \mathcal{A}_3(G) - \mathcal{N}$ consists of order 3 subgroups acting by diagonal automorphisms on $L_1 \cong \mathrm{L}_3(2^2)$. By Lemma 2.9, $\mathcal{A}_3(G) \simeq \mathcal{A}_3(H)$, so $\tilde{H}_*(\mathcal{A}_3(G), \mathbb{Q}) = \tilde{H}_*(\mathcal{A}_3(H), \mathbb{Q}) \neq 0$ by $(\mathrm{H1})_{\mathbb{Q}}$.

This concludes the proof of the p-rank 4 case. □

Appendix A. Groups with a strongly p-embedded subgroup

In this appendix, we summarize some of the main results on the classification of the groups with a strongly p-embedded subgroup, so that the reader can consult them directly here. For further details see [2, 3, 8].

Recall that a finite group G has a *strongly p-embedded subgroup* if there exists a proper subgroup $M < G$ such that M contains a Sylow p-subgroup of G and $M \cap M^g$ is a p'-group for all $g \in G - M$. By [16, Proposition 5.2], G has a strongly p-embedded subgroup if and only if $\mathcal{A}_p(G)$ is disconnected. In the following theorem we state the classification of the groups with this property.

Theorem A.1 ([2, (6.2)]). *The finite group G has a strongly p-embedded subgroup (i.e. $\mathcal{A}_p(G)$ is disconnected) if and only if either $O_p(G) = 1$ and $m_p(G) = 1$, or $\Omega_1(G)/O_{p'}(\Omega_1(G))$ is one of the following groups:*

(1) Simple of Lie type of Lie rank 1 and characteristic p,
(2) $\mathbb{A}_{2p}$ with $p \geq 5$,
(3) $Aut(\mathrm{L}_2(2^3))$, $\mathrm{L}_3(2^2)$ or M_{11} with $p = 3$,
(4) $Aut(\mathrm{Sz}(2^5))$, ${}^2F_4(2)'$, McL, or Fi_{22} with $p = 5$,
(5) J_4 with $p = 11$.

In Table 1 we summarize some properties on the almost-simple groups listed in Theorem A.1. For more details on these assertions, see §7, §9 and §10 of [8, Part I].

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Table 1. Properties of almost-simple groups with a strongly p-embedded subgroup.

Group G	$\mathrm{Out}(G)$	$m_p(G)$	$m_p(\mathrm{Out}(G))$
p-rank 1 almost-simple groups			
G	cyclic Sylow p-subgroups	1	≤ 1
Lie type of Lie rank 1 in characteristic p			
$\mathrm{L}_2(p^a)$	$C_{\gcd(2,p^a-1)} \rtimes C_a$	a	$m_p(C_a) \leq 1$
$\mathrm{U}_3(p^a)$	$C_{\gcd(3,p^a+1)} \rtimes C_{2a}$	$\begin{cases} a & p=2 \\ 2a & p \neq 2 \end{cases}$	$m_p(C_{2a}) \leq 1$
$\mathrm{Sz}(2^a)$, $a \geq 3$ odd	C_a	a	0
$\mathrm{Ree}(3^a)$, $a \geq 3$ odd	C_a	$2a$	$m_p(C_a) \leq 1$
Alternating groups, $p \geq 5$			
$\mathbb{A}_{2p}$	C_2	2	0
$p = 3$ exceptions			
$\mathrm{Aut}(\mathrm{L}_2(2^3))$	1	2	0
$\mathrm{L}_3(2^2)$	D_{12}	2	1
M_{11}	1	2	0
$p = 5$ exceptions			
$\mathrm{Aut}(\mathrm{Sz}(2^5))$	1	2	0
${}^2F_4(2)'$	C_2	2	0
McL	C_2	2	0
Fi_{22}	C_2	2	0
$p = 11$ exception			
J_4	1	1	0

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