Fixed points on contractible spaces

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Assume from now on that G is a **finite** group.

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- (P.-Smith '22, P. '23) For $p = 2$, idem but for the linear, unitary, symplectic and orthogonal groups.

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- This yields the lists of the previous theorems.

Back to discs: Smith theory and Oliver's classification

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Theorem (Oliver '75)

A group G admits a fixed point free action on a disc if and only if $G \notin \mathcal{G}$.

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- The group A_5 acts **without fixed points** on the 2-skeleton X_P of the Poincaré homology 3-sphere, which is an **acyclic** 2-**complex**.
- Thus A_5 acts without fixed points on $A_5 * X_P$, which is a compact contractible 3-dimensional (simplicial) complex.

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A finite group G acting on a compact contractible 2-complex has a fixed point.

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A major work by Oliver-Segev '02 completely classifies the groups that can act without fixed points on acyclic 2-dimensional complexes.

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Then X^N is acyclic and the action of G/N on X^N is essential.

Fixed points and compact contractible 2-complexes

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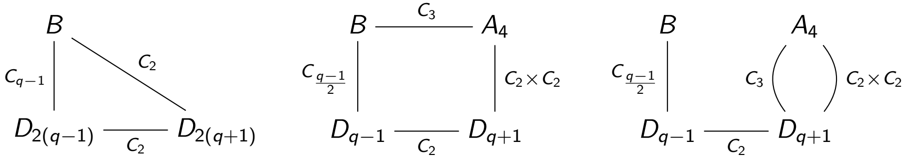

$\mathrm{PSL}_2(2^n)$ $\quad$ $\mathrm{PSL}_2(3^n)$, odd n $\quad$ $\mathrm{PSL}_2(q), q \equiv 11 \pmod{24}$.

Similar diagrams for $\mathrm{PSL}_2(q)$ with $q \equiv 5, 13, 19 \pmod{24}$ and $\mathrm{Sz}(2^{2n+1})$.

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Goal

Show that f is homotopic to a non-surjective map, so $\deg f = 0$.

Thank you very much!