

ACYCLIC 2-DIMENSIONAL COMPLEXES AND QUILLEN'S CONJECTURE

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ABSTRACT. Let G be a finite group and $\mathcal{A}_p(G)$ be the poset of nontrivial elementary abelian p -subgroups of G . Quillen conjectured that $O_p(G)$ is nontrivial if $\mathcal{A}_p(G)$ is contractible. We prove that $O_p(G) \neq 1$ for any group G admitting a G -invariant acyclic p -subgroup complex of dimension 2. In particular, it follows that Quillen's conjecture holds for groups of p -rank 3. We also apply this result to establish Quillen's conjecture for some particular groups not considered in the seminal work of Aschbacher–Smith.

1. INTRODUCTION

The study of the poset $\mathcal{S}_p(G)$ of nontrivial p -subgroups of a finite group G started when K.S. Brown proved that the Euler characteristic $\chi(\mathcal{K}(\mathcal{S}_p(G)))$ of its order complex is 1 modulo the greatest power of p dividing the order of G [Bro75]. Recall that the order complex $\mathcal{K}(X)$ of a poset X is the simplicial complex whose simplices are the finite nonempty totally ordered subsets of X . Some years later, D. Quillen studied the homotopy properties of $\mathcal{K}(\mathcal{S}_p(G))$ [Qui78]. In that article, Quillen considered the subposet $\mathcal{A}_p(G)$ of nontrivial elementary abelian p -subgroups and proved that its order complex is homotopy equivalent

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to $\mathcal{K}(\mathcal{S}_p(G))$ [Qui78, Proposition 2.1]. Quillen also proved that, if the largest normal p -subgroup $O_p(G)$ of G is nontrivial, then $\mathcal{K}(\mathcal{A}_p(G))$ is contractible [Qui78, Proposition 2.4] and conjectured that the converse should hold.

In this paper we study the following version of Quillen's conjecture. Recall that the homology of a poset is the homology of its order complex.

Quillen's conjecture. *If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$.*

Aschbacher and Smith's formulation relates rational acyclicity of $\mathcal{K}(\mathcal{A}_p(G))$ with nontriviality of $O_p(G)$ [AS93b]. Thus our integral homology version is stronger than Quillen's original statement but weaker than the Aschbacher–Smith version.

Quillen proved the conjecture for solvable groups [Qui78, Theorem 12.1]. In [AS93b], M. Aschbacher and S.D. Smith made a huge progress on the study of this conjecture. By using the classification of finite simple groups, they proved that Quillen's conjecture holds if $p > 5$ and G does not contain certain unitary components. Previously, Aschbacher and Kleidman [AK90] had proved Quillen's conjecture for almost simple groups (i.e. finite groups G such that $L \leq G \leq \text{Aut}(L)$ for some non-abelian simple group L).

The main result of our paper, which depends on the classification of the finite simple groups, is the following.

Theorem 3.2. *If X is an acyclic and 2-dimensional G -invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, then $O_p(G) \neq 1$.*

Recall that the action of G on $\mathcal{S}_p(G)$ is by conjugation. The previous result provides then a convenient tool to prove that a group verifies Quillen's conjecture.

Corollary 3.3. *Let G be a finite group. Suppose that $\mathcal{K}(\mathcal{S}_p(G))$ admits a 2-dimensional and G -invariant subcomplex homotopy equivalent to itself. Then Quillen's conjecture holds for G .*

In particular, it follows that Quillen's conjecture holds for groups of p -rank 3. Recall that the p -rank of G , usually denoted by $m_p(G)$, is the maximum possible rank of an elementary abelian p -subgroup of G . The p -rank 2 case was considered by Quillen [Qui78, Proposition 2.10] and is a consequence of Serre's result: an action of a finite group on a tree has a fixed point.

In Section 4 we make an extensive use of Corollary 3.3 to establish Quillen's conjecture for some particular groups (of p -ranks 3 and 4) for which the hypotheses of the results of Aschbacher–Smith [AS93b] do not hold.

A related conjecture, due to C. Casacuberta and W. Dicks, is that a finite group acting on a contractible 2-complex has a fixed point [CD92]. This conjecture was studied by Aschbacher and Segev in [AS93a]. Posteriorly Oliver and Segev classified the groups which admit a fixed point free action on an acyclic (finite) 2-complex [OS02]. Our proof of Theorem 3.2 is built upon the results of [OS02], which depend on the classification of finite simple groups. Theorem 3.2 can also be seen as a special case of the Casacuberta–Dicks conjecture.

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2. THE RESULTS OF OLIVER AND SEGEV

In this section we review the results of [OS02] needed in the proof of Theorem 3.2. By a G -complex we mean a G -CW complex. Note that the order complex of a G -poset is always a G -complex.

Definition 2.1 ([OS02]). A G -complex X is *essential* if there is no normal subgroup $1 \neq N \triangleleft G$ such that for each $H \subseteq G$, the inclusion $X^{HN} \rightarrow X^H$ induces an isomorphism on integral homology.

The main results of [OS02] are the following two theorems.

Theorem 2.2 ([OS02, Theorem A]). *For any finite group G , there is an essential fixed point free 2-dimensional (finite) acyclic G -complex if and only if G is isomorphic to one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod{8}$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Furthermore, the isotropy subgroups of any such G -complex are all solvable.*

Theorem 2.3 ([OS02, Theorem B]). *Let G be any finite group, and let X be any 2-dimensional acyclic G -complex. Let N be the subgroup generated by all normal subgroups $N' \triangleleft G$ such that $X^{N'} \neq \emptyset$. Then X^N is acyclic; X is essential if and only if $N = 1$; and the action of G/N on X^N is essential.*

The set of subgroups of G will be denoted by $\mathcal{S}(G)$.

Definition 2.4 ([OS02]). By a *family* of subgroups of G we mean any subset $\mathcal{F} \subseteq \mathcal{S}(G)$ which is closed under conjugation. A nonempty family is said to be *separating* if it has the following three properties: (a) $G \notin \mathcal{F}$; (b) if $H' \subseteq H$ and $H \in \mathcal{F}$ then $H' \in \mathcal{F}$; (c) for any $H \triangleleft K \subseteq G$ with K/H solvable, $K \in \mathcal{F}$ if $H \in \mathcal{F}$.

For any family \mathcal{F} of subgroups of G , a (G, \mathcal{F}) -*complex* will mean a G -complex all of whose isotropy subgroups lie in \mathcal{F} . A (G, \mathcal{F}) -complex is H -*universal* if the fixed point set of each $H \in \mathcal{F}$ is acyclic.

Lemma 2.5 ([OS02, Lemma 1.2]). *Let X be any 2-dimensional acyclic G -complex without fixed points. Let \mathcal{F} be the set of subgroups $H \subseteq G$ such that $X^H \neq \emptyset$. Then \mathcal{F} is a separating family of subgroups of G , and X is an H -universal (G, \mathcal{F}) -complex.*

If G is not solvable, the separating family of solvable subgroups of G is denoted by \mathcal{SLV} .

Proposition 2.6 ([OS02, Proposition 6.4]). *Assume that L is one of the simple groups $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $G \subseteq \mathrm{Aut}(L)$ be any subgroup containing L , and let \mathcal{F} be a separating family for G . Then there is a 2-dimensional acyclic (G, \mathcal{F}) -complex if and only if $G = L$, $\mathcal{F} = \mathcal{SLV}$, and q is a power of 2 or $q \equiv \pm 3 \pmod{8}$.*

Definition 2.7 ([OS02, Definition 2.1]). For any family \mathcal{F} of subgroups of G define

$$i_{\mathcal{F}}(H) = \frac{1}{[N_G(H) : H]} (1 - \chi(\mathcal{K}(\mathcal{F}_{>H}))).$$

Lemma 2.8 ([OS02, Lemma 2.3]). *Fix a separating family \mathcal{F} , a finite H -universal (G, \mathcal{F}) -complex X , and a subgroup $H \subseteq G$. For each n , let $c_n(H)$ denote the number of orbits of n -cells of type G/H in X . Then $i_{\mathcal{F}}(H) = \sum_{n \geq 0} (-1)^n c_n(H)$.*

Proposition 2.9 ([OS02, Tables 2,3,4]). *Let G be one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod{8}$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Then $i_{\mathcal{SLV}}(1) = 1$.*

3. THE TWO-DIMENSIONAL CASE

Using the results of Oliver and Segev stated in the previous section we prove the following.

Theorem 3.1. *Every acyclic 2-dimensional G -complex has an orbit with normal stabilizer.*

Proof. If $X^G \neq \emptyset$ we are done. Otherwise, G acts fixed point freely on X . Consider the subgroup N generated by the subgroups $N' \triangleleft G$ such that $X^{N'} \neq \emptyset$. Clearly N is normal in G . By Theorem 2.3 $Y = X^N$ is acyclic (in particular it is nonempty) and the action of G/N on Y is essential and fixed point free. By Lemma 2.5 $\mathcal{F} = \{H \leq G/N : Y^H \neq \emptyset\}$ is a separating family and Y is an H -universal $(G/N, \mathcal{F})$ -complex. Thus, Theorem 2.2 asserts that G/N must be one of the groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod{8}$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. In any case, by Proposition 2.6 we must have $\mathcal{F} = \mathcal{SLV}$. By Proposition 2.9, $i_{\mathcal{SLV}}(1) = 1$. Finally by Lemma 2.8, Y must have at least one free G/N -orbit. Therefore X has a G -orbit of type G/N and we are done. \square

Theorem 3.2. *If X is an acyclic and 2-dimensional G -invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, then $O_p(G) \neq 1$.*

Proof. By Theorem 3.1 there is a simplex $\sigma = (A_0 < \dots < A_j)$ of X with stabilizer $N \triangleleft G$. Since $A_0 \triangleleft N$, we deduce that $O_p(N)$ is nontrivial. On the other hand, $N \triangleleft G$ and $O_p(N) \text{ char } N$ implies that $O_p(N) \triangleleft G$. Therefore $O_p(N) \leq O_p(G)$ and $O_p(G)$ is thus nontrivial. \square

From Theorem 3.2 we deduce:

Corollary 3.3. *Let G be a finite group. Suppose that $\mathcal{K}(\mathcal{S}_p(G))$ admits a 2-dimensional and G -invariant subcomplex homotopy equivalent to itself. Then Quillen's conjecture holds for G .*

Since the p -rank of G is equal to $\dim \mathcal{K}(\mathcal{A}_p(G)) + 1$ we obtain:

Corollary 3.4. *Let G be a finite group of p -rank 3. If $\tilde{H}_*(\mathcal{A}_p(G)) = 0$ then $O_p(G) \neq 1$.*

We now apply Corollary 3.3 to obtain results for some related p -subgroup complexes. Recall that a p -subgroup $Q \leq G$ is *radical* if $Q = O_p(N_G(Q))$. The *Bouc poset* $\mathcal{B}_p(G)$ is the poset of nontrivial radical p -subgroups of G . It is well-known that $\mathcal{K}(\mathcal{B}_p(G))$ is homotopy equivalent to $\mathcal{K}(\mathcal{S}_p(G))$ [Bou84]. Then by Corollary 3.3 we have

Corollary 3.5. *Let G be a finite group such that $\mathcal{B}_p(G)$ has height 2. If $\tilde{H}_*(\mathcal{B}_p(G)) = 0$ then $O_p(G) \neq 1$.*

We say that a poset X is a *reduced lattice* if it is obtained from a finite lattice by removing its minimum and maximum. If X is a reduced lattice, $\mathfrak{i}(X)$ denotes the subposet of X given by the elements which can be written as the infimum of a set of maximal elements of X . It is a

general fact that the order complex of $i(X)$ is homotopy equivalent to the order complex of X for any reduced lattice X [Bar11, Section 9.1]. Hence by Corollary 3.3, we have

Corollary 3.6. *Let G be a finite group. If either $i(\mathcal{S}_p(G))$ or $i(\mathcal{A}_p(G))$ has height 2, then G satisfies Quillen's conjecture.*

For a detailed account of the relations between the different p -subgroup complexes, see [Smi11].

4. SOME EXAMPLES

In this section we apply the corollaries of Theorem 3.2 to establish Quillen's conjecture for some groups constructed so that the hypotheses of the results of [AS93b] are not satisfied. The main result of [AS93b] is the following.

Theorem 4.1 (Aschbacher–Smith [AS93b, Main Theorem]). *Let G be a finite group and $p > 5$ a prime number. Assume that whenever G has a unitary component $U_n(q)$ with $q \equiv -1 \pmod{p}$ and q odd, then the Quillen dimension property at p holds for all p -extensions of $U_m(q^{p^e})$ with $m \leq n$ and $e \in \mathbb{Z}$. Then G satisfies Quillen's conjecture.*

Recall that a group H satisfies the *Quillen dimension property at p* if $\tilde{H}_{m_p(H)-1}(\mathcal{A}_p(H)) \neq 0$. The presence of simple components of G isomorphic to $L_2(2^3)$ or $U_3(2^3)$ (in the $p = 3$ case) and $\text{Sz}(2^5)$ (in the $p = 5$ case) is an obstruction to extending Theorem 4.1 to $p = 3$ and $p = 5$. The case $p = 2$ is not considered in [AS93b] and would require a much more detailed analysis. One of the first steps in the proof of Theorem 4.1 is the reduction to the case $O_{p'}(G) = 1$ (see [AS93b, Proposition 1.6]). To do this, [AS93b, Theorems 2.3 and 2.4] are needed and these theorems make a strong use of the hypothesis $p > 5$. Concretely, it is not possible to apply [AS93b, Theorem 2.3] if a component of $C_G(O_{p'}(G))$ is isomorphic to $L_2(2^3)$, $U_3(2^3)$ (if $p = 3$) or $\text{Sz}(2^5)$ (if $p = 5$).

Before presenting the examples for $p = 3$ and $p = 5$, we give some motivation. Most of the groups G in these examples satisfy the following conditions. First, $O_{p'}(G) \neq 1$ and $C_G(O_{p'}(G))$ contains a component isomorphic to $U_3(2^3)$ if $p = 3$ and to $\text{Sz}(2^5)$ if $p = 5$. Thus, we cannot find nontrivial homology for $\mathcal{A}_p(G)$ in the same way it is done in the proof of [AS93b, Proposition 1.6] since we are not able to invoke [AS93b, Theorems 2.3 and 2.4]. Secondly, since there is an inclusion $\tilde{H}_*(\mathcal{A}_p(G/O_{p'}(G)); \mathbb{Q}) \hookrightarrow \tilde{H}_*(\mathcal{A}_p(G); \mathbb{Q})$ (see [AS93b, Lemma 0.12]), we

require $O_p(G/O_{p'}(G)) \neq 1$ so that $\tilde{H}_*(\mathcal{A}_p(G/O_{p'}(G))) = 0$. Finally, we require $O_p(G) = 1$.

The groups presented in Examples 4.5 and 4.7 have p -rank 3. The groups presented in Examples 4.6 and 4.8 have p -rank 4 and are constructed in the following way. We take a direct product of a group N , consisting of one or more copies of a particular simple p' -group, by a group K consisting of one or more copies of $L = U_3(2^3)$ if $p = 3$ or $L = \text{Sz}(2^5)$ if $p = 5$. Then we take two cyclic p -groups A and B and we let them act on the direct product $N \times K$ as follows. We take a faithful action of $A \times B$ on N , and we choose a representation $A \times B \rightarrow \text{Aut}(K)$ such that $O_p(K \rtimes (A \times B)) \cong O_p(C_A(K)) \neq 1$. The group $G = (N \times K) \rtimes (A \times B)$ satisfies the conditions $O_p(G) = 1$, $O_{p'}(G) = N \neq 1$, $C_G(N) = K$ and $O_p(G/N) = O_p(K \rtimes (A \times B)) \neq 1$. Moreover, since the p -rank of L is at most 2, we can construct G to have p -rank 4 by adjusting the number of copies of L in K .

For these groups we show that $\mathcal{K}(\mathcal{S}_p(G))$ has a 2-dimensional G -invariant subcomplex homotopy equivalent to itself, and thus Corollary 3.3 applies.

In Examples 4.10 and 4.11 we describe two groups of 2-rank 4 such that $\mathcal{K}(\mathcal{S}_2(G))$ admits a 2-dimensional G -invariant homotopy equivalent subcomplex.

For the claims on the structure of the automorphism group of the finite groups of Lie type we refer to [GLS98] and [GLS99].

Lemma 4.2. *If $G = H \rtimes K$ is a semidirect product then $m_p(G) \leq m_p(H) + m_p(K)$. Moreover if $A \leq K$ satisfies $m_p(A) = m_p(K)$ and $m_p(C_H(A)) = m_p(H)$ then equality is attained.*

Proof. Let A be an elementary abelian p -subgroup of G . Write $A = (A \cap H)A'$ for some complement A' of $A \cap H$ in A . Then $A' \cong A/A \cap H \cong AH/H$, which is isomorphic to a subgroup of K . Hence, $m_p(A') \leq m_p(K)$ and $m_p(A) = m_p(A \cap H) + m_p(A') \leq m_p(H) + m_p(K)$.

For the second part of the statement, take elementary abelian p -subgroups $B \leq A$ and $C \leq C_H(A)$ of p -ranks $m_p(K)$ and $m_p(H)$ respectively. Then CB is a direct product of elementary abelian p -subgroups and hence, $m_p(CB) = m_p(C) + m_p(B) = m_p(H) + m_p(K)$. \square

The following lemma will be used to obtain proper subcomplexes of $\mathcal{K}(\mathcal{A}_p(G))$ without changing the homotopy type. We write $X \simeq Y$ if the order complexes $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are homotopy equivalent.

Lemma 4.3. *Let G be a finite group and let $H \leq G$. In addition, suppose that each time $E \cap H = 1$ with $E \in \mathcal{A}_p(G)$, then $O_p(C_H(E)) \neq 1$. Then $\mathcal{A}_p(G) \simeq \mathcal{A}_p(H)$.*

Proof. Consider the subposet $\mathcal{N} = \{E \in \mathcal{A}_p(G) : E \cap H \neq 1\}$. We have order preserving maps $r: \mathcal{N} \rightarrow \mathcal{A}_p(H)$ and $i: \mathcal{A}_p(H) \hookrightarrow \mathcal{N}$ given by $r(E) = E \cap H$ and $i(E) = E$ such that $ir(E) \leq E$ and $ri(E) = E$. Therefore $\mathcal{N} \simeq \mathcal{A}_p(H)$.

Let $\mathcal{S} = \{E \in \mathcal{A}_p(G) : E \cap H = 1\}$ be the complement of \mathcal{N} in $\mathcal{A}_p(G)$. Now we take a linear extension E_1, \dots, E_r of \mathcal{S} (i.e. we enumerate the elements of \mathcal{S} so that $E_i \leq E_j$ implies $i \leq j$). Fix $E \in \mathcal{S}$ and consider $\mathcal{A}_p(G)_{>E} \cap \mathcal{N} = \{A \in \mathcal{N} : A > E\}$. It is easy to see that $r: \mathcal{A}_p(G)_{>E} \cap \mathcal{N} \rightarrow \mathcal{A}_p(C_H(E))$ defined by $r(B) = B \cap H$ is a homotopy equivalence with inverse $i(B) = BE$. Since $O_p(C_H(E)) \neq 1$, $\mathcal{A}_p(G)_{>E} \cap \mathcal{N} \simeq \mathcal{A}_p(C_H(E))$ is contractible.

Let $X^i = \mathcal{A}_p(G) - \{E_{i+1}, \dots, E_r\}$. We show that $X^i \hookrightarrow X^{i+1}$ is a homotopy equivalence for each $0 \leq i \leq r-1$.

Note that $X^{i+1} = X^i \cup \{E_i\}$. Moreover, $X_{>E_i}^{i+1} = \mathcal{A}_p(G)_{>E_i} \cap \mathcal{N}$, which is contractible by hypothesis. Therefore $X^i = X^{i+1} - \{E_i\} \hookrightarrow X^{i+1}$ is a homotopy equivalence. In consequence,

$$\mathcal{A}_p(G) = X^r \simeq X^0 = \mathcal{A}_p(G) - \mathcal{S} = \mathcal{N} \simeq \mathcal{A}_p(H).$$

□

Remark 4.4. In the above result it can be shown that if $H \triangleleft G$ then the homotopy equivalence is G -equivariant.

Example 4.5. Let $p = 3$ and let $L = L_2(2^3) \times L_2(2^3) \times L_2(2^3)$. Let A be a cyclic group of order 3 acting on L by permuting the copies of $L_2(2^3)$. Take $G = L \rtimes A$. Since $m_3(L_2(2^3)) = 1$ and $C_L(A) \cong L_2(2^3)$, we see that $m_3(G) = 3$. By Corollary 3.4, G satisfies Quillen's conjecture.

Example 4.6. Let $p = 3$, $N = \text{Sz}(2^3) \times \text{Sz}(2^3) \times \text{Sz}(2^3)$ and $U = U_3(2^3)$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be cyclic groups of order 3. We construct a semidirect product $G = (N \times U) \rtimes (A \times B)$. To do this we need to define a map $A \times B \rightarrow \text{Aut}(N \times U) = \text{Aut}(N) \times \text{Aut}(U)$.

Choose a field automorphism $\phi \in \text{Aut}(U_3(2^3))$ of order 3. By the properties of the p -group actions, there exists an inner automorphism $x \in \text{Inn}(U_3(2^3))$ of order 3 commuting with ϕ . Then $A \times B \rightarrow \text{Aut}(U_3(2^3))$ is given by $a \mapsto x$ and $b \mapsto \phi$. Choose a field automorphism $\psi \in \text{Aut}(\text{Sz}(2^3))$ of order 3. Let A act on each coordinate of N as ψ and

let B act on N by permuting its coordinates. This gives rise to a well defined map $A \times B \rightarrow \text{Aut}(N)$.

The 3-rank of G is $m_3(G) = m_3(U_3(2^3)AB)$. We can take an elementary abelian subgroup $E \leq C_U(\phi)$ of order 9 containing x since $C_U(\phi) \cong \text{PGU}_3(2) \cong ((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$ by [GLS99, Chapter 4, Lemma 3.10] and $\mathcal{A}_3(\text{PGU}_3(2))$ is connected of height 1. Then EAB is an elementary abelian subgroup of order 3^4 . Hence, $m_3(UAB) \geq 4$. Since $m_3(U_3(2^3)) = 2$ and $m_3(AB) = 2$, by Lemma 4.2 we have $m_3(G) = 4$.

By Corollary 3.3, to show that Quillen's conjecture holds for G and $p = 3$ it is enough to find a 2-dimensional G -invariant subcomplex X of $\mathcal{K}(\mathcal{S}_3(G))$ homotopy equivalent to $\mathcal{K}(\mathcal{S}_3(G))$ (or, equivalently, to $\mathcal{K}(\mathcal{A}_3(G))$).

Let $H = (N \times U) \rtimes A$. Note that $H \triangleleft G$ and $m_3(H) = 3$. Therefore, $\mathcal{K}(\mathcal{A}_3(H))$ is a 2-dimensional G -invariant subcomplex of $\mathcal{K}(\mathcal{A}_3(G))$. Now the plan is to use Lemma 4.3 to show that $\mathcal{A}_3(H) \simeq \mathcal{A}_3(G)$. Let $E \in \mathcal{A}_3(G)$ be such that $E \cap H = 1$. Then $E \cong EH/H \leq B \cong C_3$ and hence, E is cyclic generated by some element $e \in E$. Write $e = nua^i b^j$ with $n \in N$, $u \in U$ and $i, j \in \{0, 1, 2\}$. Note that $j \neq 0$ since $E \cap H = 1$. If $v \in U$, then

$$v^e = v^{nua^i b^j} = (v^{ua^i})^{b^j}.$$

Since $j \neq 0$ and e induces an automorphism of U of order 3 in $\text{Inn}(U)\phi^j$, by [GLS98, Proposition 4.9.1] and the definition of field automorphisms [GLS98, Definition 2.5.13], e is $\text{Inndiag}(U)$ -conjugate to ϕ^j and acts as a field automorphism on U . In particular, $C_U(E) = C_U(e) \cong C_U(\phi^j) = C_U(\phi)$. Note that $O_3(C_U(E)) \cong O_3(C_U(\phi)) \cong C_3 \times C_3 \neq 1$. Since $C_U(E) \triangleleft C_H(E)$ and $O_3(C_U(E)) \neq 1$, we conclude that $O_3(C_H(E)) \neq 1$. By Lemma 4.3, $\mathcal{A}_3(G) \simeq \mathcal{A}_3(H)$, which is 2-dimensional and G -invariant. In conclusion, the subcomplex $\mathcal{K}(\mathcal{A}_3(H))$ satisfies the hypothesis of Corollary 3.3 and therefore, Quillen's conjecture holds for G .

Note that $O_3(G) = 1$, $O_{3'}(G) = N$, $C_G(O_{3'}(G)) = U_3(2^3)$ and $O_3(G/O_{3'}(G)) = O_3(U_3(2^3)AB) = \langle ax^{-1} \rangle \cong C_3$.

Example 4.7. Let $p = 5$. Let r be a prime number such that $r \equiv 2$ or $3 \pmod{5}$ and let $q = r^{5^n}$ with $n \geq 2$. Let N be one of the simple groups $L_2(q)$, $G_2(q)$, ${}^3D_4(q^3)$ or ${}^2G_2(3^{5^n})$ and let $A = \langle a \rangle$ be a cyclic group of order 5^n . Note that $5 \nmid |N|$. Let a act on N as a field automorphism of order 5^n . Choose a field automorphism $\phi \in \text{Aut}(\text{Sz}(2^5))$ of order 5 and let A act on $\text{Sz}(2^5) \times \text{Sz}(2^5)$ as $\phi \times \phi$. Now consider the semidirect product $G = (N \times \text{Sz}(2^5) \times \text{Sz}(2^5)) \rtimes A$ defined by this action.

Since the Sylow 5-subgroups of $\text{Sz}(2^5)$ are cyclic of order 25, by Lemma 4.2 we have that $m_5(G) = 3$. By Corollary 3.4, Quillen's conjecture holds for G .

Moreover, $O_5(G) = 1$, $O_{5'}(G) = N$, $C_G(O_{5'}(G)) = \text{Sz}(2^5)^2$ and $O_5(G/O_{5'}(G)) = C_A(\text{Sz}(2^5)^2) = \langle a^5 \rangle \neq 1$.

Example 4.8. Let $p = 5$ and let $N = L^5$, where L is one of the simple $5'$ -groups of the previous example. Let $A = \langle a \rangle \cong C_{5^n}$ and $B = \langle b \rangle \cong C_5$. Let $G = (N \times \text{Sz}(2^5)^2) \rtimes (A \times B)$, where a acts on each copy of L as a field automorphism of order 5^n and trivially on $\text{Sz}(2^5)^2$, and b permutes the copies of L and acts as a field automorphism of order 5 on each copy of $\text{Sz}(2^5)$.

To compute the 5-rank of G we use Lemma 4.2:

$$\begin{aligned} m_5(G) &= m_5(\text{Sz}(2^5)^2 \rtimes (A \times B)) \\ &= m_5(A \times (\text{Sz}(2^5)^2 \rtimes B)) \\ &= m_5(A) + m_5(\text{Sz}(2^5)^2 \rtimes B) \\ &= 1 + 3 \\ &= 4. \end{aligned}$$

Now the aim is to apply Corollary 3.3 on G by finding a 2-dimensional G -invariant homotopy equivalent subcomplex X of $\mathcal{K}(\mathcal{S}_5(G))$.

Let $H = (N \times \text{Sz}(2^5)^2) \rtimes A = NA \times \text{Sz}(2^5)^2$. Note that $H \triangleleft G$ and $m_5(H) = 3$. Hence $\mathcal{K}(\mathcal{A}_5(H))$ is 2-dimensional and G -invariant. We will show that $\mathcal{A}_5(H) \simeq \mathcal{A}_5(G)$ by applying Lemma 4.3.

Let $E \in \mathcal{A}_5(G)$ be such that $E \cap H = 1$. Then E is cyclic generated by an element e of order 5 and $e = lsa^i b^j$ with $l \in N$, $s \in \text{Sz}(2^5)^2$, $0 \leq i \leq 5^n - 1$ and $j \in \{1, 2, 3, 4\}$. Thus E acts by field automorphisms on each copy of the Suzuki group and e is $\text{Inndiag}(\text{Sz}(2^5))$ -conjugate to the field automorphism induced by b^j on $\text{Sz}(2^5)$ (see [GLS98, Proposition 4.9.1] and Example 4.6). Hence, $C_H(E) = C_{NA}(E) \times C_{\text{Sz}(2^5)^2}(E)$. Note that $C_{\text{Sz}(2^5)^2}(E) \triangleleft C_H(E)$ and $C_{\text{Sz}(2^5)^2}(E) \cong C_{\text{Sz}(2^5)}(E)^2 \cong (C_5 \rtimes C_4)^2$ has a nontrivial normal 5-subgroup. Therefore $\mathcal{A}_5(G) \simeq \mathcal{A}_5(H)$ by Lemma 4.3 and Quillen's conjecture holds for G by Corollary 3.3 applied to the subcomplex $\mathcal{K}(\mathcal{A}_5(H))$.

Note that $O_{5'}(G) = N$ and $C_G(O_{5'}(G)) = \text{Sz}(2^5)^2$. On the other hand, $O_5(G) = 1$ and $O_5(G/O_{5'}(G)) = A \neq 1$.

We conclude with two examples of groups satisfying Quillen's conjecture for $p = 2$. We say that a finite group G has the *trivial intersection*

property at p if any two different Sylow p -subgroups of G have trivial intersection.

Proposition 4.9. *Let L_1 and L_2 be two finite groups with the trivial intersection property at p . Let $L = L_1 \times L_2$ and take an extension G of L such that $|G : L| = p$. Then $\mathfrak{i}(\mathcal{S}_p(G))$ and $\mathcal{B}_p(G)$ are at most 2-dimensional. If in addition the Sylow p -subgroups of L_1 and L_2 have abelian Ω_1 , then $\mathfrak{i}(\mathcal{A}_p(G))$ is at most 2-dimensional.*

Proof. The elements of $\mathfrak{i}(\mathcal{S}_p(L))$ are of the form $P_1 \times P_2$, $1 \times P_2$ or $P_1 \times 1$, where $P_i \leq L_i$ are Sylow p -subgroups. Hence, $\mathfrak{i}(\mathcal{S}_p(L))$ is 1-dimensional.

Now suppose that $Q_0 < Q_1 < \dots < Q_n$ is a chain in $\mathfrak{i}(\mathcal{S}_p(G))$. Then

$$Q_0 \cap L \leq Q_1 \cap L \leq \dots \leq Q_n \cap L$$

is a chain in $\mathfrak{i}(\mathcal{S}_p(L))$. We claim that there is at most one index i such that $Q_i \cap L = Q_{i+1} \cap L$. To see this note that

$$|Q_j : Q_j \cap L| = \begin{cases} 1 & \text{if } Q_j \subseteq L \\ p & \text{if } Q_j \not\subseteq L \end{cases}.$$

We have $|Q_{i+1} : Q_i| \cdot |Q_i : Q_i \cap L| = |Q_{i+1} : Q_{i+1} \cap L| \cdot |Q_{i+1} \cap L : Q_i \cap L|$. Then if $Q_i \cap L = Q_{i+1} \cap L$, since $|Q_{i+1} : Q_i| \geq p$ we must have $|Q_i : Q_i \cap L| = 1$ and $|Q_{i+1} : Q_{i+1} \cap L| = p$. Then $i = \max\{j : Q_j \subseteq L\}$.

From this we conclude that $\dim \mathfrak{i}(\mathcal{S}_p(G)) \leq 1 + \dim \mathfrak{i}(\mathcal{S}_p(L)) = 2$. It is well-known that $\mathcal{B}_p(G)$ is a subposet of $\mathfrak{i}(\mathcal{S}_p(G))$ (i.e. every radical p -subgroup is an intersection of Sylow p -subgroups). Then $\mathcal{B}_p(G)$ is at most 2-dimensional also. The same proof can be easily adapted to prove that, if the Sylow p -subgroups of L_1 and L_2 have abelian Ω_1 , $\mathfrak{i}(\mathcal{A}_p(G))$ is at most 2-dimensional. \square

In the following examples we use the fact that the groups A_5 and $U_3(2^2)$ have the trivial intersection property at 2 and that $\Omega_1(P)$ is abelian for P a Sylow 2-subgroup of either A_5 or $U_3(2^2)$.

Example 4.10. Let G be the group extension $(A_5 \times A_5) \rtimes C_2$ where the generator of C_2 acts on each coordinate as conjugation by the transposition (12). Then by Lemma 4.2, G has 2-rank 4. By Proposition 4.9, $\mathfrak{i}(\mathcal{A}_2(G))$, $\mathfrak{i}(\mathcal{S}_2(G))$ and $\mathcal{B}_2(G)$ are 2-dimensional and then Quillen's conjecture holds for G since Corollaries 3.5 and 3.6 apply.

Example 4.11. Let $G = (U_3(2^2) \times A_5) \rtimes C_2$ be the semidirect product constructed in the following way. Let $L = U_3(2^2) \times A_5$. Then $\text{Out}(L) \cong \text{Aut}(U_3(2^2))/\text{Inn}(U_3(2^2)) \times \text{Aut}(A_5)/\text{Inn}(A_5) \cong C_4 \times C_2$. Take

$t \in \text{Out}(L)$ to be the involution which acts nontrivially on both factors. Then $G = L \rtimes \langle t \rangle$. By Lemma 4.2 G has 2-rank 4, and just as before, Quillen's conjecture holds for G .

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