Posets associated to vector spaces with non-degenerate forms

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Basic preliminaries

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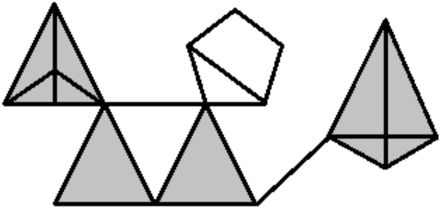

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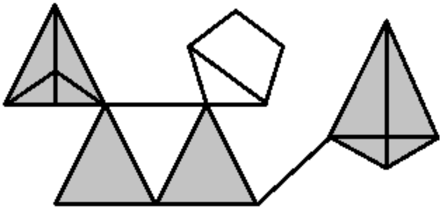

A topological space Y is a wedge of spheres if

$$Y \simeq \mathbb{S}^{n_1} \vee \mathbb{S}^{n_2} \vee \ldots \vee \mathbb{S}^{n_k},$$

where $\mathbb{S}^j$ is the sphere of dimension j.

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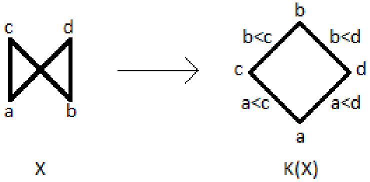

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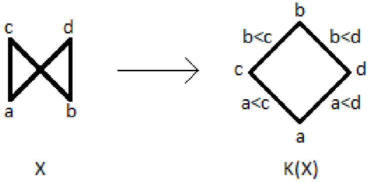

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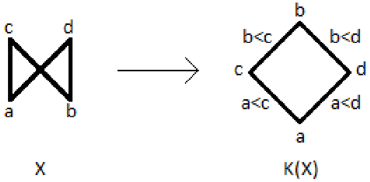

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3. We say that X is a wedge of spheres if $\mathcal{K}(X)$ is.

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If R is a ring, and we replace $(d-1)$-connected above by $\widetilde{H}_m(K, R) = 0$ for all $m \leq d-1$, then we can define spherical over R and Cohen-Macaulay over R.

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4. $\widetilde{H}_{n-2}(\mathring{\mathrm{T}}(V), \mathbb{Z})$ gives rise to the "Steinberg module" of $\mathrm{GL}_n(\mathbb{K})$.

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- Moreover, we assume that Ψ is non-degenerate:

$$V^{\perp} = \mathrm{Rad}(V, \Psi) = \{v \in V : \Psi(v, w) = 0 \text{ for all } w \in V\} = 0.$$

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3. Unitary forms are unique: $|\mathbb{K}| = q^2$, $\tau(x) = x^q$ and

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We want to understand the Cohen-Macaulay property on these posets.

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Question

For which fields and forms Ψ is $\mathrm{TI}(V)$ Cohen-Macaulay?

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Corollary

$\mathcal{S}(V)$ is Cohen-Macaulay (over R) $\Leftrightarrow$ $\mathcal{D}(V)$ is Cohen-Macaulay (over R).

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Note that

$$\dim(\mathcal{F}(V)) = \dim(\mathcal{D}(V)){\color{red}+1} = \dim(\mathcal{S}(V)){\color{red}+1}.$$

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If $\hat{\mathcal{F}}(V)$ is Cohen-Macaulay (over R) then $\mathcal{S}(V)$ and $\mathcal{D}(V)$ are too.

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Question

If $|\mathbb{K}| \geq 4^2$, $n = 4$, is $\mathcal{F}(V)$ simply connected?

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- $L(G) = \mathrm{Id} - D^{-1}A$ is the normalised Laplacian.

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We apply Garland's method to the frame complex $\mathcal{F}(V)$ when $\mathbb{K} = \mathbb{F}_{q^2}$ is a finite field:

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Key idea

We show that there are at most 4 eigenvalues by proving that the minimal polynomial has degree at most 4.

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- Compute the powers A, A^2, A^3, A^4 of the adjacency matrix $A = A(n,q)$ of $\mathcal{G}(V)$.

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(iv) *If $n > 3$ and $q = 2$, the eigenvalues are μ_1, μ_3 and $(-1)^n \mu_4$.*

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$$P_j(q) = \begin{cases} \frac{q^{j-1}-(-1)^{j-1}}{q+1} + j - 1 & j \text{ odd if } q = 2, \\ \frac{q^j+q}{q+1} + j - 1 & q = 2 \text{ and } j \text{ even.} \end{cases}$$

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Corollary

If $n < P_j(q)$ then $\widetilde{H}_i(\mathcal{F}(V), \mathbb{Q}) = 0$ for all $i \leq n - j$.

Higher connectivity for $H_*(\mathcal{F}(V), \mathbb{Q})$.

Recall that $\mathcal{F}(V) \simeq \hat{\mathcal{F}}(V)$, of dimension $n-2$.

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Let $X = \hat{\mathcal{F}}(V)$, $\mathcal{S}(V)$ or $\mathcal{D}(V)$, and $\Bbbk$ a field of characteristic 0.

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Corollary

If $n < q+1$ then $\hat{\mathcal{F}}(V)$, $\mathcal{S}(V)$ and $\mathcal{D}(V)$ are Cohen-Macaulay over $\Bbbk$.

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Failure of Cohen-Macaulayness

If $q(q-1)+1 \leq n \leq q(q(q-1)+1)$ then

$$\widetilde{H}_{n-3}(\hat{\mathcal{F}}(V), \mathbb{Q}) \neq 0 \quad \text{and} \quad \tilde{\chi}(\hat{\mathcal{F}}(V))(-1)^{n-2} < 0.$$

Thus $\hat{\mathcal{F}}(V)$ is not Cohen-Macaulay (not even over some ring R) if $n \geq q(q-1)+1$.

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1. Is X homotopically Cohen-Macaulay if $n < q + 1$?

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5. If $|\mathbb{K}| \geq 4^2$, is $\mathcal{D}(V)$ Cohen-Macaulay?

Thank you very much for your attention!