A categorical approach to study posets of decompositions into subobjects

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We will mainly work with bounded posets of finite height and look for *highly-connectedness* properties such as sphericity or the Cohen-Macaulayness.

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Direct sum decompositions, partial basis complexes, frame complexes, etc.

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The general poset-approach

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1. If $G = F_n$ is the free group of rank n, we would like to take $\mathcal{S} = \mathcal{S}(F_n)$ the poset of free factors and $\mathcal{D}(\mathcal{S}(F_n))$ the poset of decompositions into free factors.
2. If M is a f.g. free module over a PID, then $\mathcal{S}(M)$ should be the poset of direct summands and $\mathcal{D}(\mathcal{S}(M))$ the poset of direct sum decompositions of M.
3. If V is a vector space with a non-degenerate sesquilinear form Ψ, then $\mathcal{S}(V, \Psi)$ should be the poset of non-degenerate subspaces, and $\mathcal{D}(\mathcal{S}(V, \Psi))$ the poset of orthogonal decompositions of V.

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- If the unordered version is Cohen-Macaulay then the ordered version is (the converse also holds for $\mathcal{F},\mathcal{PB}$).

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2. (Hatcher-Vogtmann) $\mathcal{D}(F_n)$ is Cohen-Macaulay of dimension $n-1$.
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In particular, $\mathcal{OD}(F_n)$, $\mathcal{F}(F_n, *)$, $\mathcal{OF}(F_n, *)$ and $\mathcal{OPB}(F_n)$ are Cohen-Macaulay of dimension $n-1$.

Back to examples: free groups

Let F_n be the free group of rank n.

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Question. What is the homotopy type of $\mathcal{PD}(F_n)^*$?

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Question. What is the behaviour of $\hat{\mathcal{F}}((V,\Psi),\sqcup)$ and $\mathcal{PB}(V,\Psi)$ for infinite fields?

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Thank you very much!