The frame complex of a vector space with a Hermitian form

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Under these conditions, we say that (V, Ψ) is a unitary space of dimension n over K.

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If $K = \mathbb{F}_r$, then $r = q^2$ is a square and $\sigma(x) = x^q$. Moreover, (V, Ψ) is uniquely determined:

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We work with the topology of their order complexes.

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Orthogonal decompositions

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- $\mathcal{D}(V) =$ poset of non-trivial orthogonal decompositions of V. Order given by refinement: $\pi \leq \pi'$ if π is finer than π'.

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3. Then the face-poset of $\mathcal{F}(V)$ is homotopy equivalent to

$$\hat{\mathcal{F}}(V) = \{ \text{ frames of size } \neq n - 1\},$$

with $\dim \hat{\mathcal{F}}(V) = n - 2$.

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4. If $(n, K) = (6, \mathbb{F}_{2^2})$ then $\pi_1\big(\mathcal{F}(V)\big) = C_2 \times C_2$.

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Question

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- $L(G) = \mathrm{Id} - D^{-1}A$ is the normalised Laplacian.

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In particular, the smallest non-zero eigenvalue of $L(\mathcal{G}(V))$ is $1 - d_n^{-1} q^{n-2}$.

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Corollary

Similar results are transferred to $\mathcal{S}(V)$ and $\mathcal{D}(V)$ via the poset map

$$\mathcal{F}(V)^{n-2} \longrightarrow \mathcal{S}(V), \quad \{V_1, V_2, \dots, V_r\} \mapsto V_1 \oplus V_2 \oplus \dots \oplus V_r,$$

and the wedge relation between $\mathcal{S}(V)$ and $\mathcal{D}(V)$.

¡Muchas gracias!